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MULTIPLICATIVE PERTURBATION BY CONTRACTIONS AND UNIFORM STABILITY

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Abstract. Let T be an arbitrary bounded linear transformation of a Hilbert space into itself. We investigate classes of contractions S for which the spectral radius $r(ST)$ of the product ST is less than one. The main result gives a collection of necessary and sufficient conditions for $r(ST) < 1$ when T is multiplicatively perturbed by compact contractions S. We also give either necessary or sufficient conditions for perturbation by other classes of Hilbert space contractions, such as those that include the symmetries (e.g., involutions, unitary operators, self-adjoint, normal and normaloid contractions) or the orthogonal projections (e.g., nonnegative contractions).

1. Introduction

A Hilbert space operator T (i.e., a bounded linear transformation of a Hilbert space into itself) is uniformly stable if the power sequence $\{T^n\}$ converges uniformly (or converges in the uniform topology, or in the operator norm topology) to the null operator (i.e., if $\|T^n\| \to 0$). The term "stable operator" is reminiscent of discrete-time dynamical systems: a discrete time-invariant free bounded linear system modeled by the following autonomous homogeneous difference equation

$$x_{n+1} = Tx_n, \quad \text{with } x_0 = x, \quad \text{for every integer } n \geq 0$$

is uniformly stable if the Hilbert-space-valued state sequence $\{x_n\}$ converges to zero uniformly for all initial conditions x. That is, if $\sup_{\|x\|=1} \|T^n x\| \to 0$, which means that $\|T^n\| \to 0$. In this case, both the above (linear) model and the (linear) operator T are said to be uniformly (asymptotically) stable.

Uniform stability for infinite-dimensional systems has been under the spotlight for more than three decades (for finite-dimensional systems many more decades are to be added on the top of it). Necessary and sufficient conditions for uniform stability in an infinite-dimensional setup can be found in, for instance, [15], [10], [14] and the references therein. More recently, still motivated by (asymptotic) stability of discrete-time (linear) systems, attention has been directed to multiplicative perturbations that stabilize the system. This may be synthesized by the question: given a model as above, which class of operators S is able to make the system

$$x_{n+1} = STx_n, \quad \text{with } x_0 = x, \quad \text{for every integer } n \geq 0$$

uniformly stable? In other words, which class of operators S ensures uniform stability for ST (i.e., ensures that $\|(ST)^n\| \to 0$)? In an infinite-dimensional setting, this problem has been investigated in [4] and [2], where the original operator T is

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perturbed by some familiar classes of operators S. Generalizations of the finite-dimensional results from [1] were considered in [3], where specific convex sets of operators have been considered.

The present paper addresses multiplicative perturbations. Notational preliminaries are considered in Section 2. Section 3 focuses on sup properties for operators perturbed by contractions, which provide the main tools for proving multiplicative perturbation results. There we consider the classes of plain, nonstrict, strict and uniformly stable contractions (Theorem 1), the class of partial isometries (Theorem 2), and the classes of finite-rank or compact contractions (Theorem 3). The classes of symmetries, orthogonal projections, and nonnegative operators are also considered among other classes of Hilbert space perturbing contractions. Uniform stability for multiplicatively perturbed operators (perturbed by those classes of contractions investigated in Section 3) are obtained in Section 4. The paper closes with a collection of necessary and sufficient conditions for compact perturbation (Corollary 7), after exhibiting a counterexample on the perturbation by symmetries (Remark 3).

2. Preliminaries

Notation, terminology and basic results are posed in this section. By basic results we mean well-known standard propositions from single operator theory that will be required in the sequel. These are summarized below in order to settle the pertinent definitions and recall the necessary elementary facts only. Proofs of all stated assertions (on single operator theory) can be found in current literature (see e.g., [5], [8] or [11] among other sources).

Let $\langle\;;\;\rangle$ denote the inner product in a complex infinite-dimensional (not necessarily separable) Hilbert space $\mathcal{H}$. By a subspace of $\mathcal{H}$ we mean a closed linear manifold of $\mathcal{H}$, and by an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. For any operator T on $\mathcal{H}$ put $\mathcal{N}(T) = T^{-1}\{0\}$ (the kernel or null space of T, which is a subspace of $\mathcal{H}$) and $\mathcal{R}(T) = T(\mathcal{H})$ (the range of T, which is a linear manifold of $\mathcal{H}$). Let $\mathcal{B}[\mathcal{H}]$ be the unital Banach algebra of all operators on $\mathcal{H}$ and let $T^* \in \mathcal{B}[\mathcal{H}]$ stand for the adjoint of $T \in \mathcal{B}[\mathcal{H}]$. The same notation $\|\;\|$ is used for the norm on $\mathcal{H}$ and for the induced (uniform) norm on $\mathcal{B}[\mathcal{H}]$. An invertible element from $\mathcal{B}[\mathcal{H}]$ is an operator T with an inverse in $\mathcal{B}[\mathcal{H}]$ which, by the Inverse Mapping Theorem, means that T is injective and surjective (i.e., $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) = \mathcal{H}$).

A self-adjoint is an operator T on $\mathcal{H}$ for which $T^* = T$. A self-adjoint operator Q is nonnegative ($O \le Q$), positive ($O < Q$) or strictly positive ($O \prec Q$) if $0 \le \langle Qx\,;x\rangle$ for every x in $\mathcal{H}$, $0 < \langle Qx\,;x\rangle$ for every nonzero x in $\mathcal{H}$, or $\alpha\|x\|^2 \le \langle Qx\,;x\rangle$ for every x in $\mathcal{H}$ and some positive α, respectively — i.e., $Q \in \mathcal{B}[\mathcal{H}]$ is strictly positive if and only if it is positive and invertible, which means that it has a (bounded) strictly positive inverse in $\mathcal{B}[\mathcal{H}]$. Recall that every nonnegative operator Q in $\mathcal{B}[\mathcal{H}]$ has a unique nonnegative square root $Q^{\frac{1}{2}}$ in $\mathcal{B}[\mathcal{H}]$ and $\|Q^{\frac{1}{2}}\|^2 = \|Q\|$. Also recall that T^*T is a nonnegative operator and $\|T\|^2 = \|T^*T\|$ for every $T \in \mathcal{B}[\mathcal{H}]$ (standard notation: $|T| = (T^*T)^{\frac{1}{2}}$ so that $\|\,|T|\,\| = \|T\|$).

A contraction is an operator T on $\mathcal{H}$ such that $\|T\| \le 1$ (i.e., $\|Tx\| \le \|x\|$ for every x in $\mathcal{H}$; equivalently, $T^*T \le I$). If $\|Tx\| < \|x\|$ for every nonzero x in $\mathcal{H}$ (equivalently, if $T^*T < I$), then T is a proper contraction. A strict contraction is an operator T such that $\|T\| < 1$ (i.e., $\sup_{\|x\|=1}\|Tx\| < 1$ or, equivalently, $T^*T \prec I$, which means that $T^*T \leq \alpha I$ for some $\alpha \in (0,1)$). These are related by proper inclusion: Strict Contraction $\subset$ Proper Contraction $\subset$ Contraction. A nonstrict contraction is a contraction T with $\|T\| = 1$ and a nonproper contraction is a contraction T for with $\|Tx\| = \|x\|$ for some nonzero x in $\mathcal{H}$ (equivalently, for some x in $\mathcal{H}$ with $\|x\| = 1$). Every nonproper contraction is nonstrict. A nonnegative (or positive, or strictly positive) contraction is precisely an operator Q on $\mathcal{H}$ such that $O \leq Q \leq I$ (or $O \leq Q < I$, or $O \leq Q \prec I$), where I stands for the identity of $\mathcal{B}[\mathcal{H}]$.

An isometry on $\mathcal{H}$ is an operator V such that $\|Vx\| = \|x\|$ for every x in $\mathcal{H}$ or, equivalently, such that $V^*V = I$. Every isometry is injective. A coisometry is an operator whose adjoint is an isometry. A unitary operator is an invertible isometry or, equivalently, a surjective isometry, which means that an operator U on $\mathcal{H}$ is unitary if and only if it is invertible and $U^* = U^{-1}$ or, still equivalently, if and only if it is an isometry and also a coisometry. An orthogonal projection P on $\mathcal{H}$ is an idempotent operator (i.e., $P^2 = P$) whose range and kernel are orthogonal to each other, which is precisely a nonnegative idempotent (i.e., $O \leq P = P^2$); equivalently, a self-adjoint idempotent (i.e., $P^* = P = P^2$). An involution on $\mathcal{H}$ is an invertible operator J that coincides with its inverse (i.e., $J = J^{-1}$), which means that $J^2 = I$. A partial isometry is an operator W on $\mathcal{H}$ that acts isometrically on the orthogonal complement of its kernel (i.e., $\|Wx\| = \|x\|$ for every x in $\mathcal{N}(W)^\perp$). These are nonstrict contractions: isometries (and coisometries), unitaries, (nonzero) orthogonal projections, involutions and (nonzero) partial isometries all have norm equal to 1 (i.e., $\|V\| = \|U\| = \|P\| = \|J\| = \|W\| = 1$). Moreover, these operators also have a closed range (i.e., their ranges are subspaces of $\mathcal{H}$). A symmetry S is a unitary involution or, equivalently, a self-adjoint involution or, still equivalently, a self-adjoint unitary (i.e., $S^* = S = S^{-1}$), which is a nonstrict contraction with closed range as well. It is readily verified that S is a symmetry (i.e., S is a self-adjoint involution) if and only if $S = I - 2P$ for some orthogonal projection P (i.e., for some self-adjoint idempotent P).

Let $\rho(T) = \big\{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ has an inverse in } \mathcal{B}[\mathcal{H}]\big\}$ be the resolvent set of an operator T on $\mathcal{H}$. Its complement $\sigma(T) = \mathbb{C}\backslash\rho(T)$ is the spectrum of T and the spectral radius is the nonnegative number $r(T) = \sup_{\lambda\in\sigma(T)} |\lambda| = \max_{\lambda\in\sigma(T)} |\lambda|$. The Gelfand–Beurling formula says that $r(T) = \lim_n \|T^n\|^{\frac{1}{n}}$. If $r(T) = 0$, then T is called quasinilpotent. The numerical radius $w(T) = \sup_{\|x\|=1} |\langle Tx\,; x\rangle|$ is a positive number ($0 < w(T)$ if $T \neq O$, defining another norm in $\mathcal{B}[\mathcal{H}]$) with $w(T^*T) = \|T\|$. Spectral radius, numerical radius and the induced (uniform) norm are related as follows: $0 \leq r(T) \leq w(T) \leq \|T\| \leq 2w(T)$ for every T in $\mathcal{B}[\mathcal{H}]$. An operator T is normaloid if $r(T) = \|T\|$, and spectraloid if $r(T) = w(T)$. Every normaloid is spectraloid and T is normaloid if and only if $w(T) = \|T\|$. Among the normaloid are the self-adjoint and, in particular, the nonnegative operators so that $r(T^*T) = \|T\|^2$, and hence $r(|T|) = \|T\|$. An operator T is uniformly stable if $\lim_n \|T^n\| = 0$, which is precisely an operator $T \in \mathcal{B}[\mathcal{H}]$ with $r(T) < 1$. The point spectrum is the subset $\sigma_P(T) = \big\{\lambda \in \mathbb{C}\colon\ \mathcal{N}(\lambda I - T) \neq \{0\}\big\}$ of $\sigma(T)$ consisting of all eigenvalues of T.

3. Sup Properties

Recall that $r(ST) = r(TS)$ for every pair of operators T, S in $\mathcal{B}[\mathcal{H}]$ but, in general, $r(ST) \not\leq r(S)\,r(T)$ [7, p.43,48]. However, $r(ST) \leq \|ST\| \leq \|S\|\,\|T\|$ for every

T, S in $\mathcal{B}[\mathcal{H}]$. In particular, $r(ST) \leq \|T\|$ for every operator T and every contraction S. Thus $\sup_{S\in\mathcal{S}} r(ST) \leq \|T\|$ for every $T \in \mathcal{B}[\mathcal{H}]$ and every class of contractions $\mathcal{S}$.

Definition 1. A class $\mathcal{S}$ of contractions in $\mathcal{B}[\mathcal{H}]$ has the *norm sup property* if

$$\sup_{S\in\mathcal{S}} r(ST) = \|T\|$$

for every operator T in $\mathcal{B}[\mathcal{H}]$.

The purpose of the first part of this section is to investigate norm sup property by assuming that the perturbing operator S belongs to some specific classes of contractions (plain, strict, nonstrict and uniformly stable contractions; symmetries; involutions; unitary operators; self-adjoint, normal and normaloid contractions; partial isometries). In particular, by assuming that the perturbing operator S belongs to the classes of finite-rank or compact contractions. No assumption is imposed on the operator T.

Theorem 1. *The classes of all nonstrict contractions and of all contractions in $\mathcal{B}[\mathcal{H}]$ have the norm sup property. In fact, more is true. For every $T \in \mathcal{B}[\mathcal{H}]$,*

$$\max_{\|S\|=1} r(ST) = \max_{\|S\|\leq 1} r(ST) = \|T\|.$$

Moreover, the classes of all strict contractions and of all uniformly stable contractions in $\mathcal{B}[\mathcal{H}]$ also have the norm sup property.

Proof. Nonstrict contractions are considered in part (a), contractions are considered in part (b), strict contractions in part (c), and uniformly stable contractions in part (d). Since the results are all trivial for $T = O$, take an arbitrary $T \neq O$ in $\mathcal{B}[\mathcal{H}]$.

(a) Put $S_1 = \|T\|^{-1}T^*$ in $\mathcal{B}[\mathcal{H}]$, which is a nonstrict contraction ($\|S_1\| = 1$), and get $\|T\| = \|T\|^{-1}r(T^*T) = r(S_1T)$. Thus the identity $\max_{\|S\|=1} r(ST) = \|T\|$ follows at once because $r(ST) \leq \|T\|$ for every contraction S.

(b) According to (a) we get $\|T\| = \max_{\|S\|=1} r(ST) \leq \max_{\|S\|\leq 1} r(ST) \leq \|T\|$.

(c) Take an arbitrary $\alpha \in (0, 1)$, put $S_\alpha = \alpha\|T\|^{-1}T^*$, which is a strict contraction ($\|S_\alpha\| = \alpha < 1$), and get $r(S_\alpha T) = \alpha\|T\|^{-1}r(T^*T) = \alpha\|T\|$. Therefore,

$$\|T\| = \sup_{\alpha\in(0,1)} \alpha\|T\| = \sup_{\alpha\in(0,1)} r(S_\alpha T) \leq \sup_{\|S\|<1} r(ST) \leq \|T\|.$$

(d) This follows by items (b) and (c):

$$\|T\| = \sup_{\|S\|<1} r(ST) \leq \sup_{r(S)<1\,,\,\|S\|\leq 1} r(ST) \leq \sup_{\|S\|\leq 1} r(ST) \leq \|T\|. \qquad \square$$

There are smaller classes of contractions that also have the norm sup property. For instance, the class of symmetries, a proper subclass of nonstrict contractions. The argument of the proof below is similar to that in [4, Lemmas 4.11 and 4.12].

Proposition 1. *The class of all symmetries has the norm sup property.*

Proof. Take an arbitrary operator T in $\mathcal{B}[\mathcal{H}]$ and an arbitrary unit vector x in $\mathcal{H}$ (i.e., x in $\mathcal{H}$ such that $\|x\| = 1$). If $Tx = 0$, then $0 = \|Tx\| \in \sigma_P(T)$ — i.e., $\|Tx\|$ is an eigenvalue of T — and the trivial symmetry $S_x = I$ is such that

$$\|Tx\| \le r(S_xT).$$

Now suppose $Tx \neq 0$ and put $y_x = \|Tx\|^{-1}Tx$ in $\mathcal{H}$, which also is a unit vector. If $y_x = \gamma x$ for some γ in the unit circle (i.e., $\gamma \in \mathbb{C}$ with $|\gamma| = 1$), then $Tx = \gamma\|Tx\|x$ so that $\gamma\|Tx\| \in \sigma_P(T)$ and again, for the trivial symmetry $S_x = I$,

$$\|Tx\| \le r(S_xT).$$

If $y_x \neq \gamma x$ for all γ in the unit circle, then put $\gamma_x = |\langle y_x\,;x\rangle|^{-1}\langle y_x\,;x\rangle$ if $\langle y_x\,;x\rangle \neq 0$ or $\gamma_x = 1$ otherwise, which lies in the unit circle. Set $u_x = y_x - \gamma_x x \neq 0$, and consider the unit vector $e_x = \|u_x\|^{-1}u_x$ in $\mathcal{H}$. Let P_{e_x} in $\mathcal{B}[\mathcal{H}]$ be the (rank-one) orthogonal projection whose range is the one-dimensional space spanned by $\{e_x\}$ (i.e., $P_{e_x}z = \langle z\,;e_x\rangle e_x$ for every $z \in \mathcal{H}$), and consider the symmetry $S_x = I - 2P_{e_x}$. Set $v_x = y_x + \gamma_x x$. Since $\gamma_x = |\langle y_x\,;x\rangle|^{-1}\langle y_x\,;x\rangle$ if $\langle y_x\,;x\rangle \neq 0$, we get $v_x \perp u_x$ and so $v_x \perp e_x$. This ensures that $2S_x y_x = S_x u_x + S_x v_x = v_x - u_x = 2\gamma_x x$, and hence $\gamma_x x = S_x y_x = \|Tx\|^{-1}S_xTx$ so that $\gamma_x\|Tx\| \in \sigma_P(S_xT)$. Therefore S_x is such that

$$\|Tx\| \le r(S_xT).$$

Outcome: *For every unit vector $x \in \mathcal{H}$ there is a symmetry $S_x \in \mathcal{B}[\mathcal{H}]$ for which $\|Tx\| \le r(S_xT)$.* Thus (with $\mathcal{S}$ denoting the class of all symmetries from $\mathcal{B}[\mathcal{H}]$),

$$\|T\| \;=\; \sup_{\|x\|=1}\|Tx\| \;\le\; \sup_{\|x\|=1} r(S_xT) \;\le\; \sup_{S\in\mathcal{S}} r(ST) \;\le\; \|T\|. \qquad \square$$

Remark 1. *The classes of all involutions, of all unitaries, of all self-adjoint nonstrict contractions, and of all self-adjoint contractions from $\mathcal{B}[\mathcal{H}]$ also have the norm sup property.* Indeed, recall that a symmetry is precisely a unitary involution, which coincides with the class of all self-adjoint involutions, which in turn coincides with the class of all self-adjoint unitary operators. Since symmetries have the norm sup property, and since $r(ST) \le \|T\|$ for all contractions S, it follows that every class of contractions that contains the symmetries also has the norm sup property.

Remark 2. If the classes of all self-adjoint contractions and of all unitary operators from $\mathcal{B}[\mathcal{H}]$ have the norm sup property, then so has every class of contractions that includes any of these classes. For instance, *the classes of all normal nonstrict contractions and of all normal contractions* (and so any class of contractions including them, such as *the classes of all normaloid nonstrict contractions* ($r(S) = \|S\| = 1$) *and of all normaloid contractions* ($r(S) = \|S\| \le 1$) *also have the norm sup property.* Similarly, *the classes of all isometries, and of all coisometries have the norm sup property.*

Since the class of isometries is included in the class of partial isometries (an isometry is precisely an injective partial isometry), it follows that the class of all partial isometries also has the norm sup property. However, for partial isometries the norm sup property holds with "sup" strengthened to "max".

Theorem 2. *The class of all partial isometries from $\mathcal{B}[\mathcal{H}]$ has the norm sup property. In fact, more is true. If $\mathcal{S}$ stands for the class of all partial isometries from $\mathcal{B}[\mathcal{H}]$ then, for every $T \in \mathcal{B}[\mathcal{H}]$,*

$$\max_{S\in\mathcal{S}} r(ST) = \|T\|.$$

Proof. Take an arbitrary T in $\mathcal{B}[\mathcal{H}]$ and let $T = W|T|$ be its polar decomposition, where W is a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$ is nonnegative, both in $\mathcal{B}[\mathcal{H}]$. Recall that $|T| = W^*T$. Now put $S = W^*$ and also recall that W^* is a partial isometry whenever W is. Thus $r(ST) = r(W^*T) = r(|T|) = \|T\|$ and we get the claimed result once partial isometries are contractions and $r(ST) \le \|T\|$ whenever S is a contractions for every operator T. □

It is worth noticing that the *norm max property*, as in Theorems 1 and 2, was recently investigated for Banach space operators in [9], where it was shown that if $\mathcal{X}_n$ are finite-dimensional normed spaces, then every operator on the Banach space $\big(\bigoplus_n \mathcal{X}_n\big)_p,\ p > 1$, has the norm max property attained for a nonstrict contraction.

An operator is finite-rank if its range is finite-dimensional and compact if it maps bounded sets into relatively compact sets. Since every finite-rank operator in $\mathcal{B}[\mathcal{H}]$ is compact, it follows that if finite-rank contractions have the norm sup property, then so do the compact contractions (because plain contractions have it). The next result deals with finite-rank (and compact) contractions. Note that the symmetry $I - 2P_{e_x}$ in the proof of Proposition 1 is never finite-rank (P_{e_x} is a finite-rank projection and the identity I is not even compact on an infinite-dimensional space).

Theorem 3. *The class of all finite-rank contractions has the norm sup property, and so does the class of all compact contractions.*

Proof. Take any T in $\mathcal{B}[\mathcal{H}]$ and an arbitrary unit vector x in $\mathcal{H}$. If $Tx = 0$, then for any finite-rank contraction (actually, for any operator) S_x in $\mathcal{B}[\mathcal{H}]$,

$$\|Tx\| \le r(S_xT).$$

If $Tx \ne 0$, then put $y_x = \|Tx\|^{-1}Tx$. If $y_x = \gamma x$ for some γ in the unit circle, then set $S_x = P_{y_x}$ in $\mathcal{B}[\mathcal{H}]$, the orthogonal projection whose range is the one-dimensional space spanned by $\{y_x\}$, which is a rank-one nonnegative contraction. Thus $S_xTx = \langle Tx\,;y_x\rangle y_x = Tx = \|Tx\|y_x = \gamma\|Tx\|x$ so that $\gamma\|Tx\|$ lies in $\sigma_P(S_xT)$, and therefore

$$\|Tx\| \le r(S_xT).$$

If $y_x \ne \gamma x$ for all γ in the unit circle, then put $\gamma_x = |\langle y_x\,;x\rangle|^{-1}\langle y_x\,;x\rangle$ if $\langle y_x\,;x\rangle \ne 0$ or $\gamma_x = 1$ otherwise, which lies in the unit circle. Set $u_x = y_x - \gamma_x x \ne 0$ and $v_x = y_x + \gamma_x x \ne 0$, and put $e_x = \|u_x\|^{-1}u_x$ and $f_x = \|v_x\|^{-1}v_x$. Let P_{e_x} and P_{f_x} be the orthogonal projections whose ranges are the one-dimensional spaces spanned by $\{e_x\}$ and $\{f_x\}$, respectively ($P_{e_x}z = \langle z\,;e_x\rangle e_x$ and $P_{f_x}z = \langle z\,;f_x\rangle f_x$ for every $z \in \mathcal{H}$). Set $S_x = P_{f_x} - P_{e_x}$ in $\mathcal{B}[\mathcal{H}]$. Since $u_x \perp v_x$, and so $e_x \perp f_x$, it follows that P_{e_x} and P_{f_x} are mutually orthogonal (i.e., $P_{e_x}P_{f_x} = P_{f_x}P_{e_x} = O$). This implies that $S_x^2 = P_{f_x} + P_{e_x}$, the orthogonal projection whose range is the two-dimensional space spanned by $\{e_x, f_x\}$. However, S_x itself is just a rank-two (self-adjoint, but not nonnegative) contraction. Indeed, since P_{e_x} and P_{f_x} are mutually orthogonal, $\|S_xz\|^2 = \|P_{f_x}z\|^2 + \|P_{e_x}z\|^2 \le \|z\|^2$ for every $z \in \mathcal{H}$. Mutual orthogonality also implies that $2S_xy_x = S_xu_x + S_xv_x = v_x - u_x = 2\gamma_x x$, and hence $\gamma_x x = S_x y_x = \|Tx\|^{-1}S_xTx$ so that $\gamma_x\|Tx\| \in \sigma_P(S_xT)$. Therefore,

$$\|Tx\| \le r(S_xT).$$

Outcome: *For every unit vector x in $\mathcal{H}$ there is a finite-rank contraction S_x such that $\|Tx\| \le r(S_xT)$.* Thus (with $\mathcal{S}$ denoting the class of all finite-rank contractions from $\mathcal{B}[\mathcal{H}]$) it follows that

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \le \sup_{\|x\|=1} r(S_xT) \le \sup_{S\in\mathcal{S}} r(ST) \le \|T\|. \qquad \square$$

Norm sup property will be strengthened to numerical radius sup property below. Now we assume that the perturbing operators S belong to the classes of orthogonal projections, nonnegative, positive and strictly positive contractions. As before, no assumption is imposed on the operator T.

Definition 2. A class $\mathcal{S}$ of contractions from $\mathcal{B}[\mathcal{H}]$ has the *numerical radius sup property* if

$$\sup_{S\in\mathcal{S}} r(ST) = w(T)$$

for every operator T in $\mathcal{B}[\mathcal{H}]$.

The argument in parts (b) and (c) of the proof below are similar to those in the infinite-dimensional versions of [4, Lemmas 4.4 and 4.6].

Proposition 2. *The classes of all orthogonal projections, of all nonnegative contractions, of all positive contractions, and of all strictly positive contractions have the numerical radius sup property.*

Proof. Take an arbitrary T in $\mathcal{B}[\mathcal{H}]$. We shall show in part (a) that $w(T) \le r(PT)$ for all orthogonal projections P in $\mathcal{B}[\mathcal{H}]$, in part (b) that $w(T) \le r(QT)$ for all strictly positive contractions Q in $\mathcal{B}[\mathcal{H}]$ and, in part (c), that $r(QT) \le w(T)$ for all nonnegative contractions Q in $\mathcal{B}[\mathcal{H}]$. Thus

$$w(T) \le \sup_{O\le P=P^2} r(PT) \le \sup_{O\le Q\le I} r(QT) \le w(T),$$

$$w(T) \le \sup_{O\prec Q\le I} r(QT) \le \sup_{O< Q\le I} r(QT) \le \sup_{O\le Q\le I} r(QT) \le w(T).$$

(a) Take an arbitrary unit vector x in $\mathcal{H}$ and consider the (unique) orthogonal projection P_x in $\mathcal{B}[\mathcal{H}]$ whose range is the one-dimensional space spanned by $\{x\}$; that is, $P_xy = \langle y\,;x\rangle x$ for every $y \in \mathcal{H}$ so that $P_xTx = \langle Tx\,;x\rangle x$. Thus $\langle Tx\,;x\rangle \in \sigma_P(P_xT)$ — i.e., $\langle Tx\,;x\rangle$ is an eigenvalue of P_xT — and therefore $|\langle Tx\,;x\rangle| \le r(P_xT)$. Hence,

$$w(T) = \sup_{\|x\|=1} |\langle Tx\,;x\rangle| \le \sup_{\|x\|=1} r(P_xT) \le \sup_{O\le P=P^2} r(PT).$$

(b) Consider the setup of item (a) where x is an arbitrary unit vector in $\mathcal{H}$. Set $Q_x(n) = \frac{n}{n+1}P_x + \frac{1}{n+1}I$ in $\mathcal{B}[\mathcal{H}]$ for each positive integer n. $\{Q_x(n)\}$ is a sequence of strictly positive contractions ($O \prec \frac{1}{n+1}I \le Q_x(n) \le I$) such that $Q_x(n) \to P_x$ in the uniform topology ($\|Q_x(n) - P_x\| \le \frac{2}{n+1}$) and hence $Q_x(n)T \to P_xT$. Since the spectrum of P_xT is totally disconnected (P_xT is a rank-one operator because P_x is, and so $\sigma(P_xT)$ is finite), it follows that P_xT is a point of spectral continuity. Then the above uniform convergence implies that $r(Q_x(n)T) \to r(P_xT)$ [7, p.57]. Therefore, since $\langle Tx\,;x\rangle \in \sigma_P(P_xT)$,

$$|\langle Tx\,;x\rangle| \le r(P_xT) = \lim_n r(Q_x(n)T) \le \sup_{O\prec Q\le I} r(QT).$$

Since this holds for all unit vectors x in $\mathcal{H}$, we get

$$w(T) = \sup_{\|x\|=1} |\langle Tx\,;x\rangle| \le \sup_{O\prec Q\le I} r(QT).$$

(c) Take an arbitrary nonnegative contraction Q in $\mathcal{B}[\mathcal{H}]$. If $r(QT)=0$, then $r(QT)\leq w(T)$ trivially. Thus suppose $r(QT)\neq 0$ and let $R=Q^{\frac{1}{2}}$ be its unique nonnegative square root (which is again a contraction) so that $r(QT)=r(R^2T)=r(RTR)$. Since the spectrum of any operator in $\mathcal{B}[\mathcal{H}]$ is compact, there exists a λ in $\sigma(QT)$ such that $|\lambda|=r(QT)=r(RTR)$. This λ lies in the boundary of $\sigma(RTR)$, and hence in the approximate point spectrum of RTR, which means that there is a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $(\lambda I-RTR)x_n\to 0$. Therefore,

$$\lambda=\lim_n\langle RTRx_n\,;x_n\rangle$$

because $|\lambda-\langle RTRx_n\,;x_n\rangle| = |\langle(\lambda I-RTR)x_n\,;x_n\rangle| \leq \|(\lambda I-RTR)x_n\|$ (since $\|x_n\|=1$), which in turn implies that

$$\liminf_n\|Rx_n\|\neq 0$$

(reason: if $\liminf_n\|Rx_n\|=0$, then $\lim_n\langle RTRx_n\,;x_n\rangle=0$ so that $\lambda=0$, which contradicts the assumption $r(QT)\neq\{0\}$). Thus $0<\|Rx_n\|$ for every $n\geq n_0$, for some integer n_0. Since R is self-adjoint, $\|Rx_n\|^{-2}\langle RTRx_n\,;x_n\rangle=\langle Tu_n\,;u_n\rangle$ where each $u_n=\|Rx_n\|^{-1}Rx_n$ also is a unit vector in $\mathcal{H}$, for each $n\geq n_0$. Hence, recalling that $\|Rx_n\|\leq 1$ (because R is a contraction), we get

$$\begin{aligned} r(QT)=|\lambda| \;&=\; \lim_n|\langle RTRx_n\,;x_n\rangle|\leq\limsup_n\|Rx_n\|^2\,|\langle Tu_n\,;u_n\rangle| \\ &\leq\;\limsup_n|\langle Tu_n\,;u_n\rangle|\;\leq\;\sup_{\|x\|=1}|\langle Tx\,;x\rangle|=w(T). \end{aligned}$$ □

Definition 3. A class $\mathcal{S}$ of contractions from $\mathcal{B}[\mathcal{H}]$ has the *spectral radius sup property* if

$$\sup_{S\in\mathcal{S}} r(ST)=r(T)$$

for every operator T in $\mathcal{B}[\mathcal{H}]$.

Let $\{T\}'$ denote the commutant of an operator $T\in\mathcal{B}[\mathcal{H}]$ (the unital subalgebra of $\mathcal{B}[\mathcal{H}]$ consisting of all operators that commute with T). A direct consequence of the Gelfand–Beurling formula for the spectral radius is that $r(ST)\leq r(S)r(T)$ whenever $S\in\{T\}'$ (see e.g., [7, p.48]). Recall that a scalar operator is a multiple of the identity (i.e., an operator $S=\alpha I$ on $\mathcal{H}$ for some $\alpha\in\mathbb{C}$). The class of all scalar operators is a unital subalgebra of the commutant of *every* operator T, and it is trivially verified that *the class of all scalar contractions from $\mathcal{B}[\mathcal{H}]$ has the spectral radius sup property.* Indeed, for every $T\in\mathcal{B}[\mathcal{H}]$,

$$\max_{S=\alpha I,\,|\alpha|\leq 1} r(ST) \;=\; r(T).$$

Does there exist another class of contractions (not a subclass of scalar contractions) that has the spectral radius sup property? Note that the class of all quasinilpotent (in particular, of all nilpotent) operators from $\mathcal{B}[\mathcal{H}]$ does not have any of the sup properties of Definitions 1, 2 and 3. Indeed, if $T=I\in\mathcal{B}[\mathcal{H}]$, then $r(T)=w(T)=\|T\|=1$ and $r(ST)=r(S)=0$ for all quasinilpotent operator $S\in\mathcal{B}[\mathcal{H}]$.

4. Uniform Stability

Recall that an operator T in $\mathcal{B}[\mathcal{H}]$ is uniformly stable if the power sequence $\{T^n\}$ converges uniformly to the null operator (i.e., if $\|T^n\| \to 0$). There is a myriad of equivalent conditions for uniform stability (see e.g., [10] and the references therein); among them, $\|T^n\| \to 0$ if and only if $r(T) < 1$: an operator is uniformly stable if an only if its spectrum is included in the open unit disc.

The theorems were proved in the previous section. Now we harvest the corollaries by providing necessary and/or sufficient conditions for ST to be uniformly stable when T is perturbed by operators S belonging to those classes of contractions considered in Section 3.

Corollary 1. *Take S, T in $\mathcal{B}[\mathcal{H}]$. The following assertions are equivalent.*

(a) *ST is uniformly stable for every contraction S.*

(b) *ST is uniformly stable for every nonstrict contraction S.*

(c) *ST is uniformly stable for every partial isometry S.*

(d) *T is a strict contraction.*

Proof. Let $\mathcal{S}$ be any class of contractions such that

$$\max_{S\in\mathcal{S}} r(ST) = \|T\|$$

for every operator T in $\mathcal{B}[\mathcal{H}]$. If $T \in \mathcal{B}[\mathcal{H}]$, then

$$r(ST) < 1 \;\; \forall\, S \in \mathcal{S} \quad \iff \quad \|T\| < 1.$$

Therefore, according to Theorems 1 and 2, it follows that (a), (b), (c) and (d) are pairwise equivalent. □

Corollary 2. *Let S and T be operators in $\mathcal{B}[\mathcal{H}]$. The assertion*

(d) *T is a strict contraction*

implies each of the assertions below.

(e) *ST is uniformly stable for every S in any class of contractions that includes all strict contractions.*

(f) *ST is uniformly stable for every S in any class of contractions that includes all finite-rank contractions.*

(g) *ST is uniformly stable for every S in any class of contractions that includes all symmetries.*

Moreover, each of the above assertions implies that

(h) *T is a contraction.*

Proof. Let $\mathcal{C}$ be the class of all contractions in $\mathcal{B}[\mathcal{H}]$. Let $\mathcal{S}$ be any class of contractions that includes a class $\mathcal{S}_0$ of contractions that has the norm sup property. That is, $\mathcal{S}_0 \subseteq \mathcal{S} \subseteq \mathcal{C}$ where $\|T\| = \sup_{S\in\mathcal{S}_0} r(ST)$. Since $\sup_{S\in\mathcal{S}} r(ST) \leq \|T\|$,

$$\sup_{S\in\mathcal{S}_0} r(ST) = \sup_{S\in\mathcal{S}} r(ST) = \|T\|$$

for every operator T in $\mathcal{B}[\mathcal{H}]$. Thus take any $T \in \mathcal{B}[\mathcal{H}]$ and observe that

$$\|T\|<1 \;\Leftrightarrow\; \sup_{S\in\mathcal{S}} r(ST)<1 \;\Rightarrow\; r(ST)<1 \;\;\forall\, S\in\mathcal{S} \;\Rightarrow\; \sup_{S\in\mathcal{S}} r(ST)\leq 1 \;\Leftrightarrow\; \|T\|\leq 1.$$

Therefore, (d) implies that ST is uniformly stable for every S in every class $\mathcal{S}$ that includes $\mathcal{S}_0$, which implies that ST is uniformly stable for every S in some class $\mathcal{S}$ that includes $\mathcal{S}_0$, which in turn implies (h). In particular, according to Theorems 1, 3 and Proposition 1, this holds whenever $\mathcal{S}_0$ is the class of all strict contractions, the class of all symmetries, or the class of all finite-rank contractions. □

Corollary 3. *Take S, T in $\mathcal{B}[\mathcal{H}]$. The following assertions are equivalent.*

(i) *ST is uniformly stable for every strict contraction S.*

(h) *T is a contraction.*

Proof. If $\|T\| \leq 1$ and $\|S\| < 1$, then $r(ST) \leq \|ST\| \leq \|S\|\,\|T\| < \|T\| \leq 1$ so that (h) implies (i). By Corollary 2 (e) implies (h). In particular (i) implies (h). □

Corollary 4. *Let S and T be operators in $\mathcal{B}[\mathcal{H}]$. The assertion*

(f) *ST is uniformly stable for every S in any class of contractions that includes all finite-rank contractions*

implies that

(j) *T is a proper contraction.*

Proof. By Corollary 2 (f) implies (h). Thus T is a contraction ($\|T\| \leq 1$) if (f) holds. The outcome in the proof of Theorem 3 says that for every unit vector x in $\mathcal{H}$ there is a finite-rank contraction S_x in $\mathcal{B}[\mathcal{H}]$ for which $\|Tx\| \leq r(S_xT)$. If the contraction T is not proper, then there is a unit vector u in $\mathcal{H}$ such that $1 = \|u\| = \|Tu\|$, and so $1 \leq r(S_uT) < 1$, which is a contradiction: $r(S_xT) < 1$ for every unit vector x if (f) holds. Thus (j): T is a proper contraction. □

Corollary 5. *Let S and T be operators in $\mathcal{B}[\mathcal{H}]$. The assertion*

(g) *ST is uniformly stable for every S in any class of contractions that includes all symmetries*

implies that

(k) *$w(T) < 1$ and T is a proper contraction.*

Proof. Suppose assertion (g) holds. Since the identity I is a symmetry, it follows that $r(T) < 1$. Moreover, $\|T\| \leq 1$ because (g) implies (h) in Corollary 2. The outcome in the proof of Proposition 1 reads as follow: for every unit vector x in $\mathcal{H}$ there is a symmetry S_x in $\mathcal{B}[\mathcal{H}]$ for which $\|Tx\| \leq r(S_xT)$. But $r(S_xT) < 1$ for every unit vector x if (g) holds. If the contraction T is not proper, then there is a unit vector u in $\mathcal{H}$ such that $1 = \|u\| = \|Tu\|$, and hence $1 \leq r(S_uT) < 1$, which is a contradiction. Thus T is a proper contraction. Recall that $w(T) \leq 1$ (because $\|T\| \leq 1$). If $w(T) = 1$, then $w(T) = \|T\| = 1$, which implies that T is normaloid so that $r(T) = \|T\| = 1$; another contradiction (since $r(T) < 1$). Therefore, $w(T) < 1$. Thus (g) implies (k). □

Remark 3. Assertion (k) says that T is either a strict contraction or a non-normaloid nonstrict proper contraction. This is a necessary condition for uniform stability of multiplicative perturbation by symmetries. However, assertion (k) is not sufficient. In other words, according to Corollary 5 the assertion

(l) ST is uniformly stable for every symmetry S

implies (k) but, as we shall see next, (k) does not imply (l). Indeed, we shall exhibit a nonnormaloid nonstrict proper contraction T and a symmetry S such that $r(ST)=1$. Take $T_0=\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}-1 & -1\\ 1 & 1\end{smallmatrix}\right)$ and $S_0=\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ in $\mathcal{B}[\mathbb{C}^2]$, where S_0 is a symmetry, T_0 is a quasinilpotent nonstrict contraction ($r(T_0)=0$ and $\|T_0\|=1$), and $S_0T_0=\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ is a nonnegative (thus normaloid) nonstrict contraction,

$$r(S_0T_0)=w(S_0T_0)=\|S_0T_0\|=\|T_0\|=1.$$

Then $w(T_0)\neq\|T_0\|$ (otherwise T_0 would be normaloid) so that

$$0=r(T_0)<w(T_0)<w(S_0T_0)=\|T_0\|=1.$$

Now consider the direct sums $T=\bigoplus_{n=1}^{\infty}\frac{n}{n+1}T_0$ and $S=\bigoplus_{n=1}^{\infty}S_0$ on $\mathcal{H}=\ell_+^2(\mathbb{C}^2)$, the Hilbert space made up of all square summable $\mathbb{C}^2$-valued sequences. Clearly, S is a symmetry and $\|T\|=\sup_{n\geq 1}\big\|\frac{n}{n+1}T_0\big\|=1$. Note that $ST=\bigoplus_{n=1}^{\infty}\frac{n}{n+1}S_0T_0$ is again a nonnegative nonstrict contraction,

$$r(ST)=w(ST)=\|ST\|=\|T\|=1.$$

Claim: $r(T)=0$. In fact, take $n\geq 1$ and $\lambda\neq 0$ arbitrary. Since $\sigma\big(\frac{n}{n+1}T_0\big)=\{0\}$ we get $\lambda\in\rho\big(\frac{n}{n+1}T_0\big)$, and so $\big(\lambda I-\frac{n}{n+1}T_0\big)$ has a bounded inverse, namely,

$$\big(\lambda I-\tfrac{n}{n+1}T_0\big)^{-1}=\tfrac{n}{\sqrt{2}\,(n+1)\lambda^2}\begin{pmatrix}\frac{\sqrt{2}\,(n+1)\lambda}{n}-1 & -1\\ 1 & \frac{\sqrt{2}\,(n+1)\lambda}{n}+1\end{pmatrix}$$

(since $A^{-1}=\frac{1}{\beta\alpha^2}\left(\begin{smallmatrix}\alpha-1 & -1\\ 1 & \alpha+1\end{smallmatrix}\right)$ whenever $A=\beta\left(\begin{smallmatrix}\alpha+1 & 1\\ -1 & \alpha-1\end{smallmatrix}\right)$ for any nonzero α and β). Observe that $\big\|\big(\lambda I-\frac{n}{n+1}T_0\big)^{-1}\big\|\leq\frac{n}{\sqrt{2}\,(n+1)|\lambda|^2}\big(\frac{\sqrt{2}\,(n+1)|\lambda|}{n}+2\big)<\frac{|\lambda|+\sqrt{2}}{|\lambda|^2}$ and hence $\sup_{n\geq 1}\big\|\big(\lambda I-\frac{n}{n+1}T_0\big)^{-1}\big\|<\infty$. This implies that $\lambda I-T=\bigoplus_{n=1}^{\infty}\big(\lambda I-\frac{n}{n+1}T_0\big)$ also has a *bounded* inverse $(\lambda I-T)^{-1}=\bigoplus_{n=1}^{\infty}\big(\lambda I-\frac{n}{n+1}T_0\big)^{-1}$. Thus $\lambda\in\rho(T)$, and therefore T is quasinilpotent: $\sigma(T)=\{0\}$. Again, since T is a quasinilpotent nonstrict contraction, $w(T)\neq\|T\|$ so that

$$0=r(T)<w(T)<w(ST)=\|T\|=1.$$

Finally, we verify that T is a proper (nonstrict) contraction. Actually, for an arbitrary nonzero $x=\{x_n\}_{n=1}^{\infty}$ in $\ell_+^2(\mathbb{C}^2)$,

$$\|Tx\|^2=\sum_{n=1}^{\infty}\big\|\tfrac{n}{n+1}T_0\,x_n\big\|^2<\sum_{n=1}^{\infty}\|T_0\,x_n\|^2\leq\sum_{n=1}^{\infty}\|x_n\|^2=\|x\|^2.$$

Then T satisfies (k) but (l) fails once $r(ST)=1$ for the symmetry S.

What is behind the above example is the fact that the numerical radius $w(ST)$ of a product ST may be larger than the product $w(T)\|S\|$. (For more on numerical radius properties see e.g., [6], [7] and [8].)

Remark 4. If $w(T)<1$, then $r(ST)<1$ for every S in any class of contractions that has the numerical radius sup property. Thus, by Proposition 2,

(m) $w(T)<1$

implies that

(n) ST is uniformly stable for every orthogonal projection S,

(o) ST is uniformly stable for every nonnegative contraction S,

(p) ST is uniformly stable for every positive contraction S,

(q) ST is uniformly stable for every strictly positive contraction S,

and each of these, in turn, implies that

(r) $w(T) \leq 1$ and T is uniformly stable.

Indeed, if any of (n) to (q) holds, then $w(T) \leq 1$ by Proposition 2 and, since the identity is an orthogonal projection (thus nonnegative) as well as a strictly positive (thus positive) contraction, it also follows that $r(T) < 1$. However, unlike the case of Remark 3, the necessary condition (r) is sufficient for uniform stability of multiplicative perturbation by strictly positive contractions. This is a result from [2] which we summarize below for completeness and for the reader's convenience.

Corollary 6. [2] *Take S, T in $\mathcal{B}[\mathcal{H}]$. The following assertions are equivalent.*

(q) *ST is uniformly stable for every strictly positive contraction S.*

(r) *$w(T) \leq 1$ and T is uniformly stable.*

Proof. [2] We saw in Remark 4 that (q) implies (r). Conversely, suppose (r) holds for an operator T. If (q) fails for this T, then there exists an operator Q such that $O \prec Q \leq I$ and $r(QT) = 1$. Consider the setup in the proof of Proposition 2, part (c), where $R = Q^{\frac{1}{2}}$ (so that $O \prec R \leq I$), $\lambda \in \sigma(QT)$ with $|\lambda| = r(QT) = 1$, and

$$(\lambda I - RTR)x_n \to 0 \quad \text{which implies} \quad \langle RTRx_n\,; x_n\rangle \to \lambda$$

for a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ (see proof of Proposition 2). Write

$$(*) \qquad \begin{aligned} \|(\lambda I - T)Rx_n\| &= \|R^{-1}\big[\lambda(Q - I) + (\lambda I - RTR)\big]x_n\| \\ &\leq \|R^{-1}\|\,\big(\|(I - Q)x_n\| + \|(\lambda I - RTR)x_n\|\big). \end{aligned}$$

If $w(T) \leq 1$, then $|\langle Tx\,; x\rangle| \leq \|x\|^2$ for every nonzero x in $\mathcal{H}$, and so

$$|\langle RTRx_n\,; x_n\rangle| = |\langle TRx_n\,; Rx_n\rangle| \leq \|Rx_n\|^2 \leq \|R\| \leq 1.$$

Recall that $|\langle RTRx_n\,; x_n\rangle| \to |\lambda| = 1$. Thus $\|Rx_n\| \to 1$ by above inequality, which in turn implies that $\|(I - Q)x_n\| \to 0$ because

$$\begin{aligned} \|(I - Q)x_n\|^2 &= \|x_n\|^2 - 2\,\mathrm{Re}\,\langle Qx_n,\ x_n\rangle + \|Qx_n\|^2 \\ &\leq 1 - 2\|Rx_n\|^2 + \|R\|^2\|Rx_n\|^2 \leq 1 - \|Rx_n\|^2. \end{aligned}$$

Since $\|(I - Q)x_n\| \to 0$ and $\|(\lambda I - RTR)x_n\| \to 0$, we get $\|(\lambda I - T)Rx_n\| \to 0$ according to $(*)$. Set $u_n = \|Rx_n\|^{-1}Rx_n$, which is again a unit vector in $\mathcal{H}$. Since $\|Rx_n\| \to 1$, it also follows that $\|(\lambda I - T)u_n\| \to 0$. This means that λ lies in the approximate point spectrum of T, and hence $\lambda \in \sigma(T)$ so that $1 = |\lambda| \leq r(T)$, which is a contradiction ($r(T) < 1$ if (r) holds). Thus (r) implies (q). □

We close the paper with a collection of necessary and sufficient conditions for uniform stability of multiplicative perturbations by compact contractions. Recall that an operator T in $\mathcal{B}[\mathcal{H}]$ is strongly stable if the power sequence $\{T^n\}$ converges strongly to the null operator (i.e., $\|T^n x\| \to 0$ for every $x \in \mathcal{H}$); and weakly stable if $\{T^n\}$ converges weakly to the null operator (i.e., if $\langle T^n x\,;y\rangle \to 0$ for every $x, y \in \mathcal{H}$ or, equivalently, $\langle T^n x\,;x\rangle \to 0$ for every $x \in \mathcal{H}$ since $\mathcal{H}$ is a complex Hilbert space). Uniform stability implies strong stability, which implies weak stability. The next result ensures that *ST is uniformly stable for every compact contraction S if and only if T is a proper contraction.*

Corollary 7. *Take S, T in $\mathcal{B}[\mathcal{H}]$. The following assertions are equivalent.*

(s) *ST is weakly stable for every compact contraction S.*
(t) *ST is uniformly stable for every compact contraction S.*
(u) *Eigenvalues of ST lie in the open unit disc for every compact contraction S.*
(j) *T is a proper contraction.*
(v) *ST is a proper contraction for every compact contraction S.*
(w) *ST is a strict contraction for every compact contraction S.*

Proof. Since the class of all compact operators from $\mathcal{B}[\mathcal{H}]$ is a two-sided ideal of $\mathcal{B}[\mathcal{H}]$, it follows that ST is compact for every operator T whenever S is compact. However, the concepts of weak, strong and uniform stabilities coincide for a compact operator on a complex Hilbert space (see e.g., [12, p.80]). Therefore ST is uniform stable if and only if ST is weakly stable whenever S is compact contraction so that (s) and (t) are equivalent. Moreover, $\sigma(ST)\backslash\{0\} = \sigma_P(ST)\backslash\{0\}$ whenever ST is compact (Fredholm Alternative) so that $\sigma(ST) \subseteq \mathbb{D}$ if and only if $\sigma_P(ST) \subseteq \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc (centered about the origin of the complex plane). Hence (t) and (u) are equivalent. That (t) implies (j) is a particular case of Corollary 4 (which is a consequence of Theorem 3), since every finite-rank operator in $\mathcal{B}[\mathcal{H}]$ is compact. But if (j) holds, then ST is a proper contraction for every contraction S, and hence the compact contraction ST is, in fact, a strict contraction: the concepts of proper and strict contractions coincide for compact operators [13]. Thus (v) and (w) are equivalent assertions, which are implied by (j). Finally, (w) implies (t) since $r(ST) \leq \|ST\|$. □

Note that the above equivalent assertions imply their finite-rank counterpart, which are still equivalent according to the same argument.

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