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WEYL'S THEOREM FOR DIRECT SUMS

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Abstract. Let T and S be Hilbert space operators such that Weyl's theorem holds for both of them. In general, it does not follow that Weyl's theorem holds for the direct sum $T \oplus S$. We give asymmetric sufficient conditions on T and S to ensure that the direct sum $T \oplus S$ satisfies Weyl's theorem. It is assumed that just one of the direct summands satisfies Weyl's theorem but is not necessarily isoloid, while the other has no isolated point in its spectrum.

1. Introduction

By an operator we mean a bounded linear transformation of a Hilbert space into itself. The Weyl spectrum of an operator T is the set $\sigma_w(T)$ of all scalars λ such that $\lambda I - T$ is not a Fredholm operator of index zero, which is a subset of the whole spectrum $\sigma(T)$ of T. An operator T satisfies Weyl's theorem (or Weyl's theorem holds for T) if the complement of $\sigma_w(T)$ in $\sigma(T)$ coincides with the set of all isolated eigenvalues of T of finite multiplicity. Weyl's theorem has been much investigated over the past forty years. In this paper we focus on Weyl's theorem for direct sums. (Throughout this paper, all direct sums are orthogonal.) It is easy to give examples (Sections 4 and 5) of operators T and S for which Weyl's theorem holds for both of them, while their direct sum $T \oplus S$ does not satisfy Weyl's theorem. However, there are classical cases where Weyl's theorem is transferred from direct summands to the direct sum. A particularly important case reports to hyponormal operators. It is well known that normal and purely hyponormal operators satisfy Weyl's theorem. Moreover, every hyponormal operator is the direct sum of a normal operator and a pure hyponormal operator. In this case, Weyl's theorem for the direct summands is transferred to the direct sum: Weyl's theorem holds for large classes of operators that include the hyponormal ones.

We show in Theorem 1 that, if T has no isolated point in its spectrum and S satisfies Weyl's theorem, then $T \oplus S$ satisfies Weyl's theorem whenever $\sigma_w(T \oplus S) = \sigma(T) \cup \sigma_w(S)$. We also give sufficient conditions on T or S to ensure that such spectral identity holds. As a consequence, we show in Section 5 that if each finite-dimensional eigenspace of T reduces T, then T satisfies Weyl's theorem whenever the restriction of it to the orthogonal complement of the span of those eigenspaces has no isolated point in its spectrum and its adjoint has no eigenvalue of finite multiplicity. Direct sums involving hereditarily normaloid and compact operators are also considered in Section 5.

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2. Notation and Terminology

Throughout the paper $\mathcal{H}$ and $\mathcal{K}$ will be nonzero complex Hilbert spaces. Let $\mathcal{B}[\mathcal{H}]$ be the unital Banach algebra of all operators on $\mathcal{H}$ (bounded linear transformations of $\mathcal{H}$ into itself). Take any $T \in \mathcal{B}[\mathcal{H}]$. Put $\mathcal{N}(T) = T^{-1}\{0\}$ and $\mathcal{R}(T) = T(\mathcal{H})$; the kernel (or null space) and range of T, respectively. Recall the following elementary properties (where T^* stands for the adjoint of T and $\mathcal{R}(T)^\perp$ for the orthogonal complement of $\mathcal{R}(T)$): $\mathcal{N}(T^*) = \mathcal{H} \ominus \mathcal{R}(T)^- = \mathcal{R}(T)^\perp$ and $\mathcal{R}(T^*)$ is closed if and only if $\mathcal{R}(T)$ is closed. An operator in $T \in \mathcal{B}[\mathcal{H}]$ is Fredholm if it has a closed range and the kernels of both T and T^* are finite-dimensional. Let $\mathcal{F}$ denote the class of all Fredholm operators on $\mathcal{H}$:

$$\mathcal{F} = \big\{T \in \mathcal{B}[\mathcal{H}]\colon\; \mathcal{R}(T) \text{ is closed, } \dim\mathcal{N}(T) < \infty \text{ and } \dim\mathcal{N}(T^*) < \infty\big\}.$$

The Fredholm index of T in $\mathcal{F}$ is the integer $\operatorname{ind}(T) = \dim\mathcal{N}(T) - \dim\mathcal{N}(T^*)$. A Weyl operator is a Fredholm operator with null Fredholm index. Let $\mathcal{W}$ denote the class of all Weyl operators from $\mathcal{B}[\mathcal{H}]$:

$$\mathcal{W} = \big\{T \in \mathcal{F}\colon\; \operatorname{ind}(T) = 0\big\}.$$

Since T^* and T lie in $\mathcal{F}$ together, it follows that $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$, and so $T \in \mathcal{W}$ if and only if $T^* \in \mathcal{W}$. The essential spectrum of T is the set of all λ such that $\lambda I - T$ is not Fredholm,

$$\sigma_e(T) = \big\{\lambda \in \mathbb{C}\colon\; (\lambda I - T) \notin \mathcal{F}\big\},$$

which coincides with the spectrum of the natural image of T in the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$, the quotient algebra of $\mathcal{B}[\mathcal{H}]$ modulo the ideal $\mathcal{B}_\infty[\mathcal{H}]$ of compact operators. Note that $\sigma_e(T) = \sigma_{\ell e}(T) \cup \sigma_{re}(T)$, where $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$ are the left and right essential spectrum of T, respectively, which are given by

$$\sigma_{\ell e}(T) = \big\{\lambda \in \mathbb{C}\colon\; \mathcal{R}(\lambda I - T) \text{ is not closed or } \dim\mathcal{N}(\lambda I - T) = \infty\big\},$$
$$\sigma_{re}(T) = \big\{\lambda \in \mathbb{C}\colon\; \mathcal{R}(\lambda I - T) \text{ is not closed or } \dim\mathcal{N}(\overline{\lambda} I - T^*) = \infty\big\}.$$

The Weyl spectrum of T is the set of all λ for which $\lambda I - T$ is not Weyl,

$$\sigma_w(T) = \big\{\lambda \in \mathbb{C}\colon\; (\lambda I - T) \notin \mathcal{W}\big\}.$$

Let $\sigma(T)$ denote the spectrum of T. It is readily verified that $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma(T)$. Now let $\sigma_0(T)$ be the complement of $\sigma_w(T)$ in $\sigma(T)$,

$$\begin{aligned}\sigma_0(T) = \sigma(T)\backslash\sigma_w(T) &= \big\{\lambda \in \sigma(T)\colon\; (\lambda I - T) \in \mathcal{W}\big\}\\ &= \big\{\lambda \in \sigma(T)\colon\; (\lambda I - T) \in \mathcal{F} \text{ and } \operatorname{ind}(\lambda I - T) = 0\big\},\end{aligned}$$

which is the set of all $\lambda \in \sigma(T)$ such that $\mathcal{R}(\lambda I - T)$ is closed and $\dim\mathcal{N}(\lambda I - T) = \dim\mathcal{N}(\overline{\lambda} I - T^*) < \infty$. Thus (see e.g., [10, pp. 452, 454]),

$$\begin{aligned}\sigma_0(T) \;=\; \big\{\lambda \in \sigma_P(T)\colon\; &\mathcal{R}(\lambda I - T)^- = \mathcal{R}(\lambda I - T) \neq \mathcal{H} \text{ and}\\ &\dim\mathcal{N}(\lambda I - T) = \dim\mathcal{N}(\overline{\lambda} I - T^*) < \infty\big\},\end{aligned}$$

where $\sigma_P(T)$ denotes the point spectrum (i.e., the set of all eigenvalues) of T. Let $\sigma_{\rm iso}(T)$ denote the set of isolated points of $\sigma(T)$ and put

$$\pi_0(T) = \sigma_{\rm iso}(T) \cap \sigma_0(T).$$

This is the set of Riesz points of T (some authors refer to it as the set of isolated eigenvalues of T of finite algebraic multiplicity). Now let $\sigma_{PF}(T)$ denote the set of all eigenvalues of T of finite multiplicity,

$$\sigma_{PF}(T)=\big\{\lambda\in\sigma_P(T):\ \dim\mathcal{N}(\lambda I-T)<\infty\big\}.$$

It is clear that $\sigma_0(T)\subseteq\sigma_{PF}(T)$. Finally, let $\pi_{00}(T)$ denote the set of all isolated eigenvalues of T of finite multiplicity,

$$\pi_{00}(T)=\sigma_{\rm iso}(T)\cap\sigma_{PF}(T),$$

which is sometimes also referred to as the set of isolated eigenvalues of T of finite geometric multiplicity.

Remark 1. Since $\sigma_0(T)\subseteq\sigma_{PF}(T)$, it follows that $\pi_0(T)\subseteq\pi_{00}(T)$. Actually,

$$\begin{aligned}\pi_0(T)&=\big\{\lambda\in\pi_{00}(T):\ \mathcal{R}(\lambda I-T)\text{ is closed}\big\}\\&=\pi_{00}(T)\backslash\sigma_w(T)=\pi_{00}(T)\backslash\sigma_e(T)=\sigma_{\rm iso}(T)\backslash\sigma_w(T)=\sigma_{\rm iso}(T)\backslash\sigma_e(T).\end{aligned}$$

Indeed, if $\lambda\in\pi_0(T)\subseteq\sigma_0(T)$, then $\lambda\in\pi_{00}(T)$ and $\mathcal{R}(\lambda I-T)$ is closed. Now recall that, if $\lambda\in\sigma_{\rm iso}(T)$, then $\lambda\in\sigma_0(T)$ if and only if $\lambda\notin\sigma_{\ell e}(T)\cap\sigma_{re}(T)$ [4, p. 366]; which ensures the converse: if $\mathcal{R}(\lambda I-T)$ is closed and $\lambda\in\pi_{00}(T)$, then $\lambda\notin\sigma_{\ell e}(T)$ (for $\mathcal{R}(\lambda I-T)$ is closed and $\dim\mathcal{N}(\lambda I-T)<\infty$), and hence $\lambda\in\pi_0(T)$. Thus,

$$\pi_0(T)=\big\{\lambda\in\pi_{00}(T):\ \mathcal{R}(\lambda I-T)\text{ is closed}\big\}.$$

Moreover, it is also readily verified that

$$\pi_0(T)=\sigma_{\rm iso}(T)\backslash\sigma_w(T)=\sigma_{\rm iso}(T)\backslash\sigma_e(T).$$

In fact, since $\sigma_0(T)=\sigma(T)\backslash\sigma_w(T)$, it follows that

$$\sigma_{\rm iso}(T)\backslash\sigma_w(T)=\sigma_{\rm iso}(T)\cap\big(\sigma(T)\backslash\sigma_w(T)\big)=\sigma_{\rm iso}(T)\cap\sigma_0(T)=\pi_0(T).$$

But $\sigma_w(T)=\sigma_e(T)\cup G(T)$ where $G(T)$ is a union of open subsets of $\sigma(T)$ [4, p. 367]. (Actually, $G(T)$ is the part of the spectral picture of T [14] consisting of the union of all holes of the essential spectrum with nonzero indices.) Therefore,

$$\sigma_{\rm iso}(T)\backslash\sigma_w(T)=\sigma_{\rm iso}(T)\backslash\big(\sigma_e(T)\cup G(T)\big)=\big(\sigma_{\rm iso}(T)\backslash\sigma_e(T)\big)\cap\big(\sigma_{\rm iso}(T)\backslash G(T)\big).$$

Since $G(T)$ is a subset of $\sigma(T)$ that is open in $\mathbb{C}$, it follows that $\sigma_{\rm iso}(T)\cap G(T)=\varnothing$. Hence $\sigma_{\rm iso}(T)\backslash G(T)=\sigma_{\rm iso}(T)$, and so

$$\sigma_{\rm iso}(T)\backslash\sigma_w(T)=\sigma_{\rm iso}(T)\backslash\sigma_e(T).$$

Finally, since $\pi_0(T)\subseteq\pi_{00}(T)\subseteq\sigma_{\rm iso}(T)$, $\pi_0(T)\subseteq\sigma(T)\backslash\sigma_w(T)$ and $\sigma_e(T)\subseteq\sigma_w(T)$, it follows by the above identities that

$$\begin{aligned}\pi_{00}(T)\backslash\sigma_w(T)&\subseteq\pi_{00}(T)\backslash\sigma_e(T)\subseteq\sigma_{\rm iso}(T)\backslash\sigma_e(T)\\&=\sigma_{\rm iso}(T)\backslash\sigma_w(T)=\pi_0(T)=\pi_0(T)\backslash\sigma_w(T)\subseteq\pi_{00}(T)\backslash\sigma_w(T),\end{aligned}$$

and hence

$$\pi_0(T)=\pi_{00}(T)\backslash\sigma_w(T)=\pi_{00}(T)\backslash\sigma_e(T).$$

Remark 2. Take any $T\in\mathcal{B}[\mathcal{H}]$. The following assertions are pairwise equivalent.

(a) $\sigma(T)\backslash\sigma_w(T)=\pi_{00}(T)$.

(b) $\sigma_0(T)=\pi_{00}(T)$.

(c) $\sigma(T)\backslash\pi_{00}(T)=\sigma_w(T)$.

Indeed, since $\{\sigma_w(T), \sigma_0(T)\}$ forms a partition of the spectrum of T (i.e., $\sigma(T) = \sigma_w(T) \cup \sigma_0(T)$ and $\sigma_w(T) \cap \sigma_0(T) = \varnothing$), it follows that (a) and (b) are equivalent, (b) implies (c) and, since $\pi_{00}(T) \subseteq \sigma(T)$, (c) implies (b) as well:

$$\sigma_0(T) = \sigma(T)\backslash\sigma_w(T) = \sigma(T)\backslash\big(\sigma(T)\backslash\pi_{00}(T)\big) = \pi_{00}(T).$$

Remark 3. Moreover, if any of the above equivalent assertions holds, then

$$\pi_0(T) = \pi_{00}(T).$$

In fact, since all isolated points of $\sigma(T)$ in $\sigma_0(T)$ lie in $\pi_0(T)$, it follows that, if $\sigma_0(T) = \pi_{00}(T)$, then $\pi_{00}(T) \subseteq \pi_0(T)$; but $\pi_0(T) \subseteq \pi_{00}(T)$ always. Therefore,

$$\sigma_0(T) = \pi_{00}(T) \quad \text{if and only if} \quad \sigma_0(T) = \pi_0(T) \;\text{ and }\; \pi_0(T) = \pi_{00}(T).$$

(This equivalence is commonly rephrased by saying that *Weyl's theorem holds for T if and only if Browder's theorem holds for T and $\pi_0(T) = \pi_{00}(T)$.*) Finally, note that, according to Remark 1,

$$\pi_0(T) = \pi_{00}(T) \quad\iff\quad \pi_{00}(T) \cap \sigma_w(T) = \varnothing \quad\iff\quad \pi_{00}(T) \cap \sigma_e(T) = \varnothing.$$

3. Weyl's Theorem

It is usual to say that *Weyl's theorem holds for T* (or *T satisfies Weyl's theorem*) if any of the equivalent assertions (a), (b) or (c) of Remark 2 holds (for further equivalent conditions see [8], and recently [6]). It is easy to show that every operator T on a finite-dimensional space satisfies Weyl's theorem with $\sigma_0(T) = \pi_{00}(T) = \sigma(T)$ (this extends to finite-rank but not to compact operators) and, on the other hand, every operator T without eigenvalues ($\sigma_P(T) = \varnothing$) also satisfies Weyl's theorem with $\sigma_0(T) = \pi_{00}(T) = \varnothing$. These are the trivial cases. Weyl proved in [18] that Weyl's theorem holds for self-adjoint operators, which was extended to normal operators in [17], to hyponormal operators in [3], and to seminormal operators in [1]. Recall that $T \in \mathcal{B}[\mathcal{H}]$ is hyponormal if $T^*T - TT^* \geq O$ and cohyponormal if T^* is hyponormal, so that T is normal if it is both hyponormal and cohyponormal. If T is either hyponormal or cohyponormal, then it is called seminormal. The next result from [1] yields a generalization that extends many of the previous results (also see [2]).

Lemma 1. [1] *If finite-dimensional eigenspaces of a Hilbert space operator are reducing and every direct summand of it is isoloid, then it satisfies Weyl's theorem.*

By a subspace we mean a closed linear manifold of $\mathcal{H}$. A subspace $\mathcal{M}$ is invariant for T if $T(\mathcal{M}) \subseteq \mathcal{M}$, and reducing if it is invariant for both T and T^*. Also recall that an operator is isoloid if every isolated point of its spectrum is an eigenvalue (i.e., T is isoloid if $\sigma_{\rm iso}(T) \subseteq \sigma_P(T)$).

A Hilbert space operator T is dominant if $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda} I - T^*)$ for every $\lambda \in \mathbb{C}$. This implies that $\mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*)$ for every $\lambda \in \mathbb{C}$, which in turn implies that $\mathcal{N}(\lambda I - T)$ reduces T for each $\lambda \in \sigma_P(T)$. That is, every eigenspace of a dominant operator is reducing and so is, in particular, every finite-dimensional eigenspace of it. Therefore, a straightforward corollary of Lemma 1 says that *if $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda} I - T^*)$ and the restriction $T|_{\mathcal{M}}$ of T to each reducing subspace $\mathcal{M}$ is such that $\sigma_{\rm iso}(T|_{\mathcal{M}}) \subseteq \sigma_P(T|_{\mathcal{M}})$, then T satisfies Weyl's theorem.* We restate this consequence of Lemma 1 below.

Lemma 2. *If a Hilbert space operator is dominant and every direct summand of it is isoloid, then it satisfies Weyl's theorem.*

An operator T is normaloid if its spectral radius coincides with its norm (i.e., $r(T) = \|T\|$), and transaloid if $\lambda I - T$ is normaloid for every $\lambda \in \mathbb{C}$. Observe that a direct summand of a transaloid operator need not be transaloid (in fact, it does not need to be even normaloid). For instance, take $T = Q \oplus S$, where Q is a quasinilpotent nonzero contraction (i.e., $r(Q) = 0 \neq \|Q\| \leq 1$ — e.g., $Q = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ on $\mathbb{C}^2$) and S is a transaloid contraction whose spectrum is the closed unit disc (e.g., S is any unilateral shift, which is hyponormal, and so $\lambda I - S$ is hyponormal for every λ, thus normaloid, and hence S is transaloid). Since S is transaloid and $r(S) = 1$,

$$\|\lambda I - S\| = r(\lambda I - S) = |\lambda| + r(S) = |\lambda| + 1$$

(reason: $\sigma(\lambda I - S) = \lambda - \sigma(S)$ and $\sigma(S) = \mathbb{D}^-$, the closed unit disc), and therefore

$$\|\lambda I - T\| = \|(\lambda I - Q) \oplus (\lambda I - S)\| = \|\lambda I - S\| = r(\lambda I - S),$$

since $\|\lambda I - Q\| \leq \|\lambda I - S\|$ because $\|\lambda I - Q\| \leq |\lambda| + \|Q\| \leq |\lambda| + 1$. Moreover,

$$\sigma(\lambda I - T) = \lambda - \sigma(T) = \lambda - \sigma(Q) \cup \sigma(S) = \lambda - \sigma(S) = \sigma(\lambda I - S)$$

and so $r(\lambda I - T) = r(\lambda I - S)$, which implies $r(\lambda I - T) = \|\lambda I - T\|$. Thus the direct sum $T = Q \oplus S$ is transaloid but the direct summand Q (a nonzero quasinilpotent) is not even normaloid.

A part of an operator is a restriction of it to an invariant subspace. We say that T is *hereditarily transaloid* (abbreviated HT) if every part of it is transaloid (i.e., if the restriction $T|_{\mathcal{M}}$ of T to every invariant subspace $\mathcal{M}$ is transaloid), and *totally hereditarily transaloid* (abbreviated THT) if it is hereditarily transaloid and every invertible part of it has a transaloid inverse. Clearly, hereditarily transaloid operators are transaloid, and hence normaloid. Actually, HT is a large class of isoloid and normaloid operators that includes the hyponormal operators.

Proposition 1. Hyponormal $\subset$ THT $\subset$ HT $\subset$ Isoloid.

Proof. Recall that hyponormal operators are normaloid, $\lambda I - T$ is hyponormal whenever T is hyponormal for every scalar λ, parts of a hyponormal operator are again hyponormal, and the inverse of an invertible hyponormal operator also is hyponormal (see e.g., [11, pp. 68, 86, 67, 99]), so that hyponormal operators are totally hereditarily transaloid, thus ensuring the first inclusion. To verify the third inclusion we need the Riesz Decomposition Theorem, which says that *if* $\sigma(T) = \sigma_1 \cup \sigma_2$, *where* σ_1 *and* σ_2 *are disjoint nonempty and closed sets, then* T *has a pair of complementary nontrivial invariant subspaces* $\{\mathcal{M}_1, \mathcal{M}_2\}$ *such that* $\sigma(T|_{\mathcal{M}_1}) = \sigma_1$ *and* $\sigma(T|_{\mathcal{M}_2}) = \sigma_2$. Take an arbitrary $\lambda \in \sigma_{\text{iso}}(T)$ so that $\sigma(T) = \{\lambda\} \cup \sigma$ for some closed set σ that does not contain λ. The Riesz Decomposition Theorem ensures that T has a nonzero invariant subspace $\mathcal{M}$ such that $\sigma(T|_{\mathcal{M}}) = \{\lambda\}$. Put $H = T|_{\mathcal{M}}$ on $\mathcal{M} \neq \{0\}$ so that $\sigma(\lambda I - H) = \{0\}$ (by the Spectral Mapping Theorem). If T is hereditarily transaloid, then $\lambda I - H$ is a normaloid operator so that $\|\lambda I - H\| = r(\lambda I - H) = 0$, and hence $T|_{\mathcal{M}} = H = \lambda I$ in $\mathcal{B}[\mathcal{M}]$, which implies $\lambda \in \sigma_P(T)$. Thus $\sigma_{\text{iso}}(T) \subseteq \sigma_P(T)$. □

An operator is *hereditarily normaloid* (abbreviated HN) if every part of it is normaloid, and *totally hereditarily normaloid* (abbreviated THN) if it is hereditarily normaloid and every invertible part of it has a normaloid inverse. These classes were introduced in [5]. Clearly, totally hereditarily normaloid operators are hereditarily normaloid, which in turn are normaloid. THN is another large class of isoloid and normaloid operators that also includes the hyponormal operators. Recall that an operator T is paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every x in $\mathcal{H}$. The counterpart of Proposition 1 to hereditarily normaloid operators reads as follows (also see [7]).

Proposition 2. Hyponormal $\subset$ Paranormal $\subset$ THN $\subset$ Isoloid.

Proof. Recall that hyponormal operators are paranormal, which are normaloid. Moreover, parts of a paranormal operator are again paranormal, and so is the inverse of any invertible paranormal operator (see e.g., [11, pp. 93, 94, 98, 99]), so that paranormal operators are totally hereditarily normaloid, thus ensuring the first inclusions. To verify the last inclusion take an arbitrary $\lambda \in \sigma_{\rm iso}(T)$ so that $\sigma(T) = \{\lambda\} \cup \sigma$ for some closed set σ that does not contain λ. Again, the Riesz Decomposition Theorem ensures that T has a nonzero invariant subspace $\mathcal{M}$ such that $\sigma(T|_{\mathcal{M}}) = \{\lambda\}$. If T is THN, then $T|_{\mathcal{M}}$ also is THN (in particular, $T|_{\mathcal{M}}$ is normaloid). If $\lambda = 0$, then $T|_{\mathcal{M}} = O$ and $\lambda \in \sigma_P(T)$ trivially. If $\lambda \neq 0$, then put $U = \lambda^{-1}T|_{\mathcal{M}}$ on $\mathcal{M} \neq \{0\}$, which is again THN. Since $\sigma(U) = \{1\}$, it follows that U is unitary (reason: THN *operators with spectrum in the unit circle are unitary* [7, Proposition 2]). Hence $U - I$ is quasinilpotent (i.e., $\sigma(U - I) = \{0\}$ by the Spectral Mapping Theorem) and normaloid (in fact, normal) so that $\|U - I\| = r(U - I) = 0$. Thus $T|_{\mathcal{M}} = \lambda U = \lambda I$ in $\mathcal{B}[\mathcal{M}]$, and so $\lambda \in \sigma_P(T)$. Therefore, $\sigma_{\rm iso}(T) \subseteq \sigma_P(T)$. □

Since every hyponormal operator is dominant, every part (and, in particular, every direct summand) of a hyponormal operator is again hyponormal, and every hyponormal operator is isoloid (according to Proposition 1 or Proposition 2), it follows by Lemma 2 that *every hyponormal operator satisfies Weyl's theorem.*

Remark 4. The above argument does not survive if hyponormal is replaced with paranormal because paranormal operators (although still isoloid according to Proposition 2) are not necessarily dominant. However, *if T is totally hereditarily normaloid, then both T and T^* satisfy Weyl's theorem* [5, Lemma 2.5]. Thus, Weyl's theorem holds for paranormal operators and their adjoints and, in particular, Weyl's theorem holds for hyponormal operators and their adjoints, so that this also gives another proof that *every seminormal operator satisfies Weyl's theorem.* In fact, since hereditarily transaloid operators are hereditarily normaloid, and totally hereditarily transaloid operators are totally hereditarily normaloid, it follows that *if T is totally hereditarily transaloid, then both T and T^* satisfy Weyl's theorem.*

4. Direct Sum

If $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$ satisfy Weyl's theorem, it does not necessarily follows that the (orthogonal) direct sum $T \oplus S \in \mathcal{B}[\mathcal{H} \oplus \mathcal{K}]$ satisfies Weyl's theorem. For instance, if S is a unilateral weighted shift on ℓ^2_+ with a positive weighting sequence that converges to zero, then $\sigma(S) = \sigma_R(S) = \{0\}$ and $\mathcal{R}(S)$ is not closed, where $\sigma_R(S) = \sigma_P(S^*)^* \backslash \sigma_P(S)$ is the residual spectrum of S (e.g., see [10, p. 471]). Therefore, since $\sigma_P(S) = \varnothing$, it follows that S satisfies Weyl's theorem (with $\sigma_0(S) = \pi_{00}(S) = \varnothing$ — note that S is not isoloid). Now consider the direct sum $0 \oplus S$ on $\mathbb{C} \oplus \ell^2_+$ so that $\sigma(0 \oplus S) = \sigma_P(0 \oplus S) = \pi_{00}(0 \oplus S) = \{0\}$. Since $\mathcal{R}(S)$ is not closed in ℓ^2_+, it follows that $\mathcal{R}(0 \oplus S) = \{0\} \oplus \mathcal{R}(S)$ is not closed in

$\mathbb{C} \oplus \ell^2_+$, and hence $0 \notin \sigma_0(0 \oplus S)$. Thus, $\varnothing = \sigma_0(0 \oplus S) \neq \pi_{00}(0 \oplus S) = \{0\}$, and so $0 \oplus S$ does not satisfy Weyl's theorem. Summing up: $0 \oplus S$ is a compact operator for which Weyl's theorem does not hold, although Weyl's theorem holds for both of its (compact) direct summands.

We shall be concerned with the problem of giving conditions on the direct summands to ensure that Weyl's theorem holds for the direct sum (e.g., see [12], [13] and the references therein). It is readily verified that the Weyl spectrum of a direct sum is included in the union of the Weyl spectra of the summands, that is,

$$\sigma_w(T \oplus S) \subseteq \sigma_w(T) \cup \sigma_w(S)$$

for every $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$, but equality does not hold in general [9]. A useful sufficient condition, namely $\sigma_w(T) \cap \sigma_w(S)$ has empty interior, was given in [13]:

$$\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S) \quad \text{whenever} \quad \big(\sigma_w(T) \cap \sigma_w(S)\big)^\circ = \varnothing.$$

This identity involving Weyl spectra plays a central role in establishing conditions for the direct sum to satisfy Weyl's theorem. The next result from [13] (also see [12]) assumes, in addition, that Weyl's theorem holds for both direct summands, which in turn are supposed to be isoloid.

Lemma 3. [13] *If both $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$ are isoloid, satisfy Weyl's theorem, and $\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S)$, then Weyl's theorem holds for $T \oplus S$.*

The above result considers symmetric assumptions on the direct summands T and S: both T and S are isoloid and both satisfy Weyl's theorem. In Theorem 1 (below) we consider asymmetric assumptions. In particular, we do not assume that S is isoloid, as we do not assume that T satisfies Weyl's theorem.

Theorem 1. *Suppose T in $\mathcal{B}[\mathcal{H}]$ has no isolated point in its spectrum and S in $\mathcal{B}[\mathcal{K}]$ satisfies Weyl's theorem. If $\sigma_w(T \oplus S) = \sigma(T) \cup \sigma_w(S)$, then Weyl's theorem holds for $T \oplus S$.*

Proof. Recall: $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$ for any pair of operators. If Weyl's theorem holds for S, then $\sigma(S)\backslash\sigma_w(S) = \pi_{00}(S)$ and, if $\sigma_w(T \oplus S) = \sigma(T) \cup \sigma_w(S)$, then

$$\begin{aligned}
\sigma(T \oplus S)\backslash\sigma_w(T \oplus S) &= \big(\sigma(T) \cup \sigma(S)\big)\big\backslash\big(\sigma(T) \cup \sigma_w(S)\big) \\
&= \sigma(S)\big\backslash\big(\sigma(T) \cup \sigma_w(S)\big) \\
&= \big(\sigma(S)\backslash\sigma_w(S)\big)\big\backslash\sigma(T) = \pi_{00}(S) \cap \rho(T),
\end{aligned}$$

where $\rho(T) = \mathbb{C}\backslash\sigma(T)$ is the resolvent set of T. On the other hand, observe that $\sigma_{\rm iso}(T \oplus S)$ is the set of isolated points of $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$. If $\sigma_{\rm iso}(T) = \varnothing$ (so that $\sigma(T) = \sigma_{\rm acc}(T)$, where $\sigma_{\rm acc}(T) = \sigma(T)\backslash\sigma_{\rm iso}(T)$ is the set of all accumulation points of $\sigma(T)$), then

$$\begin{aligned}
\sigma_{\rm iso}(T \oplus S) &= \big(\sigma_{\rm iso}(T) \cup \sigma_{\rm iso}(S)\big)\big\backslash\big[\big(\sigma_{\rm iso}(T) \cap \sigma_{\rm acc}(S)\big) \cup \big(\sigma_{\rm acc}(T) \cap \sigma_{\rm iso}(S)\big)\big] \\
&= \big(\sigma_{\rm iso}(T)\backslash\sigma_{\rm acc}(S)\big) \cup \big(\sigma_{\rm iso}(S)\backslash\sigma_{\rm acc}(T)\big) \\
&= \sigma_{\rm iso}(S)\backslash\sigma(T) = \sigma_{\rm iso}(S) \cap \rho(T).
\end{aligned}$$

Recall that $\sigma_P(T \oplus S) = \sigma_P(T) \cup \sigma_P(S)$ and $\dim \mathcal{N}(T \oplus S) = \dim \mathcal{N}(T) + \dim \mathcal{N}(S)$ for every pair of operators so that

$$\sigma_{PF}(T \oplus S) = \big\{\lambda \in \sigma_{PF}(T) \cup \sigma_{PF}(S) \colon \ \dim \mathcal{N}(\lambda I - T) + \dim \mathcal{N}(\lambda I - S) < \infty\big\}.$$

Therefore,

$$\pi_{00}(T\oplus S)=\sigma_{\rm iso}(T\oplus S)\cap\sigma_{PF}(T\oplus S)=\sigma_{\rm iso}(S)\cap\rho(T)\cap\sigma_{PF}(S)=\pi_{00}(S)\cap\rho(T).$$

Thus

$$\sigma(T\oplus S)\backslash\sigma_w(T\oplus S)=\pi_{00}(T\oplus S),$$

and hence $T\oplus S$ satisfies Weyl's theorem. □

Here are two different instances where Theorem 1 can be applied to verify that a direct sum satisfies Weyl's theorem.

Example 1. Let S be a unilateral weighted shift on ℓ^2_+ with a positive weighting sequence that converges to zero as in the first paragraph of this section. Let $T=S_+$ be the canonical unilateral shift on ℓ^2_+ (which has no isolated point in its spectrum, thus being trivially isoloid). Since S is not isoloid, Lemma 3 does not apply in this case, even though S satisfies Weyl's theorem. However, $\sigma_P(T\oplus S)$ is empty (the point spectra of both T and S are empty) and so $T\oplus S$ satisfies Weyl's theorem. This is confirmed by Theorem 1 since $\sigma(T\oplus S)=\sigma_w(T\oplus S)=\sigma(T)\cup\sigma_w(S)$.

Example 2. The above example exhibited a direct sum where both direct summands satisfy Weyl's theorem but one of them is not isoloid. Now we consider a case where both direct summands are isoloid but one of them does not satisfy Weyl's theorem. First note that if S_+ is the canonical unilateral shift on ℓ^2_+ of multiplicity one, then $\sigma_{\rm iso}(S_+^*\oplus S_+)=\varnothing$ but Weyl's theorem does not hold for $T=S_+^*\oplus S_+$ (although it does hold for both S_+^* and S_+, once S_+ is hyponormal). In fact, since $\mathcal{R}(S_+^*\oplus S_+)=\mathcal{R}(S_+^*)\oplus\mathcal{R}(S_+)$ is closed but not equal to $\ell^2_+\oplus\ell^2_+$ (as each $\mathcal{R}(S_+^*)$ and $\mathcal{R}(S_+)$ is closed in ℓ^2_+ and $\mathcal{R}(S_+)\neq\ell^2_+$), since $\mathcal{N}(S_+^*\oplus S_+)$ and $\mathcal{N}(S_+\oplus S_+^*)$ are both one-dimensional (for $\mathcal{N}(S_+)=\{0\}$ and $\dim\mathcal{N}(S_+^*)=1$) and, finally, since $\sigma_P(S_+^*\oplus S_+)=\sigma_P(S_+^*)=\mathbb{D}$, it follows that 0 lies in $\sigma_0(S_+^*\oplus S_+)$. This implies that $\sigma_0(T)\neq\pi_{00}(T)=\varnothing$, and hence T does not satisfy Weyl's theorem. Therefore, Lemma 3 does not apply in this case as well, regardless which operator S is. If $S=S_+$, which satisfies Weyl's theorem, then $\sigma_w(T\oplus S)=\sigma(T)\cup\sigma_w(S)$. Indeed, take any λ in $\sigma_P(T\oplus S)=\sigma_P(S_+^*)=\sigma_R(S_+)=\mathbb{D}$ so that $\mathcal{N}(\lambda I-S_+)=\{0\}$ and $\dim\big(\mathcal{N}(\lambda I-S_+^*)\big)=1$, and hence $\dim\mathcal{N}\big(\lambda I-(T\oplus S)\big)\neq\dim\mathcal{N}\big(\overline{\lambda}I-(T\oplus S)^*\big)$ — reason: $\mathcal{N}\big(\lambda I-(T\oplus S)\big)=\mathcal{N}(\lambda I-S_+^*)\oplus\{0\}\oplus\{0\}$ is one-dimensional and, on the other hand, $\mathcal{N}\big(\overline{\lambda}I-(T\oplus S)^*\big)=\{0\}\oplus\mathcal{N}(\overline{\lambda}I-S_+^*)\oplus\mathcal{N}(\overline{\lambda}I-S_+^*)$ is two-dimensional by the symmetry of the open unit disc $\mathbb{D}$. Then $\sigma_0(T\oplus S)=\varnothing$ so that $\sigma_w(T\oplus S)=\sigma(T\oplus S)=\sigma(T)\cup\sigma(S)=\sigma(T)\cup\sigma_w(S)$ (since $\sigma_0(S)=\sigma_P(S_+)$ is empty). Moreover, recall that $\sigma_{\rm iso}(T)=\varnothing$. Thus $T\oplus S$ satisfies Weyl's theorem by Theorem 1.

Examples 1 and 2 motivate an immediate consequence of Theorem 1. Suppose S satisfies Weyl's theorem. If $\sigma_{\rm iso}(S)\cap\sigma_{PF}(S)=\varnothing$, then $\sigma(S)=\sigma_w(S)$. If, in addition, $\sigma_w(T\oplus S)=\sigma(T\oplus S)$, then $\sigma_w(T\oplus S)=\sigma(T)\cup\sigma_w(S)$. Thus a straightforward application of Theorem 1 leads to the following corollary.

Corollary 1. *Take T in $\mathcal{B}[\mathcal{H}]$ with $\sigma_{\rm iso}(T)=\varnothing$. If S in $\mathcal{B}[\mathcal{K}]$ satisfies Weyl's theorem, $\sigma_{\rm iso}(S)\cap\sigma_{PF}(S)=\varnothing$ and $\sigma_0(T\oplus S)=\varnothing$, then $T\oplus S$ satisfies Weyl's theorem.*

5. Applications

We begin with an auxiliary result on normal and compact direct summands that will be applied often in the sequel.

Corollary 2. *If T in $\mathcal{B}[\mathcal{H}]$ is an isoloid operator that satisfies Weyl's theorem, then $T \oplus S$ satisfies Weyl's theorem whenever S in $\mathcal{B}[\mathcal{K}]$ is either* (a) *a normal operator or* (b) *an isoloid compact operator that satisfies Weyl's theorem.*

Proof. Schechter Theorem ([15], [16] — also see [4, p. 367]) says that

$$\sigma_w(S) = \sigma_e(S) \cup \bigcup_{k \in \mathbb{Z}\setminus\{0\}} \sigma_k(S)$$

for every operator $S \in \mathcal{B}[\mathcal{K}]$, where $\mathbb{Z}$ denotes the set of all integers and

$$\sigma_k(S) = \big\{\lambda \in \sigma(S)\colon\; (\lambda I - S) \in \mathcal{F} \text{ and } \operatorname{ind}(\lambda I - S) = k\big\}$$

for each $k \in \mathbb{Z}$. Since $\sigma_k(S) \cap \sigma_e(S) = \varnothing$ for every $k \in \mathbb{Z}$, it follows that

$$\bigcup_{k \in \mathbb{Z}\setminus\{0\}} \sigma_k(S) = \varnothing \quad \text{if and only if} \quad \sigma_e(S) = \sigma_w(S).$$

Moreover, it was shown in [9] that, for every $T \in \mathcal{B}[\mathcal{H}]$,

$$\sigma_e(S) = \sigma_w(S) \quad \text{implies} \quad \sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S).$$

If S is a normal, then $\mathcal{N}(\lambda I - S) = \mathcal{N}(\overline{\lambda} I - S^*)$ for every $\lambda \in \mathbb{C}$ and so $\sigma_k(S) = \varnothing$ for all $k \neq 0$. If S is compact, then again $\sigma_k(S) = \varnothing$ for all $k \neq 0$ (by the Fredholm alternative). Therefore, in both cases, $\sigma_e(S) = \sigma_w(S)$ so that

$$\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S)$$

whenever S is either normal or compact. Thus the stated result follows by Lemma 3 since every normal operator is isoloid and satisfies Weyl's theorem. □

Recall that T *satisfies Weyl's theorem if and only if* $\sigma_0(T) = \sigma_{\text{iso}}(T) \cap \sigma_{PF}(T)$. If $\sigma_{\text{iso}}(T) = \varnothing$, then T satisfies Weyl's theorem if and only if $\sigma_0(T) = \varnothing$. Thus the assumption $\sigma_{\text{iso}}(T) = \varnothing$ of Theorem 1 does not imply that T satisfies Weyl's theorem (for instance, $\sigma_{\text{iso}}(S_+^* \oplus S_+) = \varnothing$ but Weyl's theorem does not hold for $S_+^* \oplus S_+$ as we saw in Example 2). However, if $\sigma_0(T) \cup \sigma_{\text{iso}}(T) = \varnothing$, then T satisfies Weyl's theorem. Indeed,

$$\sigma_0(T) \cup \sigma_{\text{iso}}(T) = \varnothing \quad \textit{if and only if}$$
$$T \in \mathcal{B}[\mathcal{H}] \textit{ satisfies Weyl's theorem and has no isolated point in its spectrum.} \quad (*)$$

Since $\sigma_0(T) = \varnothing$ if and only if $\sigma_w(T) = \sigma(T)$ we get the following corollary of Theorem 1 (where S is not assumed to be isoloid).

Corollary 3. *Suppose T in $\mathcal{B}[\mathcal{H}]$ is such that $\sigma_0(T) \cup \sigma_{\text{iso}}(T) = \varnothing$ and S in $\mathcal{B}[\mathcal{K}]$ satisfies Weyl's theorem. If $\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S)$, then Weyl's theorem holds for $T \oplus S$.*

Here is a first consequence of the above corollaries. It is well known that a pure hyponormal operator H_p (i.e., a hyponormal operator that has no normal direct summand) is such that $\sigma_{\text{iso}}(H_p) = \sigma_P(H_p) = \varnothing$ (see e.g., [10, pp. 503, 508]). Since $\sigma_P(H_p) = \varnothing$, it follows that $\sigma_0(H_p) = \varnothing$, and hence

$$\sigma_0(H_p) \cup \sigma_{\text{iso}}(H_p) = \varnothing.$$

Thus a pure hyponormal operator H_p (which is trivially isoloid) satisfies Weyl's theorem with $\sigma_0(H_p) = \pi_{00}(H_p) = \varnothing$ and $\sigma_w(H_p) = \sigma(H_p)$. Moreover,

$$\sigma_w(H_p \oplus N) = \sigma_w(H_p) \cup \sigma_w(N)$$

for every normal operator N (cf. proof of Corollary 2) so that either Corollary 2(a) or Corollary 3 give still another proof that *every hyponormal operator satisfies Weyl's theorem* (as originally shown in [3]). Indeed, every normal operator N satisfies Weyl's theorem and every hyponormal operator H is of the form $H = H_p \oplus N$, where H_p is a pure hyponormal and N is normal.

Remark 5. If $T \in \mathcal{B}[\mathcal{H}]$ is such that $\sigma_{PF}(T) = \sigma_{PF}(T^*) = \varnothing$, then

$$\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S) = \sigma(T) \cup \sigma_w(S)$$

for every $S \in \mathcal{B}[\mathcal{K}]$. Indeed, take any $T \in \mathcal{B}[\mathcal{H}]$ and consider the definition of $\sigma_k(T)$ as in the proof of Corollary 2. It is easy to show that $\sigma_k(T) \subseteq \sigma_{PF}(T) \cup \sigma_{PF}(T^*)^*$ for all $k \in \mathbb{Z}\backslash\{0\}$ and $\sigma_0(T) \subseteq \sigma_{PF}(T) \cap \sigma_{PF}(T^*)^*$. Since $\sigma(T) = \sigma_w(T) \cup \sigma_0(T)$,

$$\sigma_{PF}(T) = \sigma_{PF}(T^*) = \varnothing \quad \text{implies} \quad \sigma_e(T) = \sigma_w(T) = \sigma(T).$$

Therefore (cf. proof of Corollary 2), $\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S) = \sigma(T) \cup \sigma_w(S)$ for every $S \in \mathcal{B}[\mathcal{K}]$ whenever $\sigma_{PF}(T) = \sigma_{PF}(T^*) = \varnothing$.

Remark 5 leads to the following further corollary of Theorem 1.

Corollary 4. *Take any $T \in \mathcal{B}[\mathcal{H}]$ and put*

$$\mathcal{M} \;=\; \bigvee_{\lambda \in \sigma_{PF}(T)} \mathcal{N}(\lambda I - T).$$

Suppose each finite-dimensional eigenspace of T reduces T. If $\mathcal{M}$ is trivial, then T satisfies Weyl's theorem. If $\mathcal{M}$ is nontrivial and $\sigma_{\mathrm{iso}}(T|_{\mathcal{M}^\perp}) = \varnothing$, then T satisfies Weyl's theorem whenever either $\sigma_{PF}(T^*|_{\mathcal{M}^\perp}) = \varnothing$ or $\sigma_0(T|_{\mathcal{M}^\perp}) = \varnothing$.

Proof. We shall split the proof into four parts.

(i) If $\mathcal{M} = \{0\}$, then $\sigma_{PF}(T) = \varnothing$ (the empty span is null). Therefore, if $\mathcal{M} = \{0\}$, then $\sigma_0(T) = \pi_{00}(T) = \varnothing$ and so T satisfies Weyl's theorem. Thus suppose $\mathcal{M} \neq \{0\}$.

(ii) If $\mathcal{N}(\lambda I - T)$ reduces T for each $\lambda \in \sigma_{PF}(T)$, then $\mathrm{span}_{\lambda \in \sigma_{PF}(T)} \mathcal{N}(\lambda I - T)$ reduces T, and so does its closure $\mathcal{M} = \bigvee_{\lambda \in \sigma_{PF}(T)} \mathcal{N}(\lambda I - T)$. Then consider the decomposition

$$T = T|_{\mathcal{M}^\perp} \oplus T|_{\mathcal{M}}.$$

Observe that $T|_{\mathcal{M}}$ satisfies Weyl's theorem. Indeed, if $\mathcal{N}(\lambda I - T)$ reduces T and u lies in $\mathcal{N}(\lambda I - T)$ for some $\lambda \in \sigma_{PF}(T)$, then T^*u lies in $\mathcal{N}(\lambda I - T)$ and so $TT^*u = \lambda T^*u = T^*Tu$. This implies that $TT^*v = T^*Tv$ for every v in $\mathrm{span}_{\lambda \in \sigma_{PF}(T)} \mathcal{N}(\lambda I - T)$, which in turn extends by continuity for every v in $\mathcal{M}$ so that $T|_{\mathcal{M}}(T|_{\mathcal{M}})^* = (T|_{\mathcal{M}})^* T|_{\mathcal{M}}$ because $\mathcal{M}$ reduces T. Hence $T|_{\mathcal{M}}$ is normal, and therefore satisfies Weyl's theorem.

(iii) If $\mathcal{M} = \mathcal{H}$, then $T = T|_{\mathcal{H}}$ satisfies Weyl's theorem trivially. Thus suppose $\mathcal{M}$ is nontrivial (i.e., $\{0\} \neq \mathcal{M} \neq \mathcal{H}$). Clearly, $\sigma_{PF}(T|_{\mathcal{M}^\perp}) = \varnothing$. Since $\mathcal{M}$ reduces T we get $T^*|_{\mathcal{M}^\perp} = (T|_{\mathcal{M}^\perp})^*$. Thus, if $\sigma_{PF}(T^*|_{\mathcal{M}^\perp}) = \varnothing$, then $\sigma_w(T) = \sigma(T|_{\mathcal{M}^\perp}) \cup \sigma_w(T|_{\mathcal{M}})$ according to Remark 5. Therefore, if $\sigma_{\rm iso}(T|_{\mathcal{M}^\perp}) = \varnothing$, then Theorem 1 ensures that T satisfies Weyl's theorem.

(iv) If $\sigma_0(T|_{\mathcal{M}^\perp}) = \sigma_{\rm iso}(T|_{\mathcal{M}^\perp}) = \varnothing$, then $T|_{\mathcal{M}^\perp}$ is isoloid and satisfies Weyl's theorem according $(*)$, and so T satisfies Weyl's theorem by Corollary 2(a). □

Corollary 3 leads to a version of Corollary 2(b), where the compact direct summand S is not necessarily isoloid.

Corollary 5. *If T in $\mathcal{B}[\mathcal{H}]$ satisfies Weyl's theorem and has no isolated point in its spectrum, then $T \oplus S$ satisfies Weyl's theorem for every compact S in $\mathcal{B}[\mathcal{K}]$ that satisfies Weyl's theorem.*

Proof. If S is compact, then $\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S)$ — cf. proof of Corollary 2. Thus, by Corollary 3 and the equivalence in $(*)$, $T \oplus S$ satisfies Weyl's theorem if both T and S satisfy Weyl's theorem and $\sigma_{\rm iso}(T) = \varnothing$. □

We close the paper with a version of Corollaries 2(b) and 5 for totally hereditarily normaloid operators. Recall that this is a large class that includes the paranormal (and so the hyponormal) operators (cf. Proposition 2),

Corollary 6. *If $T \in \mathcal{B}[\mathcal{H}]$ is a totally hereditarily normaloid and $S \in \mathcal{B}[\mathcal{K}]$ is a compact that satisfies Weyl's theorem, then $T \oplus S$ satisfies Weyl's theorem whenever either* (a) *S is isoloid or* (b) *T is has no isolated point in its spectrum.*

Proof. Every totally hereditarily normaloid operator is isoloid (Proposition 2) and satisfies Weyl's theorem [5, Lemma 2.5]. Thus Corollary 2(b) ensures part (a) and Corollary 5 ensures part (b). □

Since T is isoloid, $\sigma_P(T) = \varnothing$ implies the hypothesis in (b); that is, $\sigma_{\rm iso}(T) = \varnothing$.

Example 3. Let T and S be unilateral weighted shifts on ℓ^2_+. Suppose T has a weighting sequence with all entries equal to 1, except the second entry which lies in $(0, 1)$. This is a totally hereditarily normaloid operator [7] (thus an isoloid that satisfies Weyl's theorem) which is not hyponormal, not even paranormal (see e.g., [11, p. 95]). Observe that $\sigma(T) = \mathbb{D}^-$ and $\sigma_P(T) = \varnothing$. Both identities ensure that $\sigma_{\rm iso}(T) = \varnothing$. Suppose S has a positive weighting sequence that converges to zero. This is a compact operator that satisfies Weyl's theorem but is not isoloid (Example 1). However, according to Corollary 6(b), $T \oplus S$ satisfies Weyl's theorem.

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