Advances in Mathematical Sciences and Applications **16** (2006) 643–660

ABSTRACT WAVELETS GENERATED BY HILBERT SPACE SHIFT OPERATORS

CARLOS S. KUBRUSLY AND NHAN LEVAN

Abstract. An abstract wavelet is a nonzero vector w in a Hilbert space $\mathcal{H}$ that generates an orthonormal basis for $\mathcal{H}$ in terms of a pair $\{D,T\}$ of noncommuting bilateral shifts of infinite multiplicity on $\mathcal{H}$ such that $DT^2 = TD$. Properties of such an abstract wavelet, as well as decompositions of the Hilbert space $\mathcal{H}$ on which w lives, are investigated. It turns out that these properties are precisely those of $\mathcal{L}^2(\mathbb{R})$-wavelets.

1. Introduction

We show in this paper that the concept of $\mathcal{L}^2(\mathbb{R})$-wavelets can be extended to abstract wavelets — to be defined as wavelets on abstract Hilbert spaces. An $\mathcal{L}^2(\mathbb{R})$-wavelet is a function $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ that generates a double indexed orthonormal basis $\{\psi_{m,n}(\cdot)\}$, $(m,n) \in \mathbb{Z}\times\mathbb{Z}$, for the separable Hilbert space $\mathcal{L}^2(\mathbb{R})$ [16] (see also [3], [19] and the references therein). The generating process is achieved by two bilateral shifts of infinite multiplicity. It will be seen that an abstract wavelet has all properties of an $\mathcal{L}^2(\mathbb{R})$-wavelet.

Basic notions, notation and terminology are posed in Section 2. Results on span of images of sets under invertible operators, which will play a crucial role in the sequel, are considered in Section 3. Equivalent definitions and essential properties of Hilbert space bilateral shifts are discussed in Section 4. Abstract wavelets are then introduced in Section 5 as nonzero vectors that generate orthonormal bases for Hilbert spaces in terms of two noncommuting bilateral shifts of infinite multiplicity D and T such that D intertwines T^2 to T. Wavelet expansions are taken up in Section 6. There we show that an abstract wavelet vector, and the orthonormal basis of vectors generated by it, result in two families of orthogonal subspaces that decompose the underlying Hilbert space. Further properties of these subspaces are investigated in Section 7. We close the paper by considering projections on wavelets in Section 8. These, in the case of $\mathcal{L}^2(\mathbb{R})$-wavelets, result in the so-called Discrete Wavelet Transform (DWT) for Signal Processing.

2. Preliminaries

Let $\mathcal{H}$ be a (complex, infinite-dimensional, but not necessarily separable) Hilbert space. By a subspace of $\mathcal{H}$ we mean a *closed* linear manifold of $\mathcal{H}$, and by an operator on $\mathcal{H}$ we mean a *bounded linear* transformation of $\mathcal{H}$ into itself. Let $\mathcal{B}[\mathcal{H}]$ be the unital Banach algebra of all operators on $\mathcal{H}$. Take any $L \in \mathcal{B}[\mathcal{H}]$ and let $L^* \in \mathcal{B}[\mathcal{H}]$

Date: July 18, 2005.
2000 *Mathematics Subject Classification.* Primary 47A15; Secondary 42C40.
Keywords. Bilateral shifts, Hilbert space, invariant subspaces, orthogonal decomposition, unitary operators, wavelets, wandering subspaces.

denote the adjoint of L. The linear manifold $\operatorname{ran} L = L(\mathcal{H})$ is the range of L, and the subspace $\ker L = L^{-1}(\{0\})$ is the kernel (or null space) of L. An isometry is an operator $V \in \mathcal{B}[\mathcal{H}]$ such that $\|Vx\| = \|x\|$ for every $x \in \mathcal{H}$ or, equivalently, such that $V^*V = I$, where I denotes the identity in $\mathcal{B}[\mathcal{H}]$. Recall that isometries preserve inner product. An invertible operator is one that has a bounded inverse, and a unitary operator is an invertible isometry (equivalently, an invertible operator U such that $U^{-1} = U^*$). The orthogonal complement $\mathcal{H} \ominus \mathcal{M}$ of a subspace $\mathcal{M}$ of $\mathcal{H}$ will be denoted by $\mathcal{M}^\perp$. If $\{\mathcal{M}_\gamma\}$ is any indexed family of *pairwise orthogonal* subspaces of $\mathcal{H}$ (i.e., $\mathcal{M}_\alpha \perp \mathcal{M}_\beta$ whenever $\alpha \neq \beta$), then their direct sum $\bigoplus_\gamma \mathcal{M}_\gamma$ (the Hilbert space consisting of all square-summable families of vectors in $\mathcal{H}$ with each vector in each $\mathcal{M}_\gamma$) is unitarily equivalent to their topological sum; that is,

$$\bigoplus_\gamma \mathcal{M}_\gamma \cong \Big(\sum_\gamma \mathcal{M}_\gamma\Big)^- = \Big(\operatorname{span}\Big\{\bigcup_\gamma \mathcal{M}_\gamma\Big\}\Big)^- = \bigvee_\gamma \mathcal{M}_\gamma,$$

where $\cong$ stands for unitary equivalence. (As usual, we shall write $=$ for $\cong$.) Throughout the paper, indices m, n, j, k will always run over the set of all integers $\mathbb{Z}$, unless otherwise specified.

3. Span of Images

The (linear) span of a subset A of $\mathcal{H}$, denoted by $\operatorname{span} A$, is the linear manifold of $\mathcal{H}$ consisting of all (finite) linear combinations of vectors in A; its closure $(\operatorname{span} A)^-$ (also referred to as the *span of* A) is a subspace of $\mathcal{H}$, usually denoted by $\bigvee A$.

Lemma 1. *If A is any set of vectors in $\mathcal{H}$ and L is an operator on $\mathcal{H}$, then*

$$\Big(L\bigvee A\Big)^- = \bigvee LA.$$

Proof. Since L is continuous,

$$L(\operatorname{span} A)^- \subseteq \big(L \operatorname{span} A\big)^-$$

(see e.g., [11, Problem 3.46]). Thus

$$L\bigvee A = L(\operatorname{span} A)^- \subseteq \big(L \operatorname{span} A\big)^- \subseteq \big(L(\operatorname{span} A)^-\big)^- = \Big(L\bigvee A\Big)^-,$$

and hence

$$\Big(L\bigvee A\Big)^- = \big(L \operatorname{span} A\big)^-.$$

But L is linear and $\operatorname{span} A$ consists of finite linear combinations. Therefore,

$$L \operatorname{span} A = \operatorname{span} LA.$$

The above two identities ensure that

$$\Big(L\bigvee A\Big)^- = \big(\operatorname{span} LA\big)^- = \bigvee LA. \qquad \square$$

Remark 1. Observe that, for any set A of vectors in $\mathcal{H}$ and any operator L on $\mathcal{H}$,

$$L\bigvee A \subseteq \bigvee LA.$$

Corollary 1. *Take any set A of vectors in $\mathcal{H}$ and let L be an operator on $\mathcal{H}$.*

(a) $$L\bigvee A \ \text{ is closed}$$

if and only if

(b) $$L\bigvee A = \bigvee LA.$$

Proof. (a) implies (b) by Lemma 1. The converse is trivial. □

Corollary 2. *Take any set A of vectors in $\mathcal{H}$. If an operator L on $\mathcal{H}$ is a closed mapping* (i.e., *takes closed sets into closed sets*)*, then*

$$L\bigvee A = \bigvee LA.$$

Proof. $\bigvee A$ is closed and so is $L\bigvee A$ if L is a closed mapping. Apply Corollary 1. □

Corollary 3. *If A is any set in $\mathcal{H}$ and L is an invertible operator on $\mathcal{H}$, then*

$$L\bigvee A = \bigvee LA.$$

Proof. If L is invertible (with a bounded inverse), then it is a closed mapping (see e.g., [11, Theorem 3.24]) so that we can apply Corollary 2. □

Corollary 3 will be applied often in the sequel. We might be tempted in applying Corollary 1 for operators with closed ranges, in particular, for isometries or orthogonal projections. However, these are not necessarily closed mappings. The remark below uses orthogonal projections to show that Corollary 2 is the best we can have as far as "moving operators inside spans" is concerned.

Remark 2. Recall that the range of a continuous projection (in particular, of an orthogonal projection) is a (closed) subspace. Question: Are orthogonal projections closed mappings? In particular, is the image $P(\mathcal{N})$ of a (closed) subspace $\mathcal{N}$ of $\mathcal{H}$ under an orthogonal projection $P\colon \mathcal{H} \to \mathcal{H}$ a (closed) subspace of $\mathcal{H}$? The answer is "No". Therefore (Corollary 2), with $\mathcal{N} = \bigvee A$, it may happen that

$$P\bigvee A \neq \bigvee PA \qquad \Big(\text{i.e., } \textit{proper} \text{ inclusion: } P\bigvee A \subset \bigvee PA\Big).$$

Example. Let $\{e_k\}_{k=1}^{\infty}$ be the canonical orthonormal basis for $\mathcal{H} = \ell^2_+$, put

$$v_k = \big(1 - \tfrac{1}{k^2}\big)^{\frac{1}{2}} e_{2k-1} + \tfrac{1}{k} e_{2k}$$

for each $k \geq 1$ and consider the following subspaces of $\mathcal{H}$.

$$\mathcal{M} = \bigvee\{e_{2k-1}\}_{k=1}^{\infty} \quad \text{and} \quad \mathcal{N} = \bigvee\{v_k\}_{k=1}^{\infty}.$$

Let $P\colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{M}^\perp = \bigvee\{e_{2k}\}_{k=1}^{\infty}$ (i.e., $\operatorname{ran} P = \mathcal{M}^\perp$ and so $\ker P = \mathcal{M}$). Observe that

$$P(\mathcal{N}) \subseteq \mathcal{M} + \mathcal{N}.$$

Indeed, if $v \in \mathcal{N}$, then $v = \sum_k \alpha_k v_k$ so that

$$Pv = \sum_k \alpha_k P v_k = \sum_k \alpha_k \tfrac{1}{k} e_{2k} = \sum_k \alpha_k v_k - \sum_k \alpha_k \big(1 - \tfrac{1}{k^2}\big)^{\frac{1}{2}} e_{2k-1}.$$

Now put $x_n = \sum_{k=1}^n \frac{1}{k} e_{2k} = \sum_{k=1}^n P v_k = P\big(\sum_{k=1}^n v_k\big)$ for each $n \geq 1$ so that $\{x_n\}$ is a sequence of vectors in $P(\mathcal{N})$. Note that $\{x_n\}$ converges in ℓ^2_+ to $x = \sum_{k=1}^\infty \frac{1}{k} e_{2k}$ in ℓ^2_+. However, $x \notin \mathcal{M} + \mathcal{N}$ (cf. [11, Problem 5.13]) so that $x \notin P(\mathcal{N})$. Thus the Closed Set Theorem ensures that $P(\mathcal{N})$ is not closed.

4. Bilateral Shifts

There are some alternative (but equivalent) ways of defining bilateral shifts on a Hilbert space (see e.g., [2], [4], [6], [8], [10], [15], [21]). One of them goes as follows.

Definition 1. An operator S on $\mathcal{H}$ is a *bilateral shift* if there exists an infinite family $\{\mathcal{W}_k\}$ indexed by $\mathbb{Z}$ of nonzero pairwise orthogonal subspaces of $\mathcal{H}$ that spans $\mathcal{H}$ and S maps each $\mathcal{W}_k$ isometrically onto $\mathcal{W}_{k+1}$.

That is, an operator S on $\mathcal{H}$ is a bilateral shift if there exists a family $\{\mathcal{W}_k\}_{k\in\mathbb{Z}}$ of nonzero subspaces of $\mathcal{H}$ satisfying the following conditions.

(a) $\mathcal{W}_j \perp \mathcal{W}_k$ whenever $j \neq k$,

(b) $\bigvee_k \mathcal{W}_k = \mathcal{H}$ (which, according to (a), means $\bigoplus_k \mathcal{W}_k = \mathcal{H}$),

(c) $S|_{\mathcal{W}_k} : \mathcal{W}_k \to \mathcal{W}_{k+1}$ is a unitary transformation for each k.

Condition (c) ensures that $\{W_k\}$ is a family of unitarily equivalent subspaces, and so these subspaces $\mathcal{W}_k$ are all of the same dimension. (Note: here dimension means "orthogonal" dimension: cardinality of any orthonormal basis.) This common dimension (the constant $\dim \mathcal{W}_k$) is the *multiplicity* of S. With respect to the (orthogonal) direct sum decomposition $\mathcal{H} = \bigoplus_{k=-\infty}^{\infty} \mathcal{W}_k$ of conditions (a) and (b), S is identified with the following (doubly) infinite matrix of transformations (the inner parenthesis indicates the zero-zero entry),

$$S = \begin{pmatrix} \ddots & & & & & \\ & O & & & & \\ & U_{-1} & O & & & \\ & & U_0 & (O) & & \\ & & & U_1 & O & \\ & & & & & \ddots \end{pmatrix} \quad \text{and} \quad S^* = \begin{pmatrix} \ddots & & & & & \\ & O & U_{-1}^* & & & \\ & & O & U_0^* & & \\ & & & (O) & U_1^* & \\ & & & & O & \\ & & & & & \ddots \end{pmatrix},$$

where every entry directly below the main block diagonal in the matrix of S, namely, $U_k = S|_{\mathcal{W}_k}$, is unitary as in (c) and the remaining entries are all null. Thus it is readily verified that $S^*S = SS^* = I$, which means that a bilateral shift S is unitary.

A subspace $\mathcal{W}$ of $\mathcal{H}$ is *wandering* for any operator L on $\mathcal{H}$ (or *L-wandering*) if

$$\mathcal{W} \perp L^n \mathcal{W} \quad \text{for all } \textit{positive} \text{ integers } n.$$

Proposition 1. *If U is a unitary operator, then $\mathcal{W}$ is U-wandering if and only if*

$$U^j \mathcal{W} \perp U^k \mathcal{W} \quad \text{whenever } j \neq k.$$

Proof. Let U be a unitary on $\mathcal{H}$ and take arbitrary integers j and k in $\mathbb{Z}$. Since U is invertible and its inverse is unitary, it follows that U^j is unitary. Recall that unitary operators preserve inner product. Thus, for any subspaces $\mathcal{W}$ and $\mathcal{M}$ of $\mathcal{H}$,

$$\mathcal{W} \perp \mathcal{M} \quad \Longleftrightarrow \quad U^j\mathcal{W} \perp U^k\mathcal{M}.$$

Then, for each integer $n \geq 1$, $\mathcal{W} \perp U^n\mathcal{W}$ if and only if $U^j\mathcal{W} \perp U^{j+n}\mathcal{W}$ so that

$$\mathcal{W} \perp U^n\mathcal{W} \quad \Longleftrightarrow \quad U^j\mathcal{W} \perp U^k\mathcal{W} \text{ whenever } j \neq k. \qquad \square$$

If U is a unitary operator on $\mathcal{H}$ and $\mathcal{W}$ is a U-wandering subspace of $\mathcal{H}$ such that $\{U^k\mathcal{W}\}$ spans $\mathcal{H}$, then we say that $\mathcal{W}$ is a *generating* wandering subspace. In other words, a U-wandering subspace $\mathcal{W}$ of $\mathcal{H}$ (for some unitary $U \in \mathcal{B}[\mathcal{H}]$) is generating if (cf. Proposition 1)

$$\bigvee_k U^k\mathcal{W} = \mathcal{H} \quad \text{or, equivalently,} \quad \bigoplus_k U^k\mathcal{W} = \mathcal{H}.$$

Let S be a bilateral shift as in Definition 1. Since S is unitary, and

$$\mathcal{W}_j \perp \mathcal{W}_k \quad \text{and} \quad \mathcal{W}_{m+k} = S^m\mathcal{W}_k$$

for every k, m and every $j \neq k$, all in $\mathbb{Z}$, it also follows by Proposition 1 that $\mathcal{W}_0$ is S-wandering, and so a generating wandering subspace for S by (a), (b). That is,

$$S^j\mathcal{W}_0 \perp S^k\mathcal{W}_0 \text{ whenever } j \neq k \quad \text{and} \quad \bigvee_k S^k\mathcal{W}_0 = \mathcal{H}.$$

Moreover, the multiplicity of S is precisely the dimension of $\mathcal{W}_0$. According to (b) $\mathcal{H}$ is separable if and only if $\mathcal{W}_0$ (equivalently, any $\mathcal{W}_k$) is separable. In this case, a shift of countable multiplicity μ is the (countable, orthogonal) direct sum of μ shifts of multiplicity one. Recall that two shifts have the same multiplicity if and only if they are unitarily equivalent, and that a finite power n of a shift of multiplicity μ is again a shift, now of multiplicity $n\mu$ (see e.g., [6, Section 1] and [10, Chapter 2]). Thus a shift of (finite) even multiplicity has a unique square root that is again a shift, and so has a shift of countably infinite multiplicity (see e.g., [9, p.273]). Next we consider an equivalent definition of a bilateral shift (see e.g., [2, Chapter 6]).

Definition 2. A *bilateral shift* S on $\mathcal{H}$ is a unitary operator for which there is subspace $\mathcal{V}$ of $\mathcal{H}$ (called *outgoing* subspace) satisfying the following conditions.

(i) $S\mathcal{V} \subset \mathcal{V}$,
(ii) $\bigcap_k S^k\mathcal{V} = \{0\}$,
(iii) $\big(\bigcup_k S^k\mathcal{V}\big)^- = \mathcal{H}$.

It is worth noticing that condition (iii) in fact means $\bigvee_k S^k\mathcal{V} = \mathcal{H}$ — recall that $\bigvee_k S^k\mathcal{V} = \big(\text{span}\,\big\{\bigcup_k S^k\mathcal{V}\big\}\big)^-$. Indeed, the family of subspaces $\{S^k\mathcal{V}\}$ is decreasing (i.e., $S^{k+1}\mathcal{V} \subset S^k\mathcal{V}$) by condition (i), and hence $\bigcup_k S^k\mathcal{V} = \text{span}\,\big\{\bigcup_k S^k\mathcal{V}\big\}$. Moreover, since U^* is unitary whenever U is unitary, a bilateral shift can be similarly defined in terms of is adjoint $S^* = S^{-1}$ by replacing the above conditions with

(i′) $\mathcal{V}' \subset S^*\mathcal{V}'$,
(ii′) $\bigcap_k S^k\mathcal{V}' = \{0\}$,
(iii′) $\big(\bigcup_k S^k\mathcal{V}'\big)^- = \mathcal{H}$,

where the subspace $\mathcal{V}'$ of $\mathcal{H}$ is referred to as *incoming* subspace. Indeed, put

$$\mathcal{V}' = S\mathcal{V}.$$

Since S is unitary, it has a continuous inverse, and so the linear manifold $\mathcal{V}'$ is as closed as $\mathcal{V}$ ($\mathcal{V}'$ is a subspace of $\mathcal{H}$ whenever $\mathcal{V}$ is). Note that $S^*\mathcal{V}' = S^*(S\mathcal{V}) = \mathcal{V}$. Since $S\mathcal{V} \subset \mathcal{V}$ by (i), we get $\mathcal{V}' = SS^*\mathcal{V}' = S\mathcal{V} \subset \mathcal{V} = S^*\mathcal{V}'$. This shows how (i) implies (i′); the converse follows by the same argument once S is unitary. The equivalence between (ii) and (ii′), and also between (iii) and (iii′), are trivially verified since $S^k\mathcal{V}' = S^{k+1}\mathcal{V}$ and $S^k\mathcal{V} = S^kS^*\mathcal{V}' = S^{k-1}\mathcal{V}'$. Observe that the family of subspaces $\{S^{*k}\mathcal{V}'\}$ is an increasing family (i.e., $S^{*k}\mathcal{V}' \subset S^{*(k+1)}\mathcal{V}'$) according to condition (i′). Actually, conditions (i) and (i′) are equivalent to

$$\text{(i)}\quad S^{k+1}\mathcal{V} \subset S^k\mathcal{V} \qquad\qquad \text{and} \qquad\qquad \text{(i}'\text{)}\quad S^{*k}\mathcal{V}' \subset S^{*(k+1)}\mathcal{V}'$$

for every k in $\mathbb{Z}$, respectively. The link between Definitions 1 and 2 is given by the following result. (For a proof see, for instance, [12] and the references therein; in particular, see [8] and [15]).

Proposition 2. *Let S be a unitary operator on a Hilbert space $\mathcal{H}$ as in Definition 2 and let $\mathcal{V}$ be the outgoing subspace. Then*

$$\mathcal{W}_0 = \mathcal{V} \ominus S\mathcal{V}$$

is a generating wandering subspace for S such that

$$\mathcal{V} = \bigoplus_{n=0}^{\infty} S^n\mathcal{W}_0,$$

and S is a bilateral shift as in Definition 1 with $\mathcal{W}_k = S^k\mathcal{W}_0$ for each k in $\mathbb{Z}$.

Remark 3. Since the incoming space $\mathcal{V}' = S\mathcal{V}$ is such that $S^*\mathcal{V}' = \mathcal{V}$, dual expressions of those in the above proposition read as follows.

$$\mathcal{W}_0 = S^*\mathcal{V}' \ominus \mathcal{V}' \quad \text{and} \quad \mathcal{V}' = \bigoplus_{n=1}^{\infty} S^n\mathcal{W}_0 = \bigoplus_{n=-1}^{-\infty} S^{*n}\mathcal{W}_0.$$

5. Wavelet Vectors

For simplicity we shall assume, from now on, that $\mathcal{H}$ is a *separable* Hilbert space so that shifts of infinite multiplicity on $\mathcal{H}$ are necessarily of *countably* infinite multiplicity. The basic assumption is: suppose

(A_0) $\qquad D$ and T are bilateral shifts on $\mathcal{H}$ of infinite multiplicity.

Thus D and T are unitarily equivalent. Assume that *the unitarily equivalent shifts D and T do not commute but D intertwines T^2 to T so that T^2 is unitarily equivalent to T via the unitary D.* That is, suppose

(A_1) $$DT \neq TD \quad \text{and} \quad DT^2 = TD.$$

The next proposition exhibits a collection of conditions equivalent to $DT^2 = TD$ in Assumption (A_1).

Proposition 3. *The following assertions are pairwise equivalent.*

(a) $DT^2 = TD$,

(b) $D^m T^{2^m} = TD^m$ *for every* $m \in \mathbb{Z}$,

(c) $T^{\frac{1}{2^m}} D^m = D^m T$ *for every* $m \in \mathbb{Z}$,

(a′) $DT^{2n} = T^n D$ *for every* $n \in \mathbb{Z}$,

(b′) $D^m T^{n2^m} = T^n D^m$ *for every* $m \in \mathbb{Z}$ *and every* $n \in \mathbb{Z}$,

(c′) $T^{\frac{n}{2^m}} D^m = D^m T^n$ *for every* $m \in \mathbb{Z}$ *and every* $n \in \mathbb{Z}$.

Proof. Note that any of the above assertions trivially implies (a). Thus we shall show that each of them is implied by (a). To verify that (a) implies (b) proceed as follows. If $DT^2 = TD$, then $D^n T^{2^n} = TD^n$ holds tautologically for $n = 0, 1$ and, if this holds for some $n \geq 1$, then it holds for $n + 1$. Indeed,

$$D^{n+1} T^{2^{n+1}} = DD^n T^{2^n 2} = DD^n T^{2^n} T^{2^n} = DTD^n T^{2^n} = DT^2 D^n = TDD^n = TD^{n+1},$$

which proves by induction that

$$DT^2 = TD \quad \text{implies} \quad D^n T^{2^n} = TD^n \ \text{ for every } \ n \geq 0.$$

Since T is a bilateral shift of *infinite multiplicity*, it has a unique iterative square root, which means that there exists a unique square root $(T^{\frac{1}{2^n}})^{\frac{1}{2}}$ of $T^{\frac{1}{2^n}}$ for each $n \geq 0$. In this case, it also follows that,

$$DT^2 = TD \quad \text{implies} \quad D^{-n} T^{\frac{1}{2^n}} = TD^{-n} \ \text{ for every } \ n \geq 1.$$

Indeed, $T^2 = (D^{-1} T^{\frac{1}{2}} D)^2$ so that (uniqueness of the square root) $T = D^{-1} T^{\frac{1}{2}} D$, and hence the sought result holds for $n = 1$. If it holds for some $n \geq 1$, then $(T^{\frac{1}{2^{n+1}}})^2 = T^{\frac{1}{2^n}} = D^n T D^{-n} = (D^n T^{\frac{1}{2}} D^{-n})^2$ so that (uniqueness of the square root again) $T^{\frac{1}{2^{n+1}}} = D^n T^{\frac{1}{2}} D^{-n} = D^n DTD^{-1} D^{-n}$, and hence the sought result holds for $n + 1$ (i.e., $D^{-(n+1)} T^{\frac{1}{2^{n+1}}} = TD^{-(n+1)}$), which completes the proof by induction. From the above two implications we can infer that (a) implies (b). Now, to show that (b) implies (c), multiply (b) by D^{-m} both from left and from right, and replace m with $-m$ (since (b) holds for every $m \in \mathbb{Z}$) to get the form in (c). Next, to prove that (a) implies (a′), observe that (a′) is a tautology for $n = 0$, (a) implies (a′) for $n = 1$ trivially and, if (a′) holds for some $n \geq 1$ then, by (a),

$$DT^{2(n+1)} = DT^{2n} T^2 = T^n DT^2 = T^n TD = T^{n+1} D,$$

which proves by induction that

$$DT^2 = TD \quad \text{implies} \quad DT^{2n} = T^n D \ \text{ for every } \ n \geq 0,$$

and therefore $T^{-n} D = DT^{-2n}$ for every $n \geq 0$ because T is invertible. Thus (a) implies (a′). Since T^n is a bilateral shift of infinite multiplicity for every $n \geq 0$, it follows that (a′) implies (b′) as (a) implies (b), and so (b′) implies (c′) as (b) implies (c). □

Remark 4. Recall that $(L^{-1})^* = (L^*)^{-1}$ and $(L^{-1})^n = (L^n)^{-1}$ (denoted L^{-n}) for every invertible operator L and every integer $n \in \mathbb{Z}$; also $(L^n)^* = (L^*)^n$ (denoted L^{*n}) for every operator L and every integer $n \geq 0$ (or every integer $n \in \mathbb{Z}$ if L is invertible); and that $L^* = L^{-1}$ if L is unitary. Thus assertions (a), (b), (c), (a$'$), (b$'$) and (c$'$) of Proposition 3 are equivalent to

(a*) $\;D\,T^{*2} = T^*D$,

(b*) $\;D^mT^{*2^m} = T^*D^m$ for every $m \in \mathbb{Z}$,

(c*) $\;T^{*\frac{1}{2^m}}D^m = D^mT^*$ for every $m \in \mathbb{Z}$,

(a$^{*\prime}$) $\;D\,T^{*2n} = T^{*n}D$ for every $n \in \mathbb{Z}$,

(b$^{*\prime}$) $\;D^mT^{*n2^m} = T^{*n}D^m$ for every $m \in \mathbb{Z}$ and every $n \in \mathbb{Z}$,

(c$^{*\prime}$) $\;T^{*\frac{n}{2^m}}D^m = D^mT^{*n}$ for every $m \in \mathbb{Z}$ and every $n \in \mathbb{Z}$,

respectively. Therefore, assertions (a), (b), (c), (a$'$), (b$'$) and (c$'$) of Proposition 3 and the above (a*), (b*), (c*), (a$^{*\prime}$), (b$^{*\prime}$) and (c$^{*\prime}$) are all pairwise equivalent.

Moreover, also assume that $\{D^mT^nw\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ *is a* (double indexed, countable) *orthonormal basis for the* (separable) *Hilbert space* $\mathcal{H}$ *for some nonzero vector* w *in* $\mathcal{H}$. That is, suppose there exists $0 \neq w$ in $\mathcal{H}$ such that

$$(\mathrm{A}_2)\qquad \begin{cases} \text{(a)} \quad D^mT^nw \perp D^jT^kw \quad \text{whenever} \quad (m,n) \neq (j,k) \text{ in } \mathbb{Z}\times\mathbb{Z}, \\ \text{(b)} \quad \|D^mT^nw\| = 1 \quad \text{for every} \quad (m,n) \in \mathbb{Z}\times\mathbb{Z}, \\ \text{(c)} \quad \bigvee_{m,n} D^mT^nw = \mathcal{H}. \end{cases}$$

Definition 3. If D and T are operators on $\mathcal{H}$ satisfying Assumptions A_0 and A_1, then any nonzero vector w in $\mathcal{H}$ that makes $\{D^mT^nw\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ an orthonormal basis for $\mathcal{H}$ as in Assumption A_2 is a *wavelet* (or an *orthogonal wavelet*), and the vectors $w_{m,n} = D^mT^nw$ are the *wavelet vectors generated from the wavelet* w.

Remark 5. If $\mathcal{H}$ represents some concrete Hilbert space of functions, then the term "wavelet functions" is used instead of "wavelet vectors". For instance, if $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ and $x \in \mathcal{L}^2(\mathbb{R})$, then the operators D and T on $\mathcal{L}^2(\mathbb{R})$ defined by

$$y = Dx \quad \text{with} \quad y(t) = \sqrt{2}\,x(2t)$$

and

$$y = Tx \quad \text{with} \quad y(t) = x(t-1)$$

(for almost all t in $\mathbb{R}$ with respect to Lebesgue measure) are bilateral shifts on $\mathcal{L}^2(\mathbb{R})$ satisfying Assumptions A_0 and A_1 (see [5, p.4] for discussions on Assumption A_1). A well-known wavelet associated with the above operators is the so-called Haar wavelet (e.g., [19, p.248]), the function w in $\mathcal{L}^2(\mathbb{R})$ defined (almost everywhere) by

$$w(t) = \begin{cases} 1; & 0 < t \leq \frac{1}{2}, \\ -1; & \frac{1}{2} < t \leq 1, \\ 0; & t \in \mathbb{R}\backslash(0,1]. \end{cases}$$

More generally, for any real number α let T_α on $\mathcal{L}^2(\mathbb{R})$ be defined by

$$y = T_\alpha x \quad \text{with} \quad y(t) = x(t-\alpha)$$

almost everywhere in $\mathbb{R}$. It is readily verified that this is again a bilateral shift on $\mathcal{L}^2(\mathbb{R})$ — the shifting unit is now α rather than 1 — that together with D satisfy Assumption A_1. Note that any integer power, say n, of T_α coincides with $T_{n\alpha}$:

$$T_\alpha^n = T_{n\alpha} \ \text{ for every } \ n \in \mathbb{Z} \ \text{ and } \ \alpha \in \mathbb{R},$$

and hence the identities of Proposition 3(b,c) are given by

$$D^m T_{2^m\alpha} = T_\alpha D^m \quad \text{and} \quad T_{\frac{\alpha}{2^m}} D^m = D^m T_\alpha$$

for every $m \in \mathbb{Z}$ and every $\alpha \in \mathbb{R}$.

6. Wavelet Expansion

Now take an arbitrary $x \in \mathcal{H}$ and consider the following families $\{\mathcal{W}_m(x)\}_{m\in\mathbb{Z}}$ and $\{\mathcal{H}_n(x)\}_{n\in\mathbb{Z}}$ of subspaces of $\mathcal{H}$.

6.1. **The subspaces $\mathcal{W}_m(x)$.** For each vector $x \in \mathcal{H}$ let $\{\mathcal{W}_m(x)\}_{m\in\mathbb{Z}}$ be a family of subspaces of $\mathcal{H}$ defined by

$$\mathcal{W}_0(x) = \bigvee_n T^n x \quad \text{and} \quad \mathcal{W}_m(x) = D^m \mathcal{W}_0(x) = D^m \bigvee_n T^n x \quad \text{for each } \ m \in \mathbb{Z},$$

where the spans run over all integers n in $\mathbb{Z}$, so that

$$\mathcal{W}_m(x) = \mathcal{W}_m(T^k x) \quad \text{and} \quad \mathcal{W}_{m+k}(x) = D^k \mathcal{W}_m(x) \quad \text{for each } \ m, k \in \mathbb{Z}.$$

According to Corollary 3, these subspaces can also be written as

$$\mathcal{W}_m(x) = \bigvee_n D^m T^n x \quad \text{for each } \ m \in \mathbb{Z}.$$

We shall see below that if $w \in \mathcal{H}$ is a wavelet, then $\{\mathcal{W}_m(w)\}$ is a family of pairwise *orthogonal* subspaces, and this implies that each $\mathcal{W}_m(w)$ is not D-invariant nor D^*-invariant because $D\,\mathcal{W}_m(w) = \mathcal{W}_{m+1}(w)$ and $D^*\mathcal{W}_{m+1}(w) = \mathcal{W}_m(w)$.

Proposition 4. *If w is a wavelet, then $\mathcal{W}_0(w)$ is a generating wandering subspace for the bilateral shift D so that $\{\mathcal{W}_m(w)\}$ is the family of subspaces that actually define the bilateral shift D as in Definition 1. That is,*

(a) $\mathcal{W}_j(w) \perp \mathcal{W}_m(w)$ *whenever* $j \neq m$,

(b) $\bigvee_m \mathcal{W}_m(w) = \mathcal{H}$ (which, according to (a), means $\bigoplus_m \mathcal{W}_m(w) = \mathcal{H}$),

(c) $D|_{\mathcal{W}_m} : \mathcal{W}_m(w) \to \mathcal{W}_{m+1}(w)$ *is a unitary transformation for each* m.

Proof. By linearity in the first argument and continuity of the inner product, and also by Corollary 3, it follows from Assumption A_2(a) that, whenever $m \neq j$,

$$D^m \mathcal{W}_0(w) = \mathcal{W}_m(w) = \bigvee_n D^m T^n w \perp \bigvee_k D^j T^k w = \mathcal{W}_j(w) = D^j \mathcal{W}_0(w).$$

Then $\mathcal{W}_0(w)$ is a wandering subspace for the unitary operator D according to Proposition 1, which in fact is a generating subspace by Assumption A_2(c). Indeed, by unconditional convergence of the Fourier series (for the double indexed orthonormal basis $\{D^m T^n w\}$ for the Hilbert space $\mathcal{H}$ of Assumption A_2) we get

$$\bigvee_m D^m\mathcal{W}_0(w)=\bigvee_m\bigvee_n D^mT^nw=\bigvee_n\bigvee_m D^mT^nw=\bigvee_{m,n} D^mT^nw=\mathcal{H}.$$

Thus assertions (a) and (b) follow by the very definition of the subspaces $\mathcal{W}_m(w)$, and assertion (c) follows from the fact $D|_{\mathcal{W}_m}$ is and isometry (reason: restriction of an isometry to any subspace is again an isometry), which is surjective (again, by the very definition of $\mathcal{W}_m(w)$) so that $D|_{\mathcal{W}_m}$ is a unitary transformation. □

Remark 6. Consider the setup of Proposition 4. Note that the incoming space $\mathcal{V}(w)$ of Definition 2 (with respect to the bilateral shift D) is, by Proposition 2,

$$\mathcal{V}(w)=\bigoplus_{n=0}^{\infty} D^n\mathcal{W}_0(w)=\bigoplus_{n=0}^{\infty}\mathcal{W}_n(w)=\bigvee_{n=0}^{\infty}\mathcal{W}_n(w),$$

where the third identity follows from Proposition 4(a) so that, by Corollary 3,

$$D^k\mathcal{V}(w)=\bigvee_{n=0}^{\infty} D^k\mathcal{W}_n(w)=\bigvee_{n=0}^{\infty}\mathcal{W}_{n+k}(w)=\bigvee_{n=k}^{\infty}\mathcal{W}_n(w)\quad\text{for each}\quad k\in\mathbb{Z}$$

and all conditions of Definition 2 are verified, as expected. Indeed,

(i) $D^{k+1}\mathcal{V}(w)=\bigvee_{n=k+1}^{\infty}\mathcal{W}_n(w)\subset\bigvee_{n=k}^{\infty}\mathcal{W}_n(w)=D^k\mathcal{V}(w)$ for every $k\in\mathbb{Z}$,

(ii) $\bigcap_k D^k\mathcal{V}(w)=\{0\}$ (cf. matrix representation of the bilateral shift D),

(iii) $\big(\bigcup_k D^k\mathcal{V}(w)\big)^-=\bigvee_k D^k\mathcal{V}(w)=\bigvee_k\mathcal{W}_k(w)=\mathcal{H}$ (cf. Proposition 4(b)).

If $w\in\mathcal{H}$ is a wavelet, then Proposition 4(a,b) supplies an orthogonal direct sum decomposition of $\mathcal{H}$ into $\{\mathcal{W}_m(w)\}$ (also see [1], [7], [13], [14], [17], [18] and [20]),

$$\mathcal{H}=\bigoplus_m\mathcal{W}_m(w)=\bigvee_m\mathcal{W}_m(w)=\bigvee_n\bigvee_m D^mT^nw=\bigvee_{m,n} D^mT^nw,$$

which is referred to as a *wavelet expansion.* Indeed, for each pair of integers (m,n) in $\mathbb{Z}\times\mathbb{Z}$ consider the wavelet vector $w_{m,n}=D^mT^nw\in\mathcal{H}$ generated from the wavelet $w\in\mathcal{H}$. Thus, according to Assumption A_2, every vector $x\in\mathcal{H}$ has a Fourier series expansion $x=\sum_{m,n}\langle x\,;w_{m,n}\rangle w_{m,n}$ in terms of the double indexed orthonormal basis $\{w_{m,n}\}$ for $\mathcal{H}$ generated from the wavelet w as in Definition 3. Note that, as it happens for any double indexed orthonormal basis $\{w_{m,n}\}$ for a Hilbert space $\mathcal{H}$, the Fourier series expansion of every vector x in $\mathcal{H}$ has a "Fubini-like" property where the summation order can always be interchanged,

$$x=\sum_m\sum_n\langle x\,;w_{m,n}\rangle w_{m,n}=\sum_{m,n}\langle x\,;w_{m,n}\rangle w_{m,n}=\sum_n\sum_m\langle x\,;w_{m,n}\rangle w_{m,n},$$

due to the unconditional convergence of the Fourier series.

6.2. **The subspaces $\mathcal{H}_n(x)$.** For each vector $x\in\mathcal{H}$ let $\{\mathcal{H}_n(x)\}_{n\in\mathbb{Z}}$ be a family of subspaces of $\mathcal{H}$ defined by

$$\mathcal{H}_0(x)=\bigvee_m D^mx\quad\text{and}\quad\mathcal{H}_n(x)=\mathcal{H}_0(T^nx)=\bigvee_m D^mT^nx\quad\text{for each}\;\; n\in\mathbb{Z},$$

where the spans run over all integers n in $\mathbb{Z}$, so that (cf. Corollary 3)

$$\mathcal{H}_n(x)=D^k\mathcal{H}_n(x)\quad\text{and}\quad\mathcal{H}_{n+k}(x)=\mathcal{H}_n(T^kx)\quad\text{for each}\;\; n,k\in\mathbb{Z}.$$

An important feature of these subspaces is that, for any vector $x \in \mathcal{H}$ (in particular, for a wavelet $w \in \mathcal{H}$), each $\mathcal{H}_n(x)$ reduces D, which means that each $\mathcal{H}_n(x)$ is both D-invariant and D^*-invariant. In fact, $D^* = D^{-1}$ and $D^{\pm 1}\mathcal{H}_n(x) = \mathcal{H}_n(x)$ (according to the above consequence of Corollary 3) for every $n \in \mathbb{Z}$.

Recall that, if w is a wavelet, then $\{\mathcal{W}_m(w)\}$ is a family of pairwise orthogonal subspaces of $\mathcal{H}$ that are not D-invariant, and so they cannot be used to decompose the bilateral shift D. The next result explores the fact that $\{\mathcal{H}_n(w)\}$ is again a family of pairwise orthogonal subspaces spanning $\mathcal{H}$, thus another *wavelet expansion*,

$$\bigvee_{m,n} D^m T^n w = \bigvee_m \mathcal{W}_m(w) = \bigoplus_m \mathcal{W}_m(w) = \mathcal{H} = \bigoplus_n \mathcal{H}_n(w) = \bigvee_n \mathcal{H}_n(w) = \bigvee_{n,m} D^m T^n w$$

(also see [13], [14], [17] and [18]), which has the extra property of reducing D.

Theorem 1. *If $w \in \mathcal{H}$ is a wavelet, then $\{\mathcal{H}_n(w)\}$ is a family of pairwise orthogonal subspaces of $\mathcal{H}$ that spans $\mathcal{H}$:*

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n(w).$$

Moreover, each $\mathcal{H}_n(w)$ reduces D so that

$$D = \bigoplus_n D_n(w),$$

with each $D_n(w) = D|_{\mathcal{H}_n(w)}$ being a bilateral shift of multiplicity one acting on each subspace $\mathcal{H}_n(w)$.

Proof. Since $\{D^m T^n w\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}$, it follows that $\{\mathcal{H}_n(w)\}$ is a family of pairwise orthogonal subspaces of $\mathcal{H}$, and Theorem 1 from [14] ensures that $\bigoplus_m \mathcal{W}_m(w) = \bigoplus_n \mathcal{H}_n(w)$. Thus $\mathcal{H} = \bigoplus_n \mathcal{H}_n(w)$ by Proposition 4(b). Furthermore, as we saw above, each $\mathcal{H}_n(x)$ reduces D so that $D = \bigoplus_n D|_{\mathcal{H}_n(w)}$. Putting $D_n(w) = D|_{\mathcal{H}_n(w)}$ for each n we get

$$D_n(w)\, w_{m,n} = D|_{\mathcal{H}_n(w)} D^m T^n w = D^{m+1} T^n w = w_{m+1,n}.$$

Hence each $D_n(w)$ shifts the orthonormal basis $\{w_{m,n}\}_{m\in\mathbb{Z}}$ for each Hilbert space $\mathcal{H}_n$, and so each $D_n(w)$ is a bilateral shift of multiplicity one acting on $\mathcal{H}_n$. □

7. Properties of $\mathcal{W}_m(x)$ and $\mathcal{H}_n(x)$

Take an arbitrary vector (not necessarily a wavelet) $x \in \mathcal{H}$. We had already seen that, for any integers $k, m, n \in \mathbb{Z}$,

$$D^k \mathcal{W}_m(x) = \mathcal{W}_{m+k}(x) = D^{m+k}\mathcal{W}_0(x) \quad \text{and} \quad D^k \mathcal{H}_n(x) = \mathcal{H}_n(x) = \mathcal{H}_0(T^n x)$$

(the second identity showing that each $\mathcal{H}_n(x)$ reduces D) so that the actions of D on $\mathcal{W}_m(x)$ and $\mathcal{H}_n(x)$ are evident from the very definitions of these subspaces. We had also seen — again, as an immediate consequence of the definitions of $\mathcal{W}_m(x)$ and $\mathcal{H}_n(x)$ — that, for any integers $k, m, n \in \mathbb{Z}$,

$$\mathcal{W}_m(x) = \mathcal{W}_m(T^k x) \quad \text{and} \quad \mathcal{H}_{n+k}(x) = \mathcal{H}_n(T^k x),$$

which show how these subspaces are affected (or not affected) by images of all integral powers of T. Applying Proposition 3 we now prove further relations in this line. In particular, we consider the action of T on $\mathcal{W}_m(x)$ and $\mathcal{H}_n(x)$.

Proposition 5. *For any vector* $x \in \mathcal{H}$,

(a) $T\,\mathcal{W}_m(x) = \mathcal{W}_m(T^{2^m}x)$ *for every* $m \in \mathbb{Z}$,

(b) $\mathcal{W}_m(x) = T^{\frac{1}{2^m}}\mathcal{W}_m(T^*x)$ *for every* $m \in \mathbb{Z}$,

(c) $\mathcal{H}_{2^n}(x) = \mathcal{H}_1(D^n x)$ *for every* $n \geq 0$,

(d) $\mathcal{H}_{-n}(x) = \mathcal{H}_{2^n}(D^{*n}T^{*(n+1)}x)$ *for every* $n \geq 0$.

Dually,

(a*) $T^*\mathcal{W}_m(x) = \mathcal{W}_m(T^{*2^m}x)$ *for every* $m \in \mathbb{Z}$,

(b*) $\mathcal{W}_m(x) = T^{*\frac{1}{2^m}}\mathcal{W}_m(Tx)$ *for every* $m \in \mathbb{Z}$,

(c*) $\mathcal{H}_{-2^n}(x) = \mathcal{H}_{-1}(D^n x)$ *for every* $n \geq 0$,

(d*) $\mathcal{H}_n(x) = \mathcal{H}_{-2^n}(D^{*n}T^{n+1}x)$ *for every* $n \geq 0$.

Proof. Take an arbitrary $x \in \mathcal{H}$ and consider the definitions of the subspaces $\mathcal{W}_m(x)$ and $\mathcal{H}_n(x)$ for each $m, n \in \mathbb{Z}$. According to Corollary 3 and Proposition 3(b,c),

$$T\,\mathcal{W}_m(x) = \bigvee_n TD^mT^nx = \bigvee_n D^mT^{2^m}T^nx = \bigvee_n D^mT^nT^{2^m}x = \mathcal{W}_m(T^{2^m}x),$$

$$\mathcal{W}_m(x) = \bigvee_n D^mTT^nT^{-1}x = \bigvee_n T^{\frac{1}{2^m}}D^mT^nT^{-1}x = T^{\frac{1}{2^m}}\mathcal{W}_m(T^*x),$$

which proves (a) and (b). By Proposition 3(b) we get the result in (c):

$$\mathcal{H}_{2^n}(x) = \bigvee_m D^mT^{2^n}x = \bigvee_m D^mD^nT^{2^n}x = \bigvee_m D^mTD^nx = \mathcal{H}_1(D^nx)$$

for every $n \geq 0$. Moreover, by Proposition 3(c) we get

$$\mathcal{H}_n(x) = \bigvee_m D^mT^nx = \bigvee_m D^mD^nTT^{n-1}x = \bigvee_m D^mT^{\frac{1}{2^n}}D^nT^{n-1}x$$

for every $n \in \mathbb{Z}$ so that

$$\mathcal{H}_{-n}(x) = \bigvee_m D^mT^{\frac{1}{2^{-n}}}D^{-n}T^{-n-1}x = \bigvee_m D^mT^{2^n}D^{-n}T^{-(n+1)}x = \mathcal{H}_{2^n}(D^{*n}T^{*(n+1)}x)$$

for every $n \geq 0$, thus proving (d). Dually, by Corollary 3 and Remark 4(b*,c*),

$$T^*\mathcal{W}_m(x) = \bigvee_n T^*D^mT^nx = \bigvee_n D^mT^{*2^m}T^nx = \bigvee_n D^mT^nT^{*2^m}x = \mathcal{W}_m(T^{*2^m}x),$$

$$\mathcal{W}_m(x) = \bigvee_n D^mT^*T^nTx = \bigvee_n T^{*\frac{1}{2^m}}D^mT^nTx = T^{*\frac{1}{2^m}}\mathcal{W}_m(Tx),$$

which proves (a*) and (b*). By Remark 4(b*) we get the result in (c*):

$$\mathcal{H}_{-2^n}(x) = \bigvee_m D^mT^{-2^n}x = \bigvee_m D^mD^nT^{*2^n}x = \bigvee_m D^mT^{-1}D^nx = \mathcal{H}_{-1}(D^nx)$$

for every $n \geq 0$. Moreover, By Remark 4(c*) we get

$$\mathcal{H}_{-n}(x) = \bigvee_m D^mT^{-n}x = \bigvee_m D^mD^nT^*T^{-n+1}x = \bigvee_m D^mT^{*\frac{1}{2^n}}D^nT^{-n+1}x$$

for every $n \in \mathbb{Z}$ so that

$$\mathcal{H}_n(x) = \bigvee_m D^mT^{*\frac{1}{2^{-n}}}D^{-n}T^{n+1}x = \bigvee_m D^mT^{-2^n}D^{-n}T^{n+1}x = \mathcal{H}_{-2^n}(D^{*n}T^{n+1}x)$$

for every $n \geq 0$, thus proving (d*). $\square$

Observe that the full action of T on $\mathcal{W}_n(x)$ is clear from Proposition 5(a,a*). However, there is no counterpart of assertions (a) and (a*) for $\mathcal{H}_n(x)$. That is, we cannot express the action of T on $\mathcal{H}_n(x)$ in terms of $\mathcal{H}_{f(n)}(Fx)$ for some pair of functions f and F. Indeed, all we get from Corollary 3 and Proposition 3 is

$$T\,\mathcal{H}_n(x) = \bigvee_m TD^mT^nx = \bigvee_m D^mT^{2^m}T^nx = \bigvee_m D^mT^nT^{2^m}x.$$

Remark 7. Consider the concrete example with operators D and T_α on $\mathcal{L}^2(\mathbb{R})$, where $T_{n\alpha} = T_\alpha^n$ for every $n \in \mathbb{Z}$ and every $\alpha \in \mathbb{R}$, as in Remark 5. Take any function $x \in \mathcal{L}^2(\mathbb{R})$ (not necessarily a wavelet). For each real number α, consider the countable families of subspaces $\{\mathcal{W}_{m,\alpha}(x)\}$ and $\{\mathcal{H}_{n\alpha}(x)\}$ with

$$\mathcal{W}_{m,\alpha}(x) = \bigvee_n D^mT_{n\alpha}x \quad \text{and} \quad \mathcal{H}_{n\alpha}(x) = \bigvee_m D^mT_{n\alpha}x$$

for each m and n in $\mathbb{Z}$ so that, for every $\alpha \in \mathbb{R}$ and every $k, m, n \in \mathbb{Z}$,

$$D^k\mathcal{W}_{m,\alpha}(x) = \mathcal{W}_{(m+k),\alpha}(x) = D^{m+k}\mathcal{W}_{0,\alpha}(x),$$

$$D^k\mathcal{H}_{n\alpha}(x) = \mathcal{H}_{n\alpha}(x) = \mathcal{H}_0(T_{n\alpha}x),$$

$$\mathcal{W}_{m,\alpha}(x) = \mathcal{W}_{m,\alpha}(T_{k\alpha}x) \quad \text{and} \quad \mathcal{H}_{(n+k)\alpha}(x) = \mathcal{H}_{n\alpha}(T_{k\alpha}x).$$

In this particular case the expressions of Proposition 5 are given as follows.

(a) $T_\alpha\mathcal{W}_{m,\alpha}(x) = \mathcal{W}_{m,\alpha}(T_{2^m\alpha}x)$ for every $m \in \mathbb{Z}$ and every $\alpha \in \mathbb{R}$,
(b) $\mathcal{W}_{m,\alpha}(x) = T_{\frac{1}{2^m}\alpha}\mathcal{W}_{m,\alpha}(T_{-\alpha}x)$ for every $m \in \mathbb{Z}$ and every $\alpha \in \mathbb{R}$,
(c) $\mathcal{H}_{2^n\alpha}(x) = \mathcal{H}_\alpha(D^nx)$ for every $n \geq 0$ and every $\alpha \in \mathbb{R}$,
(d) $\mathcal{H}_{-n\alpha}(x) = \mathcal{H}_{2^n\alpha}(D^{-n}T_{-(n+1)\alpha}x)$ for every $n \geq 0$ and every $\alpha \in \mathbb{R}$,
(a*) $T_{-\alpha}\mathcal{W}_{m,\alpha}(x) = \mathcal{W}_{m,\alpha}(T_{-2^m\alpha}x)$ for every $m \in \mathbb{Z}$ and every $\alpha \in \mathbb{R}$,
(b*) $\mathcal{W}_{m,\alpha}(x) = T_{-\frac{1}{2^m}\alpha}\mathcal{W}_{m,\alpha}(Tx)$ for every $m \in \mathbb{Z}$ and every $\alpha \in \mathbb{R}$,
(c*) $\mathcal{H}_{-2^n\alpha}(x) = \mathcal{H}_{-\alpha}(D^nx)$ for every $n \geq 0$ and every $\alpha \in \mathbb{R}$,
(d*) $\mathcal{H}_{n\alpha}(x) = \mathcal{H}_{-2^n\alpha}(D^{-n}T_{(n+1)\alpha}x)$ for every $n \geq 0$ and every $\alpha \in \mathbb{R}$.

8. Projection on Wavelets

Let $\mathcal{M}$ be a subspace of $\mathcal{H}$ and let $P_\mathcal{M}\colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{M}$ (i.e., the unique orthogonal projection $P_\mathcal{M}$ of $\mathcal{H}$ into itself such that $\operatorname{ran} P_\mathcal{M} = \mathcal{M}$). In particular, for any unit vector y in $\mathcal{H}$ (i.e., for any $y \in \mathcal{H}$ with $\|y\| = 1$), the *projection on* y is the orthogonal projection $P_y\colon \mathcal{H} \to \mathcal{H}$ onto the one-dimensional space spanned by y (i.e., the unique orthogonal projection P_y of $\mathcal{H}$ into itself such that $\operatorname{ran} P_y = \operatorname{span}\{y\}$), which is given by

$$P_yx = \langle x\,;y\rangle y \quad \text{for every} \quad x \in \mathcal{H}.$$

Theorem 2. *Let D and T be operators on $\mathcal{H}$ satisfying Assumptions* A_0 *and* A_1*, and let w be a wavelet in $\mathcal{H}$. Every $x \in \mathcal{H}$ admits the following decompositions.*

(a) $x = \sum_m \sum_n (D^m T^n) P_w (D^m T^n)^* x$
$\quad = \sum_m D^m \sum_n P_{T^n w} D^{*m} x = \sum_m D^m P_{\mathcal{W}_0} D^{*m} x,$

(b) $x = \sum_n \sum_m (D^m T^n) P_w (D^m T^n)^* x = \sum_n \sum_m D^m P_{T^n w} D^{*m} x,$

(c) $x = \sum_n \sum_m T^{\frac{n}{2^m}} (D^m P_w D^{*m}) T^{*\frac{n}{2^m}} x = \sum_n \sum_m T^{\frac{n}{2^m}} P_{D^m w} T^{*\frac{n}{2^m}} x.$

Proof. Take an arbitrary $x \in \mathcal{H}$. Let w be a wavelet in $\mathcal{H}$. Observe that $D^m T^n w$ is a unit vector for each $m, n \in \mathbb{Z}$ and

$$\begin{aligned} P_{D^m T^n w} x &= \langle x\,; D^m T^n w\rangle D^m T^n w \\ &= \begin{cases} D^m \langle D^{*m} x\,; T^n w\rangle T^n w = D^m P_{T^n w} D^{*m} x, \\ D^m T^n \langle (D^m T^n)^* x\,; w\rangle w = (D^m T^n) P_w (D^m T^n)^* x. \end{cases} \end{aligned}$$

Recall that $\{D^m T^n w\}$ is an orthonormal family (according to Assumption A_2) and that $\mathcal{W}_0(w) = \bigvee_n T^n w = \bigoplus_n T^n w$ and, for each $m \in \mathbb{Z}$, $\mathcal{W}_m(w) = D^m \mathcal{W}_0(w) = D^m \bigvee_n T^n w = \bigvee_n D^m T^n w = \bigoplus_n D^m T^n w$. Thus

$$\begin{aligned} P_{\mathcal{W}_m(w)} x &= \sum\nolimits_n P_{D^m T^n w} x \\ &= \begin{cases} \sum_n D^m P_{T^n w} D^{*m} x, \\ \sum_n (D^m T^n) P_w (D^m T^n)^* x = D^m \sum_n T^n P_w T^{*n} D^{*m} x = D^m P_{\mathcal{W}_0(w)} D^{*m} x. \end{cases} \end{aligned}$$

Similarly, recall that $\mathcal{H}_0(w) = \bigvee_m D^m w = \bigoplus_m D^m w$ and, for each $n \in \mathbb{Z}$, $\mathcal{H}_n(w) = \mathcal{H}_0(T^n w) = \bigvee_m D^m T^n w = \bigoplus_m D^m T^n w$. Thus

$$P_{\mathcal{H}_n(w)} x = \sum\nolimits_m P_{D^m T^n w} x = \begin{cases} \sum_m D^m P_{T^n w} D^{*m} x, \\ \sum_m (D^m T^n) P_w (D^m T^n)^* x. \end{cases}$$

Since $\mathcal{H} = \bigoplus_m \mathcal{W}_m(w) = \bigoplus_n \mathcal{H}_n(w)$ (cf. Theorem 1), it follows that

$$\begin{aligned} x = \sum\nolimits_m P_{\mathcal{W}_m(w)} x &= \sum\nolimits_m \sum\nolimits_n P_{D^m T^n w} x \\ &= \begin{cases} \sum_m \sum_n D^m P_{T^n w} D^{*m} x = \sum_m D^m \sum_n P_{T^n w} D^{*m} x, \\ \sum_m \sum_n (D^m T^n) P_w (D^m T^n)^* x = \sum_m D^m P_{\mathcal{W}_0(w)} D^{*m} x, \end{cases} \end{aligned}$$

which yields the results in(a), and

$$x = \sum\nolimits_n P_{\mathcal{H}_n(w)} x = \sum\nolimits_n \sum\nolimits_m P_{D^m T^n w} x = \begin{cases} \sum_n \sum_m D^m P_{T^n w} D^{*m} x, \\ \sum_n \sum_m (D^m T^n) P_w (D^m T^n)^* x, \end{cases}$$

which yields the results in(b). Finally, recall from Proposition 3(c$'$) that $D^m T^n = T^{\frac{n}{2^m}} D^m$ for every $m, n \in \mathbb{Z}$. Thus, replacing $D^m T^n$ with $T^{\frac{n}{2^m}} D^m$ in the last of the above expressions, and recalling that $D^m P_w D^{*m} = P_{D^m w}$ for each $m \in \mathbb{Z}$ (particular case of $P_{D^m T^n w} = (D^m T^n) P_w (D^m T^n)^*$ for $m, n \in \mathbb{Z}$ — set $n = 0$), we get

$$x = \sum\nolimits_n \sum\nolimits_m T^{\frac{n}{2^m}} (D^m P_w D^{*m}) T^{*\frac{n}{2^m}} x = \sum\nolimits_n \sum\nolimits_m T^{\frac{n}{2^m}} P_{D^m w} T^{*\frac{n}{2^m}} x,$$

which yields the results in (c). □

These decompositions in (a), (b) and (c) are referred to as *scales*, *time-shifts* and *time-steps* decompositions, respectively.

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Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil
E-mail address: carlos@ele.puc-rio.br

Department of Electrical Engineering, University of California in Los Angeles, Los Angeles, CA 90024-1594, USA
E-mail address: levan@ee.ucla.edu