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ON POSINORMAL OPERATORS

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Abstract. After presenting a survey on posinormal operators we consider two classical problems restricted to this class of operators. Since transitive operators are quasiinvertible and since invertible operators are posinormal, we give a unique factorization for invertible transitive operators, and prove a characterization for transitive totally hereditarily normaloid contractions with a compact defect operator. Moreover, since dominant operators are posinormal, we give conditions for dominant operators to satisfy Weyl's theorem, and show that these conditions are tight enough. It is also exhibited counterexamples to three incorrect statements of current literature on posinormal operators.

1. Introduction

Throughout this paper the term operator means a bounded linear transformation of a Hilbert space into itself. Posinormal operators where introduced in [18] as the class of operators T for which $TT^* = T^*QT$ for some nonnegative operator Q. It was noticed then that this was a very large class that includes the hyponormal as well as all invertible operators.

The purpose of this paper is twofold. First we present a comprehensive view on posinormal operators in Section 2, 3 and 4. Equivalent conditions for posinormality are considered in Section 2, while several subclasses of posinormal operators and their connections with subclasses of normaloid operators are discussed in detail in Sections 3 and 4. Next we focus on two leading problems in Sections 5 and 6, namely, the invariant subspace problem and Weyl's theorem for subclasses of posinormal operators. Since transitive operators (i.e., operators without a nontrivial invariant subspace) are quasiinvertible and since invertible operators are posinormal, we give a unique factorization for invertible transitive operators in Theorem 1, and prove in Theorem 2 and Corollary 1 that if a transitive totally hereditarily normaloid contraction has a compact defect operator, then it is either of class $\mathcal{C}_{00}$ or of class $\mathcal{C}_{10}$, and has no supercyclic vector. (An operator is called totally hereditarily normaloid if every part of it is normaloid, as well as the inverse of their invertible parts — a class that includes the paranormal operators.) Moreover, Weyl's theorem is investigated for a subclass of posinormal operators in Theorem 3, where we prove that Weyl's theorem holds for a dominant operator if every part of it is transaloid. This condition on the parts of a dominant operator is weakened in Corollary 2, and we also show that it is tight enough by giving examples and counterexamples.

2. Posinormal Operators

Let $\mathcal{H}$ be a complex Hilbert space of dimension greater than 1. By a subspace of $\mathcal{H}$ we mean a closed linear manifold of $\mathcal{H}$, and by an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. Let $\mathcal{B}[\mathcal{H}]$ be the algebra of all

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operators on $\mathcal{H}$. For any operator $T \in \mathcal{B}[\mathcal{H}]$ put $\mathcal{N}(T) = \ker T = T^{-1}\{0\}$ (the kernel or null space of T, which is a subspace of $\mathcal{H}$) and $\mathcal{R}(T) = \operatorname{ran} T = T(\mathcal{H})$ (the range of T, which is a linear manifold of $\mathcal{H}$). Let $T^* \in \mathcal{B}[\mathcal{H}]$ stand for the adjoint of $T \in \mathcal{B}[\mathcal{H}]$, let $\mathcal{M}^\perp$ denote the orthogonal complement of a linear manifold $\mathcal{M}$, and recall that $\mathcal{M}^{\perp\perp} = \mathcal{M}^-$, where $\mathcal{M}^-$ is the closure of $\mathcal{M}$. Also recall the following well-known properties: $\mathcal{L} \subseteq \mathcal{M}$ implies $\mathcal{M}^\perp \subseteq \mathcal{L}^\perp$, and $\mathcal{R}(T)^\perp = \mathcal{N}(T^*)$, so that $\mathcal{R}(T^*)^\perp = \mathcal{N}(T)$. Moreover, $\mathcal{R}(T^*)$ is closed if and only if $\mathcal{R}(T)$ is closed.

An invertible element from $\mathcal{B}[\mathcal{H}]$ is an operator T with an inverse in $\mathcal{B}[\mathcal{H}]$ (i.e., with a bounded inverse). This means that T is injective ($\mathcal{N}(T) = \{0\}$) and surjective ($\mathcal{R}(T) = \mathcal{H}$). An operator Q in $\mathcal{B}[\mathcal{H}]$ is nonnegative ($Q \geq O$), positive ($Q > O$) or strictly positive ($Q \succ O$) if $0 \leq \langle Qx\,; x\rangle$ for every $x \in \mathcal{H}$, $0 < \langle Qx\,; x\rangle$ for every $0 \neq x \in \mathcal{H}$, or $\alpha\|x\|^2 \leq \langle Qx\,; x\rangle$ for every $x \in \mathcal{H}$ and some $\alpha > 0$, respectively — i.e., $Q \in \mathcal{B}[\mathcal{H}]$ is strictly positive if and only if it is positive and invertible, which means that it has a (bounded) strictly positive inverse in $\mathcal{B}[\mathcal{H}]$.

Proposition 1. *Take $T \in \mathcal{B}[\mathcal{H}]$. The following assertions are pairwise equivalent.*

(a) *There exists a nonnegative $Q \in \mathcal{B}[\mathcal{H}]$ such that $TT^* = T^*QT$.*

(b) *There exists a nonnegative $Q \in \mathcal{B}[\mathcal{H}]$ such that $TT^* \leq T^*QT$.*

(c) *There exists a nonnegative $\alpha \in \mathbb{R}$ such that $TT^* \leq \alpha^2 T^*T$.*

(d) $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$.

(e) *There exists $S \in \mathcal{B}[\mathcal{H}]$ such that $T = T^*S$.*

Moreover, each of the above equivalent assertions implies the next one.

(f) $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$.

Furthermore, if $\mathcal{R}(T)$ is closed, then these six assertions are pairwise equivalent.

Proof. Assertion (a) implies (b) trivially. If (b) holds, then

$$\langle TT^*x\,; x\rangle \leq \langle T^*QTx\,; x\rangle = \|Q^{\frac{1}{2}}Tx\|^2 \leq \|Q\|\,\|Tx\|^2 = \|Q\|\langle T^*Tx\,; x\rangle$$

for every $x \in \mathcal{H}$, and so (c) holds with $\alpha = \|Q\|^{\frac{1}{2}}$. On the other hand, (c) implies (b) with $Q = \alpha^2 I$. Next recall the following result from [4] (see also [10] and [1]). If A and B lie in $\mathcal{B}[\mathcal{H}]$, then the assertions below are pairwise equivalent.

(1) $AA^* \leq \alpha^2 BB^*$ for some $\alpha \geq 0$.

(2) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

(3) There exists $C \in \mathcal{B}[\mathcal{H}]$ such that $A = BC$.

Thus, by setting $A = T$ and $B = T^*$, it follows that assertions (c), (d) and (e) are pairwise equivalent. Clearly, (e) implies (a) with $Q = SS^*$. Now, $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$ implies $\mathcal{R}(T^*)^\perp \subseteq \mathcal{R}(T)^\perp$, which is equivalent to $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$. Thus (d) implies (f). Conversely, $\mathcal{R}(T^*)^\perp \subseteq \mathcal{R}(T)^\perp$ implies $\mathcal{R}(T)^{\perp\perp} \subseteq \mathcal{R}(T^*)^{\perp\perp}$, which is equivalent to $\mathcal{R}(T)^- \subseteq \mathcal{R}(T^*)^-$. Therefore, (f) implies (d) whenever $\mathcal{R}(T)$ is closed. □

An operator T in $\mathcal{B}[\mathcal{H}]$ that satisfies assertion (a) — and so any of the equivalent assertions (a) to (e) — of Proposition 1 was called *posinormal* in [18]. We note that part of the equivalent assertions in Proposition 1 were also verified in [18], and the equivalence between (a) and (b) was also shown in [12]. Proposition 1 ensures that a complex multiple of a posinormal operator is again posinormal (i.e., the class of posinormal operators is closed under scaling), and hence we might restrict the investigation of posinormal operators to posinormal contractions. Also note that, using the absolute value notation, $|T| = (T^*T)^{\frac{1}{2}}$, we get from Proposition 1(c) that an operator T is posinormal if and only if $|T^*|^2 \leq \alpha^2|T|^2$ for some $\alpha > 0$. This implies that $|T^*| \leq \alpha|T|$. The class of all operators T from $\mathcal{B}[\mathcal{H}]$ for which $|T^*| \leq \alpha|T|$ for some $\alpha > 0$ (we might call them *semiposinormal* operators) includes the class of all posinormal operators.

Remark 1. Consider the assertions of Proposition 1. Note that, although the inequality in (b) is equivalent to the identity in (a), the inclusion in (d) is not equivalent to the identity $\mathcal{R}(T) = \mathcal{R}(T^*)$. In fact, this identity may fail for a posinormal operator (sample: take the unilateral shift). Some of the equivalent definitions of a posinormal operator in Proposition 1 have an alternative reformulation. Indeed, since $\|Tx\|^2 = \langle T^*Tx\,; x\rangle$ for every $x \in \mathcal{H}$, for all $T \in \mathcal{B}[\mathcal{H}]$, it follows that assertions (a), (b) and (c) are equivalent, respectively, to the following assertions.

(a′) There exists $Q \geq O$ in $\mathcal{B}[\mathcal{H}]$ such that $\|T^*x\| = \|Q^{\frac{1}{2}}Tx\|$ for every $x \in \mathcal{H}$.

(b′) There exists $Q \geq O$ in $\mathcal{B}[\mathcal{H}]$ such that $\|T^*x\| \leq \|Q^{\frac{1}{2}}Tx\|$ for every $x \in \mathcal{H}$.

(c′) There exists $\alpha \geq 0$ such that $\|T^*x\| \leq \alpha\|Tx\|$ for every $x \in \mathcal{H}$.

Note that condition (1) in the proof of Proposition 1 is equivalent to

(1′) $\|Ax\| \leq \alpha\|Bx\|$ for every $x \in \mathcal{H}$ for some $\alpha \geq 0$,

which has been referred to by saying that B *majorizes* A. Thus condition (c′) means: T majorizes T^*. Moreover, let $S^* = WQ$ be the polar decomposition of $S^* \in \mathcal{B}[\mathcal{H}]$, where $W \in \mathcal{B}[\mathcal{H}]$ is a partial isometry and $Q = |S^*| = (SS^*)^{\frac{1}{2}} \in \mathcal{B}[\mathcal{H}]$ is nonnegative, and consider the following assertion.

(e′) There exists a partial isometry $W \in \mathcal{B}[\mathcal{H}]$ and a nonnegative $Q \in \mathcal{B}[\mathcal{H}]$ such that $T = T^*QW^*$.

It is clear that (e) and (e′) imply each other. Therefore, each condition (a′), (b′), (c′), or (e′) also is equivalent to posinormality. Observe that, if (c) holds and $T \neq O$, then $\|T\|^2 = \|TT^*\| \leq \alpha^2\|T^*T\| = \alpha^2\|T\|^2$ so that $\alpha \geq 1$. Similarly, if (c′) holds and $T \neq O$, then $\|T\| = \|T^*\| \leq \alpha\|T\|$ so that $\alpha \geq 1$. Thus the constant $\alpha \geq 0$ in (c) or (c′) is necessarily not less than 1 whenever $T \neq O$ (and it is equal to 1 for $T \neq O$ if and only if T is hyponormal (i.e., $TT^* \leq T^*T$). Hence, $T \neq O$ is posinormal if and only if any of the following equivalent assertions hold.

(c_1) There exists $\alpha \geq 1$ such that $TT^* \leq \alpha^2T^*T$.

(c_1') There exists $\alpha \geq 1$ such that $\|T^*x\| \leq \alpha\|Tx\|$ for every $x \in \mathcal{H}$.

Finally note that, according to assertions (d) and (f),

$$T \text{ and } T^* \text{ are both posinormal} \iff \mathcal{R}(T) = \mathcal{R}(T^*) \implies \mathcal{N}(T) = \mathcal{N}(T^*).$$

3. Subclasses of Posinormal Operators

Since γT is posinormal for any $\gamma \geq 0$ whenever T is posinormal (Proposition 1), it follows that the collection of all posinormal operators is a cone in $\mathcal{B}[\mathcal{H}]$. In fact, the class of posinormal operators is very large. This is confirmed in this section, where some fundamental properties of posinormal operators are revisited. To begin with, recall that a unilateral weighted shift $S = \text{shift}(\{\omega_k\}_{k=1}^{\infty})$ on ℓ^2_+ is injective if and only if the weight sequence $\{\omega_k\}$ has no zero term (i.e., $\omega_k \neq 0$ for every $k \geq 1$). Now observe that *there are injective unilateral weighted shifts that are not posinormal.* (This shows a gap in [18, Proposition 1.1] — e.g., take $\omega_k = 1$ if k is odd and $\omega_k = k^{-1}$ if k is even and get a unilateral weighted shift that is injective but not posinormal.) Actually, by Proposition 1(c):

An injective unilateral weighted shift is posinormal if and only if $\sup_k \frac{|\omega_k|}{|\omega_{k+1}|} < \infty$.

Note that the condition $\sup_k(|\omega_k|\,|\omega_{k+1}|^{-1}) < \infty$ holds if a sequence $\{\omega_k\}$ of nonzero terms converges to a nonzero limit but it may fail if $\omega_k \to 0$ (e.g., it fails for $\omega_k = k^{-k}$ but holds for $\omega_k = k^{-1}$). Moreover, we also note the following general property.

Every invertible operator is posinormal with a posinormal adjoint.

This follows by Proposition 1(d), since T is invertible if and only if T^* is invertible or, equivalently, T is invertible if and only if $\mathcal{R}(T) = \mathcal{R}(T^*) = \mathcal{H}$. Since the set of invertible operators from $\mathcal{B}[\mathcal{H}]$ is open in $\mathcal{B}[\mathcal{H}]$, this shows that the class of posinormal operators also is topologically large.

A scalar $\lambda \in \mathbb{C}$ is a normal eigenvalue of an operator T if $\{0\} \neq \mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*)$ or, equivalently, if λ is an eigenvalue of T and $(\lambda I - T)$ satisfies assertion (f) in Proposition 1. Thus the following property holds by Proposition 1(f).

If T is posinormal, then either T is injective or 0 *is a normal eigenvalue of T.*

A subspace $\mathcal{M}$ of $\mathcal{H}$ is invariant for an operator $T \in \mathcal{B}[\mathcal{H}]$ (or is T-invariant) if $T(\mathcal{M}) \subseteq \mathcal{M}$, and nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$. A subspace $\mathcal{M}$ is a reducing subspace for T (or $\mathcal{M}$ reduces T) if it is both T and T^*-invariant (equivalently, if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are T-invariant). Observe that,

λ is a normal eigenvalue of T if and only if $\{0\} \neq \mathcal{N}(\lambda I - T)$ *reduces T.*

This is what is behind the proof of the next result. Indeed, an eigenvalue λ is normal if and only if $(\lambda I - T)$ satisfies assertion (f) in Proposition 1, and a subspace $\mathcal{M}$ reduces $(\lambda I - T)$ if and only if it reduces T.

Lemma 1. *If T is posinormal, then $\mathcal{N}(T)$ reduces T. If $\mathcal{N}(T)$ reduces T and $\mathcal{R}(T)$ is closed, then T is posinormal.*

Proof. Recall that the subspace $\mathcal{N}(T)$ is T-invariant and consider the decomposition $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp$. Since $T = \left(\begin{smallmatrix} O & X \\ O & S \end{smallmatrix}\right)$ and $T^* = \left(\begin{smallmatrix} O & O \\ X^* & S^* \end{smallmatrix}\right)$, with S in $\mathcal{B}[\mathcal{N}(T)^\perp]$ and $T|_{\mathcal{N}(T)} = O$ — the null operator in $\mathcal{B}[\mathcal{N}(T)]$ — we get $T^*(u \oplus 0) = 0 \oplus X^*u$ for every $u \in \mathcal{N}(T)$, where $X^*\colon \mathcal{N}(T) \to \mathcal{N}(T)^\perp$. If T is posinormal, then $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$ by Proposition 1(f) so that $X^* = O$ (and hence $X = O$), which implies that $T = O \oplus S$; that is, $\mathcal{N}(T)$ reduces T. Conversely, if $\mathcal{N}(T)$ reduces T, then $T = O \oplus S$ and $T^* = O \oplus S^*$ so that $\mathcal{N}(T) \subseteq \mathcal{N}(T) \oplus \mathcal{N}(S^*) = \mathcal{N}(T^*)$. If $\mathcal{R}(T)$ is closed and $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$, then T is posinormal by Proposition 1. □

Recall the following standard definitions. An operator $T \in \mathcal{B}[\mathcal{H}]$ is hyponormal if $TT^* \leq T^*T$ or, equivalently, if $(\lambda I - T)(\overline{\lambda} I - T^*) \leq (\overline{\lambda} I - T^*)(\lambda I - T)$ for every λ in $\mathbb{C}$, which means $\|(\lambda I - T)^*x\| \leq \|(\lambda I - T)x\|$ for every $x \in \mathcal{H}$ and every λ in $\mathbb{C}$. It is M-hyponormal if there is a constant $M \geq 0$ such that $(\lambda I - T)(\overline{\lambda} I - T^*) \leq M(\overline{\lambda} I - T^*)(\lambda I - T)$ for every λ in $\mathbb{C}$; that is, $\|(\lambda I - T)^*x\| \leq M^{\frac{1}{2}}\|(\lambda I - T)x\|$ for every $x \in \mathcal{H}$ and every λ in $\mathbb{C}$. Proposition 1(b,c) ensures that T is M-hyponormal if and only if there is a nonnegative Q in $\mathcal{B}[\mathcal{H}]$ such that $|(\lambda I-T)^*|^2\le|Q^{\frac12}(\lambda I-T)|^2$ for every λ in C. An operator $T\in\mathcal{B}[\mathcal{H}]$ is dominant if $\mathcal{R}(\lambda I-T)\subseteq\mathcal{R}(\overline{\lambda}I-T^*)$ for every λ in $\mathbb{C}$. Thus, by Proposition 1(d), *an operator T is dominant if and only if $(\lambda I-T)$ is posinormal for every λ in* $\mathbb{C}$. Proposition 1(c,d) says that T is dominant if and only if for each λ in $\mathbb{C}$ there exists an $M_\lambda\ge 0$ such that $(\lambda I-T)(\overline{\lambda}I-T^*)\le M_\lambda(\overline{\lambda}I-T^*)(\lambda I-T)$; that is, $\|(\lambda I-T)^*x\|\le M_\lambda{}^{\frac12}\|(\lambda I-T)x\|$ for every $x\in\mathcal{H}$ and every λ in $\mathbb{C}$. By Proposition 1(b,c), T is dominant if and only if for each λ in $\mathbb{C}$ there is a nonnegative Q_λ in $\mathcal{B}[\mathcal{H}]$ such that $|(\lambda I-T)^*|^2\le|Q_\lambda{}^{\frac12}(\lambda I-T)|^2$. Every hyponormal operator is M-hyponormal (actually, hyponormal means 1-hyponormal), every M-hyponormal is dominant, and every dominant is posinormal:

$$\text{Hyponormal}\;\subset\;M\text{-Hyponormal}\;\subset\;\text{Dominant}\;\subset\;\text{Posinormal}. \tag{P}$$

It is well-known that the first two inclusions are proper; and so is the third one (take any invertible, thus posinormal, nondominant operator; sample: $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ in $\mathcal{B}[\mathbb{C}^2]$ is such that $\mathcal{R}(I-T)\not\subseteq\mathcal{R}(I-T^*)$). M-hyponormal and dominant operators were called "totally posinormal" and "conditionally totally posinormal" in [8], respectively, and dominant operators were called "totally positive-normal" in [12] and "totally posinormal" in [18] — we avoid these neologisms and will stick with the original nomenclature. Since T is dominant if and only if $\lambda I-T$ is posinormal for every $\lambda\in\mathbb{C}$, Lemma 1 yields an immediate proof for the following well-know result that will be needed in the sequel.

If T is dominant, then $\mathcal{N}(\lambda I-T)$ reduces T for each $\lambda\in\mathbb{C}$.

Since an injective unilateral weighted shift with a weight sequence $\{\omega_k\}$ such that $\sup_k(|\omega_k|\,|\omega_{k+1}|^{-1})<\infty$ is posinormal, and since every invertible operator also is posinormal, we get the following important instance of a dominant operator.

An injective unilateral weighted shift with $\sup_k\frac{|\omega_k|}{|\omega_{k+1}|}<\infty$ and $w_k\to 0$ is dominant.

Sample: $\text{shift}(\{k^{-1}\}_{k=1}^{\infty})$ is dominant. Indeed, suppose a unilateral weighted shift S is injective and $\sup_k(|\omega_k|\,|\omega_{k+1}|^{-1})<\infty$. As we saw above, S is posinormal. If $w_k\to 0$, then S also is quasinilpotent, which means that $(\lambda I-S)$ is invertible for every $\lambda\ne 0$. Hence $(\lambda I-S)$ is posinormal for every $\lambda\in\mathbb{C}$. That is, S is dominant.

A T-invariant subspace $\mathcal{M}$ is called a normal subspace for T if the restriction $T|_{\mathcal{M}}$ of T to $\mathcal{M}$ is a normal operator in $\mathcal{B}[\mathcal{M}]$. Observe that $\mathcal{N}(\lambda I-T)$ is a normal subspace for every operator T and every $\lambda\in\mathbb{C}$ once $T|_{\mathcal{N}(\lambda I-T)}=\lambda I$ whenever $\mathcal{N}(\lambda I-T)\ne\{0\}$. Thus Lemma 1 ensures the next property.

If normal subspaces for T reduce T and $\mathcal{R}(T)$ is closed, then T is posinormal.

It is worth noticing that *if T is dominant, then every normal subspace for T reduces* T [19]. This characterizes a class between dominant and posinormal operators. But there are posinormal operators with closed range for which normal subspaces do not reduce. For instance, the same nondominant posinormal operator $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ in $\mathcal{B}[\mathbb{C}^2]$ has a normal subspace that does not reduce T.

4. Subclasses of Normaloid Operators

Also recall that an operator $T\in\mathcal{B}[\mathcal{H}]$ is paranormal if $\|Tx\|^2\le\|T^2x\|\,\|x\|$ for every x in $\mathcal{H}$, and normaloid if $r(T)=\|T\|$ (where $r(T)$ denotes the spectral radius of T). A part of an operator is a restriction of it to an invariant subspace; a nontrivial part is a restriction to a nontrivial invariant subspace. An operator is hereditarily normaloid (abbreviated: HN) if every part of it is normaloid, and totally hereditarily normaloid (abbreviated: THN) if it is hereditarily normaloid and every invertible part of it has a normaloid inverse [5]. These and the hyponormal are related by proper inclusion [6]:

$$\text{Hyponormal} \subset \text{Paranormal} \subset \text{THN} \subset \text{HN} \subset \text{Normaloid}. \tag{N}$$

We refer to the chains of inclusions in (P) and (N) as the *posinormal* and *normaloid families*, respectively. The connection between them is that hyponormal operators is the only class, among the above mentioned classes, included in both families. Indeed, it was shown in [22] that the unilateral weighted shift $S = \text{shift}(\{\omega_k\}_{k=1}^{\infty})$ on ℓ_+^2 with weights $\omega_k = 1$ for all k except for $k = 2$ where $\omega_2 = 2$ is M-hyponormal. Since $\|S^n\| = 2$ for all $n \geq 1$, it follows that $r(S) = 1$ by the Gelfand–Beurling formula, and hence T is not normaloid (in particular, not hyponormal). Thus

$$M\text{-hyponormal} \not\subseteq \text{Normaloid}$$

so that posinormal operators are not necessarily normaloid. We show next that

$$\text{Paranormal} \not\subseteq \text{Posinormal}.$$

so that normaloid operators are not necessarily posinormal.

Example 1. Consider the operator

$$T = \begin{pmatrix} O & & & & \\ Q & O & & & \\ & R & O & & \\ & & R & O & \\ & & & & \ddots \end{pmatrix}$$

on $\ell_+^2(\mathbb{C}^2)$, where $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $R = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)^{\frac{1}{2}}$, and every entry not directly below the main block diagonal is null. This is a paranormal operator. In fact, it was shown in [14, Problem 9.14] that T is quasihyponormal (i.e., $T^*(T^*T - TT^*)T \geq O$ — a class that includes the hyponormal operators and is included in the paranormal class). But T is not hyponormal. Since $\mathcal{N}(T) = \mathcal{N}(Q) \oplus \mathcal{N}(R) \oplus \bigoplus_{k=3}^{\infty} \mathcal{N}(R)$, and since $\mathcal{N}(T^*) = \mathbb{C}^2 \oplus \mathcal{N}(Q) \oplus \bigoplus_{k=3}^{\infty} \mathcal{N}(R)$, it follows that $\mathcal{N}(T) \not\subseteq \mathcal{N}(T^*)$ because $\mathcal{N}(R) \not\subseteq \mathcal{N}(Q)$. Thus T is not posinormal by Proposition 1(f).

Observe that the above examples only exist on an infinite-dimensional space. Actually, compact paranormal operators are normal [17] and compact M-hyponormal operators are normal too [21]. This property (viz. compact are normal) is clearly not transferred to the class of posinormal operators because every invertible operator is posinormal and there are compact invertible operators that are not normal (sample: $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathcal{B}[\mathbb{C}^2]$ is an invertible operator with $r(T) = 1$ and $\|T\| = \sqrt{2}$.) In fact, this property does not survive even the shorter trip from M-hyponormal to dominant operators — there exist compact dominant operators that are not normal. For instance, a unilateral weighted shift S on ℓ_+^2 with a weight sequence $\{\omega_k\}$ of positive numbers that converges to zero ($0 < \omega_k \to 0$) is injective and quasinilpotent ($r(S) = 0$), which is compact, nonnormal and (as we saw in Section 3) dominant if

$\sup_k(|\omega_k|\,|\omega_{k+1}|^{-1}) < \infty$. (Also observe that these are further examples of dominant, thus posinormal operators, that are not M-hyponormal nor normaloid.)

Remark 2. Although paranormal and posinormal operators are not related by inclusion (in both directions), they share some properties (beyond including the hyponormal operators). For instance,

$$\mathcal{N}(T^2) = \mathcal{N}(T)$$

is a common property satisfied by both a posinormal or a paranormal operator T. Indeed, $\mathcal{N}(T^2) \subseteq \mathcal{N}(T)$ holds for a paranormal operator by the very definition of paranormal operators. Moreover, if T is posinormal and $x \in \mathcal{N}(T^2)$ then, by Proposition 1(f), $Tx \in \mathcal{N}(T) \subseteq \mathcal{N}(T^*)$ so that $T^*Tx = 0$, which implies $\|Tx\|^2 = \langle T^*Tx\,; x\rangle = 0$, and hence $x \in \mathcal{N}(T)$. Thus $\mathcal{N}(T^2) \subseteq \mathcal{N}(T)$ also holds for a posinormal operator (this was first pointed out in [12]). Therefore, if an operator T is either paranormal or posinormal, then $\mathcal{N}(T^2) \subseteq \mathcal{N}(T)$, and $\mathcal{N}(T) \subseteq \mathcal{N}(T^2)$ holds trivially for every operator $T \in \mathcal{B}[\mathcal{H}]$.

Such common properties might suggest the question: is a normaloid posinormal operator paranormal? No it is not. For instance, take the operator $S = \|T\|\,I \oplus T$ on $\mathbb{C}^2 \oplus \mathbb{C}^2$ with $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Since S is invertible and $r(S) = \|S\| = \|T\|$, it follows that S is a normaloid posinormal. However, S is not HN since T is not normaloid (as we saw above), and S is not dominant because $\mathcal{R}(I - T) \perp \mathcal{R}(I - T^*)$. Therefore,

$$\big(\text{Posinormal } \cap \text{ Normaloid}\big) \not\subseteq \big(\text{HN } \cup \text{ Dominant}\big).$$

In particular, this exhibits a normaloid posinormal that is not paranormal (neither M-hyponormal).

We close this section by observing that parts of a hyponormal (paranormal) operator are again hyponormal (paranormal). This well-known property extends naturally (by their own definition) to the classes of THN and HN operators. On the other hand, parts of an M-hyponormal (dominant) operator are again M-hyponormal (dominant) [23]. This was extended to posinormal operators in [12]: parts of a posinormal operator are again posinormal. Since every invertible operator is posinormal, it follows that if $\mathcal{M}$ is any invariant subspace for an operator T, then the following assertions hold true.

(a) If T lies in HN $\cap$ Posinormal, then $T|_{\mathcal{M}}$ also lies in HN $\cap$ Posinormal.

(b) If T lies in THN $\cap$ Posinormal, then $T|_{\mathcal{M}}$ also lies in THN $\cap$ Posinormal and, if $T|_{\mathcal{M}}$ is invertible, then $(T|_{\mathcal{M}})^{-1}$ is posinormal and normaloid.

5. Invariant Subspaces

An operator is transitive if it has no *nontrivial invariant subspace* (n.i.s. for short). We saw that the class of all operators T for which $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$ includes the posinormal operators (cf. Proposition 1). This class also includes the transitive operators. Indeed, recall that an operator $T \in \mathcal{B}[\mathcal{H}]$ is quasiinvertible (or a quasiaffinity) if it is injective with a dense range. That is, if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)^- = \mathcal{H}$ or, equivalently, if $\mathcal{N}(T) = \mathcal{N}(T^*) = \{0\}$ (for $\mathcal{R}(T)^- = \mathcal{N}(T^*)^\perp$). Since $\mathcal{R}(T)^-$ and $\mathcal{N}(T)$ are invariant subspaces for every operator T, if follows that if an operator has no n.i.s., then it is quasiinvertible. Therefore,

$$T \text{ has no n.i.s.} \quad \Longrightarrow \quad \mathcal{N}(T) = \mathcal{N}(T^*) = \{0\} \quad \Longrightarrow \quad \mathcal{N}(T) \subseteq \mathcal{N}(T^*).$$

The invariant subspace problem is the question that asks whether the class of all transitive operators is nonempty when $\mathcal{H}$ is infinite-dimensional and separable.

Theorem 1. *If an invertible operator $T \in \mathcal{B}[\mathcal{H}]$ has no nontrivial invariant subspace, then the following assertions hold true.*

(a) *There exists a unique S in $\mathcal{B}[\mathcal{H}]$ such that $T = T^*S$. This S is an invertible operator which is not a multiple of a unitary and does not commute with T or with T^*.*

(b) *There exists a unique nonnegative $Q = SS^*$ in $\mathcal{B}[\mathcal{H}]$ such that $TT^* = T^*QT$. This Q is strictly positive and nonscalar.*

Proof. We split the proof of (a) into (a_1) and (a_2), with (a_2) following the proof of (b). If an operator has no n.i.s., then it is quasiinvertible. In particular, if an operator T with no n.i.s. is invertible, then T and T^* are both posinormal.

(a_1) Thus Proposition 1(e) ensures the existence of operators S and S_* such that

$$T = T^*S \quad \text{and} \quad T^* = TS_*,$$

and hence $T = TS_*S$ and $T^* = T^*SS_*$ so that

$$T(I - S_*S) = O \quad \text{and} \quad T^*(I - SS_*) = O.$$

Since T and T^* have no n.i.s. (because T has no n.i.s.), it follows that they are nonzero (for $\dim \mathcal{H} > 1$) and so the above identities imply that (see e.g., [13, p.18])

$$(I - S_*S) = O \quad \text{and} \quad (I - SS_*) = O.$$

Then S is invertible and $S^{-1} = S_*$ in $\mathcal{B}[\mathcal{H}]$. This is enough to ensure that S is unique (if S_1 and S_2 are such that $T = T^*S_1 = T^*S_2$, then $S_1^{-1} = S_2^{-1} = S_*$).

(b) The equation $T = T^*S$ implies that $T^* = S^*T$, and hence $TT^* = T^*SS^*T$. If $TT^* = T^*QT$, then $T^*(SS^* - Q)T = O$. Since T has no n.i.s., it is quasiinvertible ($\mathcal{N}(T) = \mathcal{N}(T^*) = \{0\}$) and the above identity ensures that $Q = SS^*$, which is strictly positive once S is invertible. If $Q = \alpha I$ for some $\alpha > 0$ then $\alpha \neq 1$ (because $Q = I$ implies that T is normal and normal operators have a n.i.s.). But since $T \neq O$ it follows that, if $Q = \alpha I$, then $\|T\|^2 = \|TT^*\| = \alpha \|T^*T\| = \alpha \|T\|^2$ so that $\alpha = 1$; a contradiction. Thus Q is not scalar.

(a_2) Since Q is not scalar, S is not a multiple of a coisometry. Indeed, if $S = \gamma V^*$ for some nonzero $\gamma \in \mathbb{C}$ and some isometry $V \in \mathcal{B}[\mathcal{H}]$, then $Q = SS^* = |\gamma|^2 V^*V = |\gamma|^2 I$, which is again a contradiction. Since S is invertible and since a unitary operator is precisely an invertible coisometry, it follows that S is not a multiple of a coisometry if and only if it is not a multiple of a unitary operator. Moreover, since $T = T^*S$ and S is invertible, it follows that $ST = TS$ if and only is $ST^*S = T^*S^2$, which is equivalent to $ST^* = T^*S$; that is, $TS^* = S^*T$. Therefore, if S commutes with T or with T^*, then both T and T^* commute with S, which is nonscalar (since S is not a multiple of a unitary). But this means that T is reducible (see e.g., [14, p.84]); another contradiction, once T does not even have a n.i.s. Thus S does not commute with T or with T^*. $\square$

By a contraction we mean a operator $T \in \mathcal{B}[\mathcal{H}]$ such that $\|T\| \leq 1$. A contraction is *completely nonunitary* (c.n.u. for short) if it has no unitary direct summand. For any contraction T the sequence of positive numbers $\{\|T^n x\|\}$ is decreasing (thus convergent) for every $x \in \mathcal{H}$. A contraction T is of class $\mathcal{C}_{0\cdot}$ if it is strongly stable; that is, if $\{\|T^n x\|\}$ converges to zero for every $x \in \mathcal{H}$, and of class $\mathcal{C}_{1\cdot}$ if $\{\|T^n x\|\}$ does not converge to zero for every nonzero $x \in \mathcal{H}$. It is of class $\mathcal{C}_{\cdot 0}$ or of class $\mathcal{C}_{\cdot 1}$ if its adjoint T^* is of class $\mathcal{C}_{0\cdot}$ or $\mathcal{C}_{1\cdot}$, respectively. All combinations are possible, leading to the Nagy–Foiaş classes of contractions $\mathcal{C}_{00}$, $\mathcal{C}_{01}$, $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$. Recall that an operator $T \in \mathcal{B}[\mathcal{H}]$ is a contraction if and only if $I - T^*T$ is a nonnegative contraction. If T is a contraction, then the nonnegative contraction $(I - T^*T)^{\frac{1}{2}}$ is called the defect operator of T.

It was proved in [16] that a c.n.u. hyponormal contraction is of class $\mathcal{C}_{\cdot 0}$. This was extended to paranormal contractions in [15] and to dominant contractions in [20] (also see [11]). It is worth noticing that this was further extended to k-paranormal contractions and to k-quasihyponormal contractions in [7]. Recall that an operator T is k-paranormal if $\|Tx\|^{k+1} \leq \|T^{k+1}x\|\,\|x\|^k$ for some integer $k \geq 1$ and every x in $\mathcal{H}$ (a class of operators in the normaloid family (N) that includes the paranormal operators and is included in the THN class). Also recall that an operator T is k-quasihyponormal if $T^{*k}(T^*T - TT^*)T^k \geq O$ for some integer $k \geq 1$ (a class of operators that includes the quasihyponormal, thus the hyponormal operators, but is not included in any of the classes discussed in this paper).

Thus we can infer that there is no $\mathcal{C}_{01}$-contraction in any of the above mentioned classes. In particular, *there is no paranormal or dominant contraction of class* $\mathcal{C}_{01}$. Indeed, if there is a paranormal or dominant $\mathcal{C}_{01}$-contraction, then it is not c.n.u. (because it is not of class $\mathcal{C}_{\cdot 0}$) so that it has a unitary direct summand, which is a contradiction: it cannot be of class $\mathcal{C}_{0\cdot}$ with a unitary direct summand. We show in Theorem 2 below that, if there exists a THN $\mathcal{C}_{01}$-contractions with a compact defect operator, then it is not posinormal and has a n.i.s.

Recall that if $\mathcal{H}$ is an infinite-dimensional separable complex Hilbert space, then it is still unknown whether operators in any of the following classes have a n.i.s.: (1) hyponormal operators (and every class of operators that includes the hyponormals), (2) contractions of classes $\mathcal{C}_{01}$, and (3) contractions with compact defect operators. Also recall that an operator $T \in \mathcal{B}[\mathcal{H}]$ is supercyclic if there exists a nonzero vector x in $\mathcal{H}$ (called a supercyclic vector for T) for which the set of scalar multiples of the orbit $\{\gamma T^n x\}_{\gamma \in \mathbb{C},\, n \geq 0}$ is dense in $\mathcal{H}$. Supercyclic operators have a dense range, and hyponormal (and even paranormal) operators are not supercyclic [3].

Theorem 2. *Let $T \in \mathcal{B}[\mathcal{H}]$ be a* THN *contraction with a compact defect operator.*

(a) *If $T \in \mathcal{C}_{01}$, then it has a nontrivial invariant subspace and is not posinormal.*

(b) *If T is supercyclic, then it has a nontrivial invariant subspace.*

Proof. Suppose T is a contraction with a compact defect operator.

(a) If T is of class $\mathcal{C}_{01}$ and $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$, then it is not THN [6, Proposition 9]. So, for a THN $\mathcal{C}_{01}$-contraction with a compact defect operator, $\mathcal{N}(T) \not\subseteq \mathcal{N}(T^*)$.

(b) If T is THN, $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$ and nonzero isolated eigenvalues of T are normal eigenvalues, then T is not supercyclic [9, Theorem 2.1]. If T has an eigenvalue, then it has a n.i.s. Thus suppose T has no eigenvalue so that the condition "nonzero isolated eigenvalues are normal" is satisfied by vacuousness. So, for a THN supercyclic contraction with a defect compact operator and no eigenvalue, $\mathcal{N}(T) \nsubseteq \mathcal{N}(T^*)$.

If $\mathcal{N}(T) \nsubseteq \mathcal{N}(T^*)$, then T has a n.i.s. and is nonposinormal by Proposition 1(f). $\square$

We saw above that there is no paranormal or dominant (or k-paranormal, or k-quasihyponormal) $\mathcal{C}_{01}$-contraction. This is enough to ensure that if a contraction in any these classes does not have a n.i.s., then it is either a $\mathcal{C}_{00}$ or a $\mathcal{C}_{10}$-contraction. Such a conclusion can be extended to a THN contraction with a compact defect operator by using Theorem 2.

Corollary 1. *A* THN *contraction with a compact defect operator and without a nontrivial invariant subspace is either of class $\mathcal{C}_{00}$ or of class $\mathcal{C}_{10}$.*

Proof. If a contraction has no n.i.s., then it is either a $\mathcal{C}_{00}$, a $\mathcal{C}_{01}$, or a $\mathcal{C}_{10}$-contraction (see e.g., [13, p.71]). A THN contraction with a compact defect operator and without a n.i.s. is not of class $\mathcal{C}_{01}$ by Theorem 2. $\square$

6. Weyl's Theorem

According to usual terminology, Weyl's theorem is said to hold for an operator $T \in \mathcal{B}[\mathcal{H}]$, or T satisfies Weyl's theorem, if

$$\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T),$$

where $\sigma(T)$ is the spectrum of T, $\sigma_w(T)$ is the Weyl spectrum of T (the set of all $\lambda \in \mathbb{C}$ for which $(\lambda I - T)$ is not a Fredholm operator of index zero), and $\pi_{00}(T)$ is the set of all isolated eigenvalues of T of finite multiplicity. It is readily verified that the complement $\sigma_0(T)$ of $\sigma_w(T)$ in $\sigma(T)$ is given by

$$\begin{aligned}\sigma_0(T) = \sigma(T)\backslash\sigma_w(T) \;=\; \big\{\lambda \in \sigma_P(T)\colon\; &\mathcal{R}(\lambda I - T)^- = \mathcal{R}(\lambda I - T) \neq \mathcal{H} \text{ and} \\ &\dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty\big\},\end{aligned}$$

where $\sigma_P(T)$ is the point spectrum (i.e., the set of eigenvalues) of T. Let $\sigma_{\rm iso}(T)$ be the set of isolated points of $\sigma(T)$ and let $\sigma_{PF}(T)$ be the set of all eigenvalues of T of finite multiplicity, $\sigma_{PF}(T) = \big\{\lambda \in \sigma_P(T)\colon \dim \mathcal{N}(\lambda I - T) < \infty\big\}$. Thus

$$\pi_{00}(T) = \sigma_{\rm iso}(T) \cap \sigma_{PF}(T)$$

so that T satisfies Weyl's theorem if and only if

$$\sigma_0(T) = \sigma_{\rm iso}(T) \cap \sigma_{PF}(T).$$

As far as the posinormal and normaloid families in (P) and (N) are concerned, it was shown in [5] that Weyl's theorem holds for both T and T^* whenever T is THN (which includes the paranormal operators). The same sort of conclusion holds for M-hyponormal operators. Actually, Weyl's theorem holds for both $f(T)$ and $f(T^*)$ whenever T is M-hyponormal for every analytic function f defined on an open neighborhood of $\sigma(T)$ (see e.g., [8] and the reference therein). Although Weyl's theorem may fail for a dominant operator (see e.g., Remark 4 below), it has been investigated under additional hypothesis. For instance, Weyl's theorem was considered for a dominant operator T assuming that either (1) isolated points of the spectrum of T are poles of the resolvent of T [8], or (2) all parts of T belong to a class of operators which are null whenever their spectral radius is zero [12]. (As a matter of fact, assumption (2) is not enough to ensure that Weyl's theorem holds for a dominant operator, as it will be verified in Remark 4 below: it is necessary to assume that $\lambda I - T|_{\mathcal{M}}$ belongs to such a class for all parts $T|_{\mathcal{M}}$ of T and all scalars $\lambda \in \mathbb{C}$). Note that M-hyponormal operators were called "totally posinormal" in [8], and dominant operators were called "conditionally totally posinormal" in [8] and "totally positive-normal" in [12].

The next theorem gives a sufficient condition for a dominant operator to satisfy Weyl's theorem. Recall that an operator $T \in \mathcal{B}[\mathcal{H}]$ is isoloid if isolated points of the spectrum are eigenvalues (i.e., if $\sigma_{\rm iso}(T) \subseteq \sigma_P(T)$), and transaloid if $\lambda I - T$ is normaloid for every $\lambda \in \mathbb{C}$.

Theorem 3. *If every part of a dominant operator is transaloid, then it satisfies Weyl's theorem.*

Proof. Take an operator $T \in \mathcal{B}[\mathcal{H}]$. To begin with we recall a classic result from [2]. *If finite-dimensional eigenspaces of T are reducing and every direct summand of it is isoloid, then T satisfies Weyl's theorem.* As we saw in Section 3, if an operator T is dominant, then $\mathcal{N}(\lambda I - T)$ reduces T for each $\lambda \in \sigma_P(T)$ so that every eigenspace of a dominant operator is reducing and so is, in particular, every finite-dimensional eigenspace. Thus an immediate corollary of the above result reads as follows.

Claim 1. If every direct summand of a dominant operator is isoloid, then it satisfies Weyl's theorem.

Now suppose every part of an operator T is transaloid, which means that $\lambda I - T|_{\mathcal{M}}$ is normaloid for every T-invariant subspace $\mathcal{M}$ and every $\lambda \in \mathbb{C}$ (this is sometimes referred to by saying that T is hereditarily transaloid). The Riesz Decomposition Theorem says that *if $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint nonempty and closed sets, then T has a pair of complementary nontrivial invariant subspaces $\{\mathcal{M}_1, \mathcal{M}_2\}$ such that $\sigma(T|_{\mathcal{M}_1}) = \sigma_1$ and $\sigma(T|_{\mathcal{M}_2}) = \sigma_2$.* Take any λ in $\sigma_{\rm iso}(T)$ so that $\sigma(T) = \{\lambda\} \cup \sigma$ for some closed set σ that does not contain λ. Thus the Riesz Decomposition Theorem ensures that T has a nonzero invariant subspace $\mathcal{M}$ such that $\sigma(T|_{\mathcal{M}}) = \{\lambda\}$. Put $S = T|_{\mathcal{M}}$ on $\mathcal{M} \neq \{0\}$ so that $\sigma(\lambda I - S) = \{0\}$ (by the Spectral Mapping Theorem). If every part of T is transaloid, then $\lambda I - S$ is a normaloid operator so that $\|\lambda I - S\| = r(\lambda I - S) = 0$, and hence $T|_{\mathcal{M}} = S = \lambda I$ in $\mathcal{B}[\mathcal{M}]$, which implies $\lambda \in \sigma_P(T)$. Thus $\sigma_{\rm iso}(T) \subseteq \sigma_P(T)$, and we get the next result.

Claim 2. If every part of an operator is transaloid, then it is isoloid.

Therefore, if every part of an operator is transaloid, then every part of every direct summand of it is transaloid, and so every direct summand of it is isoloid by Claim 2, and hence it satisfies Weyl's theorem by Claim 1 if it is dominant. □

Note that hyponormal operators satisfy the hypothesis of Theorem 3. Indeed, since $\lambda I - T$ is hyponormal for every $\lambda \in \mathbb{C}$ whenever T is, and since every part of a hyponormal operator is again hyponormal (thus normaloid), it follows that every part of T is transaloid whenever T is hyponormal (i.e., hyponormal operators are hereditarily transaloid) so that Claim 2 in the above proof applies to hyponormal operators, which in turn are dominant. Also note that if T is a dominant operator, then $\lambda I - T$ is dominant for every $\lambda \in \mathbb{C}$, and every part of a dominant operator is again dominant, so that $\lambda I - T|_{\mathcal{M}}$ is dominant for all parts $T|_{\mathcal{M}}$ of T and all λ. But this is not enough to make dominant operators hereditarily transaloid (unlike the hyponormal operators, dominant operators are not necessarily normaloid).

Instead of focusing on the normaloid assumption (as we did when we imposed that the parts are transaloid), we might try a weaker assumption by requiring that $\big\{r(T) = 0 \implies \|T\| = 0\big\}$ instead of $\big\{r(T) = \|T\|\big\}$. This was considered in [12] and is equivalent to requiring that the operators belong to a class $\mathcal{F}$, which is defined as follows. Let $\mathcal{C}$ be any class of operators such that $\mathcal{C} \cap$ Quasinilpotent $\subseteq \{O\}$. For instance, every subclass of the normaloid operators is a class with such a property, as it is every subclass of the nonquasinilpotent operators (e.g., the invertible operators). Let $\mathcal{F}$ be union of all classes with this property; that is, $\mathcal{F}$ is the largest class of operators with the property that

$$\mathcal{F} \,\cap\, \text{Quasinilpotent} \,\subseteq\, \{O\}.$$

In other words, $\mathcal{F}$ is the class of all operators from $\mathcal{B}[\mathcal{H}]$ such that if $\sigma(T) = \{0\}$ then $T = O$ (i.e., operators that are null whenever their spectral radius is zero).

Remark 3. We show that the property of being in $\mathcal{F}$ for some T is not transferred to the translations $\lambda I - T$ neither to the parts $T|_{\mathcal{M}}$, even if $\mathcal{F}$ is intersected with dominant operators. We also exhibit a dominant operator not in $\mathcal{F}$ and a dominant operator with all parts in $\mathcal{F}$ that is not isoloid.

(a) First observe that if an operator T lies in $\mathcal{F}$, then it may happen that $\lambda I - T$ does not lie in $\mathcal{F}$ for some nonzero $\lambda \in \mathbb{C}$, even if T is dominant. For instance, take a unilateral weighted shift $S = \text{shift}(\{\omega_k\}_{k=1}^{\infty})$ on ℓ^2_+ with a weight sequence of positive numbers that converges to zero. Since $\sigma(S) = \{0\}$ with 0 in the residual spectrum of S, it follows that this is an injective quasinilpotent unilateral weighted shift. In particular, set $\omega_k = k^{-1}$ so that S is a dominant operator as we saw in Section 3. Put $T = I - S$. Since $\sigma(T) = \{1\}$, it also follows that T is a nonquasinilpotent dominant operator, and so a dominant operator in $\mathcal{F}$. But $I - T = S$ is a dominant operator not in $\mathcal{F}$ (because S is a nonzero quasinilpotent operator). Summing up: T is a dominant operator in $\mathcal{F}$ such that the dominant operator $I - T$ is not in $\mathcal{F}$.

(b) Now observe that parts of an operator in $\mathcal{F}$ are not necessarily in $\mathcal{F}$, even if they are dominant. For instance the operator S of item (a) is a dominant not in $\mathcal{F}$ but the (orthogonal) direct sum $I \oplus S$, which is clearly dominant, lies in $\mathcal{F}$ since it is not quasinilpotent ($\sigma(I \oplus S) = \{0, 1\}$).

(c) However, all parts of an invertible operator belong to $\mathcal{F}$. Indeed, it is easy to show that if $0 \notin \sigma(T)$, then $\sigma(T|_{\mathcal{M}}) \neq \{0\}$ for every nonzero part $T|_{\mathcal{M}}$ of T. Hence all parts of the dominant operator $T = I - S$ of item (a) lie in $\mathcal{F}$ (for $\sigma(T) = \{1\}$). Moreover, 1 is not an eigenvalue of T (as 0 is not an eigenvalue of S) so that T is not isoloid. Thus we have proved the following assertion.

Claim 3. A dominant operator with all parts in $\mathcal{F}$ may not be isoloid.

(This shows a gap in [12, Theorem 9], where it was proved that if an operator T is dominant and all parts of it lie in $\mathcal{F}$, then $0 \in \sigma_{\text{iso}}(T)$ implies $0 \in \sigma_P(T)$.)

If we assume that all translations of all parts of T lie in $\mathcal{F}$, instead of assuming that just the parts of T are in $\mathcal{F}$, then we get the following extension of Theorem 3.

Corollary 2. *If T is a dominant operator and $\lambda I - T|_{\mathcal{M}} \in \mathcal{F}$ for every part $T|_{\mathcal{M}}$ of T and every $\lambda \in \mathbb{C}$, then T satisfies Weyl's theorem.*

Proof. It is clear that if every part of an operator T is transaloid, then $\lambda I - T|_{\mathcal{M}}$ lies in $\mathcal{F}$ for every part $T|_{\mathcal{M}}$ and every scalar λ; and this was what we needed to get the result of Claim 2 in the proof of Theorem 3. Thus, as it is readily verified, the same proof of Claim 2 still holds for this class of operators:

Claim 2'. If $\lambda I - T|_{\mathcal{M}} \in \mathcal{F}$ for every part $T|_{\mathcal{M}}$ and every λ, then T is isoloid.

Therefore, if all translations of all parts of T lie in $\mathcal{F}$ (i.e., if $\lambda I - T|_{\mathcal{M}} \in \mathcal{F}$ for every part $T|_{\mathcal{M}}$ of T and every $\lambda \in \mathbb{C}$), then every translation of every part of every direct summand of T lies in $\mathcal{F}$ (reason: if $(T|_{\mathcal{R}})|_{\mathcal{M}}$ is a part of a direct summand $T|_{\mathcal{R}}$ of T, then $T|_{\mathcal{M}} = (T|_{\mathcal{R}})|_{\mathcal{M}}$ is a part of T), and so every direct summand of T is isoloid by Claim 2'. Hence T satisfies Weyl's theorem by Claim 1 (as in the proof of Theorem 3) if it is dominant. □

Remark 4. However, such an extended version of Theorem 3 does not hold under a weaker hypothesis where just the parts of a dominant T are required to be in $\mathcal{F}$:

Claim 3'. A dominant operator with all parts in $\mathcal{F}$ may not satisfy Weyl's theorem.

(This shows a gap in [12, Theorem 13].) Indeed, take the unilateral weighted shift $S = \text{shift}(\{k^{-1}\}_{k=1}^{\infty})$ of Remark 3 and consider the (orthogonal) direct sum $0 \oplus S$ on $\mathbb{C} \oplus \ell_+^2$. We show that $I - (0 \oplus S)$ *is a dominant operator with all parts in $\mathcal{F}$ that does not satisfy Weyl's theorem.* The proof goes as follows. Since $\sigma(S) = \{0\}$ and $\sigma_P(S) = \varnothing$,

$$\sigma(0 \oplus S) = \{0\} \cup \sigma(S) = \{0\} = \{0\} \cup \sigma_P(S) = \sigma_P(0 \oplus S),$$

so that $\sigma(I - (0 \oplus S)) = \{1\} = \sigma_P(I - (0 \oplus S))$. Moreover, since $\dim \mathcal{N}(S) = 0$, we get $\dim \mathcal{N}(I - (I - (0 \oplus S)) = \dim \mathcal{N}(0 \oplus S) = 1$ so that $\pi_{00}(I - (0 \oplus S)) = \{1\}$. Furthermore, $\mathcal{R}(S)$ is not closed in ℓ_+^2 because 0 lies in the residual spectrum of S but not in a open component of it. In other words, since $\mathcal{N}(S) = \{0\}$ and S is not bounded below, it follows by the Bounded Inverse Theorem (see e.g., [14, pp.75,76] that $\mathcal{R}(S)$ is not closed in ℓ_+^2, and therefore

$$\mathcal{R}\big(I - (I - (0 \oplus S)\big) = \mathcal{R}(0 \oplus S) = \{0\} \oplus \mathcal{R}(S)$$

is not closed in $\mathbb{C} \oplus \ell_+^2$, which implies that $1 \notin \sigma_0(I - (0 \oplus S))$. Thus $I - (0 \oplus S)$ does not satisfy Weyl's theorem. Since S is dominant, it follows that $0 \oplus S$ is dominant, and so is $I - (0 \oplus S)$. To complete the proof, observe that all parts of the operator $I - (0 \oplus S)$ are in $\mathcal{F}$ because $\sigma(I - (0 \oplus S)) = \{1\}$ (recall: all parts of an invertible operator lie in $\mathcal{F}$ — see Remark 3(c)).

Remark 5. Unlike the dominant operators, *M-hyponormal operators are of class $\mathcal{F}$* [21, Theorem 3] (i.e., the only quasinilpotent M-hyponormal is the null operator). Since translations of an M-hyponormal are trivially M-hyponormal, and parts of an M-hyponormal are again M-hyponormal [23], it then follows that $\lambda I - T|_{\mathcal{M}} \in \mathcal{F}$ for every part $T|_{\mathcal{M}}$ and every λ whenever T is M-hyponormal. This implies that

Claim 4. M-hyponormal operators are isoloid, according to Claim 2' in the proof of Corollary 2, and also that M-hyponormal operators satisfy the hypothesis of Corollary 2. Hence Corollary 2 gives still another proof that *every M-hyponormal operator satisfies Weyl's theorem.*

Finally, note that under the hypothesis of Theorem 3 (or Corollary 2), it also follows that Weyl's theorem holds for $f(T)$ for every analytic function f defined on an open neighborhood of $\sigma(T)$. Extending the proofs from T to $f(T)$ follow the same argument as in [8] or [12].

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