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TOTALLY HEREDITARILY NORMALOID OPERATORS AND WEYL'S THEOREM FOR AN ELEMENTARY OPERATOR

B.P. DUGGAL AND C.S. KUBRUSLY

ABSTRACT. A Hilbert space operator $T \in B(H)$ is *hereditarily normaloid* (notation: $T \in HN$) if every part of T is normaloid. An operator $T \in HN$ is *totally hereditarily normaloid* (notation: $T \in THN$) if every invertible part of T is normaloid. We prove that THN-operators with Bishop's property (β), also THN-contractions with a compact defect operator such that $T^{-1}(0) \subseteq T^{*-1}(0)$ and non-zero isolated eigenvalues of T are normal, are not supercyclic. Take A and B in THN and let d_{AB} denote either of the elementary operators in $B(B(H))$: $\triangle_{AB}$ and δ_{AB}, where $\triangle_{AB}(X) = AXB - X$ and $\delta_{AB}(X) = AX - XB$. We prove that if non-zero isolated eigenvalues of A and B are normal and $B^{-1}(0) \subseteq B^{*-1}(0)$ then d_{AB} is an isoloid operator such that the quasinilpotent part $H_0(d_{AB} - \lambda)$ of $d_{AB} - \lambda$ equals $(d_{AB} - \lambda)^{-1}(0)$ for every complex number λ which is isolated in $\sigma(d_{AB})$. If, additionally, d_{AB} has the *single-valued extension property* at all points not in the Weyl spectrum of d_{AB}, then d_{AB}, and the conjugate operator d^*_{AB}, satisfy Weyl's theorem.

1. Introduction

A Banach space operator $T \in B(\mathcal{X})$ is *hereditarily normaloid*, denoted $T \in HN$, if every part of T (i.e., the restriction of T to an invariant subspace) is normaloid; $T \in HN$ is said to be *totally hereditarily normaloid*, denoted $T \in THN$, if every invertible part of T is normaloid. (Recall that T is normaloid if $\|T\|$ equals the spectral radius $r(T)$ of T.) The class of THN operators is large. For example, Hilbert space operators T which are either hyponormal or p-hyponormal ($0 < p < 1$) or w-hyponormal or such that $|T|^2 \leq |T^2|$ are THN-operators. (See [13], [18] for definitions and properties of these classes of operators.) Again, paranormal (Banach space) operators are THN-operators [16, page 229]. THN-operators share many, but by no means all, of the properties of hyponormal operators. Thus the isolated points of the spectrum of a THN operator are simple poles of the resolvent of the operator, eigenspaces corresponding to distinct non-zero eigenvalues of the operator are mutually orthogonal, and the operator satisfies Weyl's theorem (see[8]). THN operators are closed under multiplication by a non-zero scalar. Structure of THN-contractions, in particular those with a compact or Hilbert-Schmidt defect operator, has been studied in [9]. This paper continues the study of THN-operators. It is proved that THN-operators with Bishop's property (β), also THN-contractions with a compact defect operator such that $T^{-1}(0) \subseteq T^{*-1}(0)$ and non-zero isolated eigenvalues of T are normal, can not be supercyclic. Of interest to us here are the elementary operators $\triangle_{AB} \in B(B(H))$, $\triangle_{AB}(X) = AXB - X$, and $\delta_{AB} \in$

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$B(B(H))$, $\delta_{AB}(X) = AX - XB$, where H is a Hilbert space and A, B are THN-operators in $B(H)$ such that their non-zero eigenvalues are normal. Letting d_{AB} denote either of these elementary operators, it is proved that if $B^{-1}(0) \subseteq B^{*-1}(0)$ then d_{AB} is an isoloid operator such that $H_0(d_{AB} - \lambda) = (d_{AB} - \lambda)^{-1}(0)$ for every complex number λ which is isolated in $\sigma(d_{AB})$, where $H_0(d_{AB} - \lambda)$ denotes the quasi-nilpotent part of $d_{AB} - \lambda$. If, additionally, d_{AB} has the *single-valued extension property* at all points not in the Weyl spectrum of d_{AB}, then it is proved that d_{AB}, and the conjugate operator d^*_{AB}, satisfy Weyl's theorem.

In the following, $\mathcal{X}$ will denote a Banach space and H will denote an infinite dimensional complex Hilbert space. $B(\mathcal{X})$ will denote the algebra of operators on $\mathcal{X}$, $\mathbf{C}$ the set of complex numbers, $\mathbf{D}$ the open unit disc in $\mathbf{C}$, $\partial\mathbf{D}$ the boundary of $\mathbf{D}$ and $\overline{\mathbf{D}}$ the closure of $\mathbf{D}$. For an operator $T \in B(\mathcal{X})$, we shall denote the spectrum, the point spectrum, the approximate point spectrum and the isolated points of the spectrum by $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$ and $\sigma_{\rm iso}(T)$, respectively. The range of T will be denoted by $T(\mathcal{X})$. Recall that T is Fredholm if $T(\mathcal{X})$ is closed and both the deficiency indices $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(\mathcal{X}/T(\mathcal{X}))$ are finite, and then the (Fredholm) index of T, ind (T), is defined by ind $(T) = \alpha(T) - \beta(T)$. T is semi-Fredholm if either $T(\mathcal{X})$ is closed and $\alpha(T) < \infty$ or $\beta(T) < \infty$. A contraction $T \in B(H)$ is of class $C_{0.}$ if the sequence $\{\|T^n x\|\}$ converges to zero for every $x \in H$, and of class $C_{1.}$ if the sequence $\{\|T^n x\|\}$ does not converge to zero for every nonzero $x \in H$. It is of class $C_{.0}$ or of class $C_{.1}$ if its adjoint T^* is of class $C_{0.}$ or $C_{1.}$, respectively. All combinations are possible, leading to classes C_{00}, C_{01}, C_{10} and C_{11}. Recall that a contraction $T \in B(H)$ is said to be completely non-unitary, shortened to cnu, if there exists no non-trivial reducing subspace M of T such that the restriction $T|_M$ of T to M is unitary. (See [21] for further information on these classes of contractions.) The rest of our notation (and terminology) will be defined progressively, on an if and when required basis.

2. Supercyclic Operators

A Banach space operator $T \in B(\mathcal{X})$ has (Bishop's) property (β) if, for every open subset $\mathcal{U}$ of $\mathbf{C}$ and every sequence of analytic functions $f_n : \mathcal{U} \to \mathcal{X}$ with the property that

$$(T - \lambda) f_n(\lambda) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly on all compact subsets of $\mathcal{U}$, it follows that $f_n(\lambda) \to 0$ as $n \to \infty$ locally uniformly on $\mathcal{U}$ [19, Definition 1.2.5]. M-hyponormal operators, p-hyponormal operators and k-quasihyponormal operators satisfy property (β) (see [6],[17] and [19]).

An operator $T \in B(H)$ is said to be *supercyclic* if, for some $x \in H$, the homogeneous orbit

$$\{\lambda T^n x : \ \lambda \in \mathbf{C} \ \text{and} \ n = 0, 1, 2, ...\}$$

is dense in H. Hyponormal operators on a Hilbert space of dimension greater than one are not supercyclic [19].

Recall that a Hilbert space operator $T \in B(H)$ is a contraction if and only if $I - T^*T$ is a nonnegative contraction. In this case, the nonnegative contraction $D_T = (I - T^*T)^{\frac{1}{2}}$ is called the defect operator of T.

Theorem 2.1. If T is a contraction in THN such that either (i) T satisfies property (β) or (ii) D_T is compact, $T^{-1}(0) \subseteq T^{*-1}(0)$ and non-zero isolated eigenvalues of T are normal, then T is not supercyclic.

Proof. Suppose, to the contrary, that T is supercyclic. Then $T \in C_{0.}$ is a cnu contraction [4, Theorem 2.2].

(i) If T satisfies property (β), then the normaloid property of $T \in THN$ implies that $\sigma(T) \subseteq \partial\mathbf{D}$ [19, Proposition 3.3.18]. Since THN contractions with spectrum in the unit circle are unitary [9, Proposition 2(a)], T is unitary. This is a contradiction.

(ii) The hypothesis D_T is compact implies that $D_T^2 = I - T^*T$ is compact, and hence that $T^*T = I - D_T^2$ is Fredholm. In particular, the range $T^*(H)$ of T^* is closed and $\alpha(T) = \alpha(T^*T) < \infty$. Hence $T(H)$ is closed and $\alpha(T) < \infty$, i.e., T is (upper) semi-Fredholm. As a $C_{0.}$-contraction, T has a triangulation

$$T = \begin{bmatrix} T_{01} & * \\ 0 & T_{00} \end{bmatrix},$$

where $T_{01} \in C_{01}$ and $T_{00} \in C_{00}$ [21, p.75]. The hypothesis D_T is compact implies T_{01} is a C_{01}-contraction with a compact defect operator, which (since $T^{-1}(0) \subseteq T^{*-1}(0)$) implies by [9, Proposition 9] that T_{01} acts on the trivial space $\{0\}$. Hence T is a C_{00}-contraction with a compact defect operator. We prove that $\operatorname{ind}(T) = 0$. Recall from [15, Proposition 3.1] (see also [4, Theorem 3.2]) that if $T \in B(H)$ is supercyclic, then $\sigma_p(T^*)$ consists at most of the singleton set $\{\lambda\}$ for some $\lambda \neq 0$. Thus 0 is not in the point spectrum of both T and T^*, which implies that $\operatorname{ind}(T) = 0$. Combining this with the fact that T is semi-Fredholm, it follows that $\sigma(T) \cap \mathbf{D} = \sigma_p(T) \cap \mathbf{D}$ consists of just one point λ for some $\lambda \neq 0$. By hypothesis, λ is a normal eigenvalue of T; hence T is the direct sum of $\lambda I|_{(T-\lambda)^{-1}(0)}$ and a THN operator T_1 such that $\sigma(T_1) \subseteq \partial\mathbf{D}$. Since THN operators with spectrum in the unit circle are unitary [9, Proposition 2(a)], T_1 acts on the trivial space $\{0\}$. But then $T = \lambda I$, and hence not supercyclic. This completes the proof. □

Evidently, THN operators are closed under multiplication by a non-zero scalar. Hence, Theorem 2.1(i) applies to hyponormal and w-hyponormal. (Observe that the argument of the proof of Theorem 2.1 applies to M-hyponormal and quasi-hyponormal operators.) A canonical example of a contraction satisfying Theorem 2.1(ii) is that of the *unilateral shift* U. Theorem 2.1(ii) however has wider applications. Observe from the proof of Theorem 2.1(ii) that if T is a contraction with a $C_{.0}$ completely non-unitary part, D_T is Hilbert-Schmidt and non-zero isolated eigenvalues of T are normal, then T is not supercyclic. Paranormal operators (i.e., operators $T \in B(H)$ such that $\|Tx\|^2 \leq \|T^2x\|$ for unit vectors $x \in H$) are THN operators such that their non-zero isolated eigenvalues are normal [25]. Since paranormal contractions have a $C_{.0}$ completely non-unitary part [10], paranormal contractions with Hilbert-Schmidt defect operator are not supercyclic.

3. Elementary Operators $\triangle_{AB}$ and δ_{AB}

An operator $T \in B(X)$ has *the single-valued extension property* at $\lambda_0 \in \mathbf{C}$, SVEP at $\lambda_0 \in \mathbf{C}$ for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f\colon \mathcal{D}_{\lambda_0} \to X$ which satisfies

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\mathbf{C} \setminus \sigma(T)$; also T has SVEP at $\lambda \in \sigma_{\rm iso}(T)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \mathbf{C}$.* The *quasinilpotent part* $H_0(T-\lambda)$ and the *analytic core* $K(T-\lambda)$ of $(T-\lambda)$ are defined by

$$H_0(T-\lambda) = \{x \in X\colon \lim_{n\to\infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\},$$

and

$$K(T-\lambda) = \{x \in X\colon \text{ there exists a sequence } \{x_n\} \subset X \text{ and } \delta > 0 \text{ for which } x = x_0,\ (T-\lambda)x_{n+1} = x_n \text{ and } \|x_n\| \le \delta^n \|x\| \text{ for all } n = 1, 2, ...\}.$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T-\lambda)$ such that $(T-\lambda)^{-p}(0) \subseteq H_0(T-\lambda)$ for all $p = 0, 1, 2, ...$ and $(T-\lambda)\,K(T-\lambda) = K(T-\lambda)$ (cf. [20], [1]).

In the following, we assume that A, B in $B(H)$ *are THN operators such that the non-zero isolated eigenvalues of A and B are normal.* An important example of such operators A and B is that of hyponormal operators: isolated points of the spectrum of a hyponormal operator are normal eigenvalues of the operator. More generally, the non-zero isolated points of the spectrum of a (Hilbert space) paranormal operator are normal eigenvalues of the operator [25]. Note that this fails for plain THN operators [9, Remark 4]. Define the *elementary operator* $\triangle_{AB} \in B(B(H))$ and the *generalised derivation* $\delta_{AB} \in B(B(H))$ by

$$\triangle_{AB}(X) = AXB - X \quad \text{and} \quad \delta_{AB}(X) = AX - XB.$$

Recall that an *isoloid* operator is one for which isolated points of the spectrum are eigenvalues of the operator. Our first observation is that $\triangle_{AB}$ retains this property for all $\lambda \in \sigma_{\rm iso}(\triangle_{AB})$ such that $\lambda \neq -1$. The following lemma is crucial to our proof of this observation.

Lemma 3.1. THN operators are isoloid.

Proof. See [9, Proposition 3(a)]. □

Theorem 3.2. $H_0(\triangle_{AB} - \lambda) = (\triangle_{AB} - \lambda)^{-1}(0)$ for all $(-1 \neq)\lambda \in \sigma_{\rm iso}(\triangle_{AB})$. In particular, $\triangle_{AB}$ is isoloid.

Proof. We start by noticing that if λ is an isolated point in $\sigma(\triangle_{AB})$, then $B(H) = H_0(\triangle_{AB} - \lambda) \oplus K(\triangle_{AB} - \lambda)$; hence if $H_0(\triangle_{AB} - \lambda) = (\triangle_{AB} - \lambda)^{-1}(0)$ then

$$(\triangle_{AB} - \lambda)(B(H)) = (\triangle_{AB} - \lambda)K(\triangle_{AB} - \lambda) = K(\triangle_{AB} - \lambda),$$

which implies

$$B(H) = H_0(\triangle_{AB} - \lambda) \oplus (\triangle_{AB} - \lambda)(B(H))$$

so that λ is a simple pole of the resolvent of $\triangle_{AB}$ [16, Proposition 50.2]. Thus to prove the theorem it would suffice to prove that $H_0(\triangle_{AB} - \lambda) = (\triangle_{AB} - \lambda)^{-1}(0)$ for all $(-1 \neq)\lambda \in \sigma_{\rm iso}(\triangle_{AB})$. Furthermore, since THN-operators are closed under multiplication by a non-zero scalar, $\frac{1}{1+\lambda}A \in THN$ for all $\lambda \neq -1$. Hence it would suffice to prove that if $0 \in \sigma_{\rm iso}(\triangle_{AB})$, then $H_0(\triangle_{AB}) = \triangle_{AB}^{-1}(0)$.

Recall from [12] that $\sigma(\triangle_{AB}) = \{\alpha\beta - 1 : \alpha \in \sigma(A),\ \beta \in \sigma(B)\}$. If 0 lies in $\sigma_{\rm iso}(\triangle_{AB})$, then there exist finite sets $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\{\beta_1, \beta_2, ..., \beta_n\}$ of distinct scalars $\alpha_i \in \sigma(A)$ and $\beta_i \in \sigma(B)$ such that $\alpha_i\beta_i = 1$ for all $1 \le i \le n$. Obviously, α_i and β_i are non-zero (for all $1 \le i \le n$), the points α_i are isolated in $\sigma(A)$ and the points β_i are isolated in $\sigma(B)$.

Since the non-zero isolated points of A and B are normal eigenvalues (by Lemma 3.1 and our hypothesis on the isolated eigenvalues of A and B), the subspace $H_1 = \bigvee_{i=1}^{n}(A-\alpha_i)^{-1}(0)$ reduces A and the subspace $H_1' = \bigvee_{i=1}^{n}(B-\beta_i)^{-1}(0)$ reduces B. Let

$$A = A|_{H_1} \oplus A|_{H\ominus H_1} = A_1 \oplus A_2 \quad \text{and} \quad B = B|_{H_1'} \oplus B|_{H\ominus H_1'} = B_1 \oplus B_2.$$

The operators A_1 and B_1 (have finite spectrum and) are normal, and $0 \notin \sigma(\triangle_{A_iB_j})$ for all $1 \le i,j \le 2$ such that $i,j \neq 1$. Let $X \in H_0(\triangle_{AB})$, $X : H_1' \oplus (H \ominus H_1') \to H_1 \oplus (H \ominus H_1)$, have the matrix representation $X = [X_{ij}]_{i,j=1}^{2}$. Then

$$\triangle_{AB}^{n}(X) = [\triangle_{A_iB_j}^{n}(X_{ij})]_{i,j=1}^{2}.$$

Since $0 \notin \sigma(\triangle_{A_iB_j})$ for all $1 \le i,j \le 2$ such that $i,j \neq 1$,

$$\lim_{n\to\infty} \|X_{ij}\|^{\frac{1}{n}} \le \lim_{n\to\infty} \|\triangle_{A_iB_j}^{-1}\| \, \|\triangle_{A_iB_j}^{n}(X_{ij})\|^{\frac{1}{n}} = 0.$$

Hence $X_{ij} = 0$ for all $1 \le i,j \le 2$ such that $i,j \neq 1$. Since the operators A_1 and B_1 are normal, and B_1 (also, A_1) is invertible,

$$\lim_{n\to\infty} \|\delta_{A_1B_1^{-1}}^{n}(X_{11})\|^{\frac{1}{n}} \le \|B_1^{-1}\| \, \|\triangle_{A_1B_1}^{n}(X_{11})\|^{\frac{1}{n}} = 0,$$

which implies that $\delta_{A_1B_1^{-1}}(X_{11}) = 0$ [23, Lemma 2]. Thus $\triangle_{A_1B_1}(X_{11}) = 0$, which implies that $H_0(\triangle_{AB}) = \triangle_{AB}^{-1}(0)$. □

The operator $\triangle_{AB} - \lambda$ reduces to the operator $\Phi(X) = AXB$ in the case in which $\lambda = -1$. Since $\sigma(\Phi) = \{\alpha\beta : \alpha \in \sigma(A), \beta \in \sigma(B)\}$, $-1 \in \sigma_{\text{iso}}(\triangle_{AB}) \iff 0 \in \sigma_{\text{iso}}(\Phi)$, so that if $-1 \in \sigma_{\text{iso}}(\triangle_{AB})$ then either $0 \in \sigma_{\text{iso}}(A)$ or $0 \in \sigma_{\text{iso}}(B)$. The example of the operators $A = I$ and $B = P \oplus Q$, where P is an invertible THN-operator and Q is a k-nilpotent operator (for some integer $k > 1$) on a finite dimensional Hilbert space, shows that the analogue of Theorem 3.2 may fail for the case in which $\lambda = -1$. (In this example every $X = \begin{bmatrix} 0 & X_{12} \\ 0 & X_{22} \end{bmatrix}$ is in $H_0(\Phi)$ but not in $\Phi^{-1}(0)$.) If, however, 0 is a normal eigenvalue of B, then one has the following. (Here we do not require the hypothesis that the isolated non-zero eigenvalues of A and B are normal.)

Theorem 3.3. If $0 \in \sigma_{\text{iso}}(\Phi)$ and $B^{-1}(0) \subseteq B^{*-1}(0)$, then $H_0(\Phi) = \Phi^{-1}(0)$.

Proof. We divide the proof into the cases (i) $0 \in \sigma_{\text{iso}}(A)$ and $0 \notin \sigma_{\text{iso}}(B)$; (ii) $0 \notin \sigma_{\text{iso}}(A)$ and $0 \in \sigma_{\text{iso}}(B)$, and (iii) $0 \in \sigma_{\text{iso}}(A)$ and $0 \in \sigma_{\text{iso}}(B)$. Note that if $0 \notin \sigma_{\text{iso}}(A)$, then $0 \notin \sigma(A)$. (Reason: if $0 \in \sigma(A)$, then exists a sequence $\{\mu_i\} \in \sigma(A)$ such that μ_i converges to 0, and then for a $\gamma \in \sigma(B)$ the sequence $\mu_i\gamma$ converges to 0 in $\sigma(\Phi)$.) A similar statement holds for B. Let $X \in H_0(\Phi)$.

(i) If $0 \in \sigma_{\text{iso}}(A)$, then ($A$ being THN, 0 is a simple pole, and hence an eigenvalue of A [8]) A has a matrix representation $A = \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where A_{22} is invertible. Letting $X = [X_{ij}]_{i,j=1}^{2}$, it then follows that

$$A^n X = \begin{bmatrix} A_{12}A_{22}^{n-1}X_{21} & A_{12}A_{22}^{n-1}X_{22} \\ A_{22}^{n}X_{21} & A_{22}^{n}X_{22} \end{bmatrix}.$$

Since $\|A^nX\| \leq \|B^{-1}\|^n\|\Phi^n(X)\|$, it follows that $\lim_{n\to\infty}\|A_{22}^nX_{2i}\|^{\frac{1}{n}} = 0$ for $i = 1,2$. This, since A_{22} is invertible, implies that

$$\lim_{n\to\infty}\|X_{2i}\|^{\frac{1}{n}} \leq \|A_{22}^{-1}\| \lim_{n\to\infty}\|A_{22}^nX_{2i}\|^{\frac{1}{n}} = 0,$$

which implies $X_{2i} = 0$ for all $i = 1,2$. But then $X \in \Phi^{-1}(0)$. Hence $H_0(\Phi) = \Phi^{-1}(0)$.

(ii) The hypothesis $B^{-1}(0) \subseteq B^{*-1}(0)$ implies that if $0 \in \sigma_p(B)$, then 0 is a normal eigenvalue of B. Consequently, if $0 \in \sigma_{\rm iso}(B)$, then (the operator B being THN, 0 is a normal eigenvalue of B and) $B = 0 \oplus B_{22}$, where B_{22} is invertible. Letting $X = [X_{ij}]_{i,j=1}^2$, it then follows that $XB^n = \begin{bmatrix} 0 & X_{12}B_{22}^n \\ 0 & X_{22}B_{22}^n \end{bmatrix}$. Since $\|XB^n\| \leq \|A^{-1}\|^n\|\Phi^n(X)\|$, it follows that

$$\lim_{n\to\infty}\|X_{i2}\|^{\frac{1}{n}} \leq \|B_{22}^{-1}\| \lim_{n\to\infty}\|X_{i2}B_{22}^n\|^{\frac{1}{n}} = 0.$$

Thus $X_{i2} = 0$, $i = 1,2$, and $X \in \Phi^{-1}(0)$, which implies that $H_0(\Phi) = \Phi^{-1}(0)$.

(iii) Arguing as in the cases above, it is seen in this case that $\Phi^n(X)$ has a representation $\Phi^n(X) = \begin{bmatrix} 0 & A_{12}A_{22}^{n-1}X_{22}B_{22}^n \\ 0 & A_{22}^nX_{22}B_{22}^n \end{bmatrix}$, where A_{22} and B_{22} are invertible. If $X \in H_0(\Phi)$, then $X_{22} = 0$, so that $X \in \Phi^{-1}(0)$ and $H_0(\Phi) = \Phi^{-1}(0)$. $\square$

The next theorem is an analogue of Theorem 3.2 for generalized derivations δ_{AB}.

Theorem 3.4. $H_0(\delta_{AB} - \lambda) = (\delta_{AB} - \lambda)^{-1}(0)$ for all non-zero $\lambda \in \sigma_{\rm iso}(\delta_{AB})$. Furthermore, if $B^{-1}(0) \subseteq B^{*-1}(0)$ and $0 \in \sigma_{\rm iso}(\delta_{AB})$, then $H_0(\delta_{AB}) = \delta_{AB}^{-1}(0)$.

Proof. Let $(0 \neq)\lambda \in \sigma_{\rm iso}(\delta_{AB})$. If we consider $(\delta_{AB} - \lambda)(X) = AX - X(B + \lambda)$ as the operator $\delta_{A(B+\lambda)}$, then $\sigma(\delta_{AB} - \lambda) = \{\alpha - (\beta + \lambda) : \alpha \in \sigma(A),\ \beta + \lambda \in \sigma(B+\lambda)\}$. Since $\lambda \in \sigma_{\rm iso}(\delta_{AB})$ if and only if $0 \in \sigma_{\rm iso}(\delta_{A(B+\lambda)})$, there exist finite sets $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\{\beta_1 + \lambda, \beta_2 + \lambda, ..., \beta_n + \lambda\}$ such that $\alpha_i \in \sigma_{\rm iso}(A)$, $\beta_i \in \sigma_{\rm iso}(B)$ and $\alpha_i = \beta_i + \lambda$ for all $1 \leq i \leq n$. We have three possible cases: (i) $0 \in \sigma_{\rm iso}(A)$ and $0 \notin \sigma_{\rm iso}(B + \lambda)$; (ii) $0 \notin \sigma_{\rm iso}(A)$ and $0 \in \sigma_{\rm iso}(B + \lambda)$; (iii) $0 \in \sigma_{\rm iso}(A)$ and $0 \in \sigma_{\rm iso}(B + \lambda)$. We consider these cases separately. Let $X \in H_0(\delta_{AB} - \lambda)$.

(i) If $0 \notin \sigma_{\rm iso}(B+\lambda)$, then $B+\lambda$ is invertible and $\delta_{A(B+\lambda)}(X) = \triangle_{A(B+\lambda)^{-1}}(X)$ $(B+\lambda)$. Since $\|\delta_{A(B+\lambda)}^n(X)\| \leq \|B+\lambda\|^n\|\triangle_{A(B+\lambda)^{-1}}^n(X)\|$ and $\|\triangle_{A(B+\lambda)^{-1}}^n(X)\| \leq \|(B + \lambda)^{-1}\|^n\|\delta_{A(B+\lambda)}^n(X)\|$, it follows from Theorem 3.2 that $H_0(\delta_{AB} - \lambda) = H_0(\triangle_{A(B+\lambda)^{-1}}) = (\triangle_{A(B+\lambda)^{-1}})^{-1}(0)$. Again, as $X \in (\triangle_{A(B+\lambda)^{-1}})^{-1}(0)$ if and only if $\{AX - X(B + \lambda)\}(B + \lambda)^{-1} = 0$ if and only if $X \in (\delta_{AB} - \lambda)^{-1}(0)$, $H_0(\delta_{AB} - \lambda) = (\delta_{AB} - \lambda)^{-1}(0)$.

(ii) If $0 \in \sigma_{\rm iso}(B + \lambda)$, then $(0 \neq) - \lambda \in \sigma_{\rm iso}(B)$, which implies that B has a representation $B = \begin{bmatrix} -\lambda I_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$ and $B + \lambda = 0 \oplus (B_{22} + \lambda)$. Here the operator $B_{22} + \lambda$ is invertible. Since $(\delta_{AB} - \lambda)(X) = A\triangle_{A^{-1}(B+\lambda)}(X)$,

$$\|\triangle_{A^{-1}(B+\lambda)}^n(X)\| \leq \|A^{-1}\|^n\|(\delta_{AB} - \lambda)^n(X)\|$$

and

$$\|(\delta_{AB} - \lambda)^n(X)\| \leq \|A\|^n\|\triangle_{A^{-1}(B+\lambda)}^n(X)\|,$$

$H_0(\delta_{AB}-\lambda) = H_0(\triangle_{A^{-1}(B+\lambda)})$. If we now let X have the representation $X = [X_{ij}]_{i,j=1}^2$, then

$$X(B+\lambda)^n = \begin{bmatrix} 0 & X_{12}(B_{22}+\lambda)^n \\ 0 & X_{22}(B_{22}+\lambda)^n \end{bmatrix}$$

and

$$\lim_{n\to\infty} \|X_{i2}(B_{22}+\lambda)^n\|^{\frac{1}{n}} = 0$$

for all $i = 1, 2$. Hence (argue as before) $H_0(\delta_{AB}-\lambda) = (\delta_{AB}-\lambda)^{-1}(0)$.

(iii) If $0 \in \sigma_{\text{iso}}(A)$ and $0 \in \sigma_{\text{iso}}(B+\lambda)$, then A has a representation $A = \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}$ and B has a representation $B = 0 \oplus (B_{22}+\lambda)$ (see case (ii)), where A_{22} and $B_{22}+\lambda$ are invertible. Letting $X = [X_{ij}]_{i,j=1}^2$, it follows that

$$(\delta_{AB}-\lambda)^n(X) = \begin{bmatrix} A_{12}A_{22}^{n-1}X_{21} & * \\ A_{22}^n X_{21} & \delta^n_{A_{22}(B_{22}+\lambda)}(X_{22}) \end{bmatrix}$$

(where, if $X_{21} = X_{22} = 0$, then the entry "$*$" equals $X_{12}(B_{22}+\lambda)^n$.) Since

$$\lim_{n\to\infty} \|X_{21}\|^{\frac{1}{n}} \leq \|A_{22}^{-1}\| \lim_{n\to\infty} \|A_{22}^n X_{21}\|^{\frac{1}{n}} = 0$$

and

$$\lim_{n\to\infty} \|X_{22}\|^{\frac{1}{n}} = \|A_{22}^{-1}\|\, \|(B_{22}+\lambda)^{-1}\| \lim_{n\to\infty} \|\delta^n_{A_{22}(B_{22}+\lambda)}(X_{22})\|^{\frac{1}{n}} = 0,$$

it follows that $X_{21} = X_{22} = 0$. But then (from the entry "$*$" in the matrix above)

$$\lim_{n\to\infty} \|X_{12}\|^{\frac{1}{n}} \leq \|(B_{22}+\lambda)^{-1}\| \lim_{n\to\infty} \|X_{12}(B_{22}+\lambda)^n\|^{\frac{1}{n}} = 0$$

implies that $X_{12} = 0$. Thus $X \in (\delta_{AB}-\lambda)^{-1}(0)$ and $H_0(\delta_{AB}-\lambda) = (\delta_{AB}-\lambda)^{-1}(0)$.

To complete the proof we note that if $B^{-1}(0) \subseteq B^{*-1}(0)$, then $0 \in \sigma_{\text{iso}}(B)$ is a normal eigenvalue of B, and the argument of cases (ii) and (iii) holds with $\lambda = 0$ (case (i) is not effected by $\lambda = 0$). □

Let $d_{AB} \in B(B(H))$ denote either of the operators $\triangle_{AB}$ and δ_{AB}. If $B^{-1}(0) \subseteq B^{*-1}(0)$, then Theorems 3.2, 3.3 and 3.4 imply that $H_0(d_{AB}-\lambda) = (d_{AB}-\lambda)^{-1}(0)$ at every $\lambda \in \sigma_{\text{iso}}(d_{AB})$. Observe that the hypotheses $A, B \in THN$ and $B^{-1}(0) \subseteq B^{*-1}(0)$ may be replaced by the hypotheses that $A, B^* \in THN$ and $B^{*-1}(0) \subseteq B^{-1}(0)$ in Theorems 3.2, 3.3 and 3.4. (This requires but obvious minor changes in the proofs of these theorems.) Examples of operators $A, B^* \in B(H)$ such that $A, B^* \in THN$ and $B^{*-1}(0) \subseteq B^{-1}(0)$ abound, the example of hyponormal A and B^* being one such example (see [7]). Summarizing, we have:

Theorem 3.5. If $A, B \in THN$ (or $A, B^* \in THN$), and $B^{-1}(0) \subseteq B^{*-1}(0)$ (resp., $B^{*-1}(0) \subseteq B^{-1}(0)$), then $H_0(d_{AB}-\lambda) = (d_{AB}-\lambda)^{-1}(0)$ at every $\lambda \in \sigma_{\text{iso}}(d_{AB})$.

Recall from [11, p.95] that a subspace $\mathcal{M}$ of the Banach space $\mathcal{X}$ is *orthogonal* to a subspace $\mathcal{N}$ of $\mathcal{X}$, denoted $\mathcal{M} \perp \mathcal{N}$, if $\|m\| \leq \|m+n\|$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$. (This asymmetric definition of orthogonality coincides with the usual definition of orthogonality in the case in which $\mathcal{X} = H$ is a Hilbert space.) Theorem 3.5 implies that if $A, B \in THN$ and $B^{-1}(0) \subseteq B^{*-1}(0)$ or $A, B^* \in THN$ and $B^{*-1}(0) \subseteq B^{-1}(0)$, then $(d_{AB}-\lambda)^{-1}(0) \perp (d_{AB}-\lambda)(B(H))$ for every $\lambda \in \sigma_{\text{iso}}(d_{AB})$.

The numerical range of $T \in B(\mathcal{X})$ is the set

$$W(B(\mathcal{X}), T) = \{f(T) \colon\ f \in B(\mathcal{X})^*,\ \|f\| = f(I) = 1\},$$

where $B(\mathcal{X})^*$ denotes the dual space of $B(\mathcal{X})$ [5]. $W(B(\mathcal{X})), T)$ is a compact convex subset of $\mathbf{C}$. A Banach space operator $T \in B(\mathcal{X})$ is said to be *semi-regular* if $T(\mathcal{X})$ is closed and $T^{-1}(0) \subseteq T^{\infty}(\mathcal{X}) = \bigcap_{n\geq 1} T^n(\mathcal{X})$. The operator T admits a *generalized Kato decomposition*, or GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $\mathcal{X} = M \oplus N$, where $T|_M$ is quasi-nilpotent and $T|_N$ is semi-regular; T is said to be of *Kato type* if T has GKD and $T|_M$ is nilpotent [3]. Obviously, the operator $d_{AB} - \lambda$, where d_{AB} is the operator of Theorem 3.5, is Kato type at every $\lambda \in \sigma_{\rm iso}(d_{AB})$. Let $\sigma_{kt}(T)$ denote the part of $\sigma(T)$ defined by

$$\sigma_{kt}(T) = \{\lambda \in \mathbf{C}\colon \ \lambda I - T \quad \text{is not of Kato type}\}.$$

Then $\sigma_{kt}(T)$ is a closed subset of $\sigma(T)$ such that every non-isolated point of the boundary of $\sigma(T)$ belongs to $\sigma_{kt}(T)$. If both T and T^* have SVEP at a point $\lambda \in \sigma_{\rm acc}(T)$, then $\lambda \in \sigma_{kt}(T)$. This follows from Theorems 2.6 and 2.9 of [2], as the following argument shows. Assume that $T - \lambda$ is Kato type. Then, since both T and T^* have SVEP at λ, both $\operatorname{asc}(T - \lambda)$ and $\operatorname{dsc}(T - \lambda)$ are finite, and hence equal. Consequently, there exists an integer $q \geq 1$ such that $B(\mathcal{X}) = (T - \lambda)^{-q}(0) \oplus (T - \lambda)^q(B(\mathcal{X}))$, which implies that λ is isolated in $\sigma(T)$, a contradiction. (See also [3].) Clearly, $\sigma_{kt}(T) \subseteq \sigma_f(T)$, where $\sigma_f(T)$ denotes the Fredholm spectrum of T. (See e.g., [1].) We remark here that the following theorem does not require the hypothesis that the isolated non-zero eigenvalues of A and B are normal.

Theorem 3.6. For each $\lambda \in \sigma(\triangle_{AB})$ such that $|1 + \lambda| = \|A\|\,\|B\|$, either λ is a simple pole of the resolvent of $\triangle_{AB}$ or a point of $\sigma_f(\triangle_{AB})$.

Proof. If we let $1 + \lambda = \exp^{i\theta} |1 + \lambda|$, and define the operators A_1 and B_1 by $A_1 = \exp(-i\theta)\frac{A}{\|A\|}$ and $B_1 = \frac{B}{\|B\|}$, then A_1, B_1 are contractions in THN. Since $(\triangle_{AB} - \lambda)(X) = (1+\lambda)\{\triangle_{A_1B_1}(X)\}$ and since $\lambda \in \sigma(\triangle_{AB})$, $0 \in \sigma(\triangle_{A_1B_1}) = \{\alpha\beta - 1 : \alpha \in \sigma(A_1), \beta \in \sigma(B_1)\}$. Let $L_{A_1} \in B(B(H))$ denote the operator of *left multiplication* by A_1, and let $R_{B_1} \in B(B(H))$ denote the operator of *right multiplication* by B_1. Then $W(B(B(H)), L_{A_1}R_{B_1}) = \{z \in \mathbf{C} : |z| \leq 1\}$, and $W(B(B(H)), \triangle_{A_1B_1}) = \{z \in \mathbf{C} : |1 + z| \leq 1\}$. Thus $0 \in \partial W(B(B(H)), \triangle_{A_1B_1})$, where $\partial(S)$ denotes the boundary of the set $S \subset \mathbf{C}$. Hence $0 \in \partial\sigma(\triangle_{A_1B_1}) \subseteq \sigma_a(\triangle_{A_1B_1})$. We have two possibilities: either $0 \in \sigma_{\rm iso}(\triangle_{A_1B_1})$ or $0 \in \sigma_{\rm acc}(\triangle_{A_1B_1})$. If $0 \in \sigma_{\rm iso}(\triangle_{A_1B_1})$, then there exist finite sets $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\{\beta_1, \beta_2, ..., \beta_n\}$ such $\alpha_i \in \sigma_{\rm iso}(A_1)$, $\beta_i \in \sigma_{\rm iso}(B_1)$ and $\alpha_i\beta_i = 1$ (so that $|\alpha_i| = |\beta_i| = 1$) for all $1 \leq i \leq n$. Since A_1 and $B_1 \in THN$ are contractions, α_i and β_i are normal eigenvalues (of A_1 and B_1, respectively). It follows that, see the proof of Theorem 3.2, 0 is a simple pole of the resolvent of $\triangle_{A_1B_1}$, which implies that $B(H) = \triangle_{A_1B_1}^{-1}(0) \oplus \triangle_{A_1B_1}(B(H))$. Notice that $\triangle_{A_1B_1}^{-1}(0) = (\triangle_{AB} - \lambda)^{-1}(0)$ and $\triangle_{A_1B_1}(B(H)) = (\triangle_{AB} - \lambda)(B(H))$. Hence λ is a simple pole of the resolvent of $\triangle_{AB}$. Assume now that $0 \in \sigma_{\rm acc}(\triangle_{A_1B_1})$. Then $0 \in \sigma_{a_{\rm acc}}(\triangle_{A_1B_1})$. The point 0 being a boundary point of $\sigma(\triangle_{A_1B_1})$, $0 \in \sigma_{kt}(\triangle_{A_1B_1}) \subseteq \sigma_f(\triangle_{A_1B_1})$. (Both $\triangle_{A_1B_1}$ and the conjugate operator $\triangle_{A_1B_1}^*$ have SVEP at 0.) Recall from the Nirschl-Schneider theorem [5] that a Banach space operator has ascent less than or equal to one at all points in the boundary of the numerical range of the operator. Hence $\operatorname{asc}(\triangle_{A_1B_1}) \leq 1$, which implies that $\operatorname{ind}(\triangle_{A_1B_1}) \leq 0$ [16, Proposition 38.5]. Thus $0 \in \sigma_f(\triangle_{A_1B_1})$ implies that either $\triangle_{A_1B_1}(B(H))$ is not closed or $\beta(\triangle_{A_1B_1}) = \infty$. But then either $(\triangle_{AB} - \lambda)(B(H))$ is not closed or $\beta(\triangle_{AB} - \lambda) = \infty$. Hence $\lambda \in \sigma_f(\triangle_{AB})$. $\square$

4. Weyl's Theorem for d_{AB}

For an operator $T \in B(\mathcal{X})$, let $\pi_{00}(T) = \{\lambda \in \sigma_{\text{iso}}(T) : 0 < \alpha(T-\lambda) < \infty\}$ denote the set of isolated eigenvalues of T of finite geometric multiplicity, and let $\pi_0(T) = \{\lambda \in \sigma(T) : \text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty\}$ denote the set of Riesz points of T. Then $\pi_0(T) \subseteq \pi_{00}(T)$. Recall that T is said to be Weyl if it is Fredholm of 0 index, and that the Weyl spectrum $\sigma_w(T)$ of T is the set $\{\lambda \in \mathbf{C} : T-\lambda \text{ is not Weyl}\}$. T satisfies Browder's theorem (Weyl's theorem) if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$ (resp., $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$) [14]. If T has SVEP, then T satisfies Browder's theorem [2, Corollary 2.17]. SVEP, however, is not sufficient to guarantee Weyl's theorem; consider for example the operator $T = T_1 \oplus T_2$, where $T_1 \in B(\ell_2^+)$ is defined by $T_1(x_1, x_2, ...) = (\frac{x_1}{2}, \frac{x_2}{3}, ...)$ and T_2 is a nilpotent on a finite dimensional space, when it is seen that $\sigma(T) = \sigma_w(T) = \{0\}$, $\pi_0(T) = \emptyset$ and $\pi_{00}(T) = \{0\}$. The following theorem shows that the elementary operator d_{AB} satisfies Weyl's theorem under a weaker SVEP condition.

Theorem 4.1. Let A and B be THN operators, If $B^{-1}(0) \subseteq B^{*-1}(0)$ and d_{AB} has SVEP at all points $\lambda \in \sigma(d_{AB}) \setminus \sigma_w(d_{AB})$, then d_{AB} and d^*_{AB} satisfy Weyl's theorem.

Proof. If $\lambda \in \pi_{00}(d_{AB})$, then $\lambda \in \sigma_{\text{iso}}(d_{AB})$ and $\alpha(d_{AB}-\lambda) < \infty$, which implies that λ is a simple pole of the resolvent of d_{AB} (such that $\alpha(d_{AB}-\lambda) < \infty$). Hence $d_{AB}-\lambda$ is Fredholm of 0 index (apply [16, Proposition 38.6]), which implies that $\pi_{00}(d_{AB}) \subseteq \sigma(d_{AB}) \setminus \sigma_w(d_{AB})$. Thus, to prove that d_{AB} satisfies Weyl's theorem it will suffice to prove that $\pi_{00}(d_{AB}) \supseteq \sigma(d_{AB}) \setminus \sigma_w(d_{AB})$. Let $\lambda \in \sigma(d_{AB}) \setminus \sigma_w(d_{AB})$. Then $d_{AB}-\lambda$ is Fredholm of 0 index. Since d_{AB} has SVEP at λ, it follows from [2, Corollary 2.10] that $\text{asc}(d_{AB}-\lambda) = \text{dsc}(d_{AB}-\lambda) < \infty$. Hence $\lambda \in \pi_{00}(d_{AB})$. We prove next that d^*_{AB} satisfies Weyl's theorem.

Since d_{AB} satisfies Weyl's theorem, both d_{AB} and d^*_{AB} satisfy Browder's theorem. (Recall from [14] that a Banach space operator T satisfies Weyl's theorem $\Longrightarrow T$ satisfies Browder's theorem $\Longleftrightarrow T^*$ satisfies Browder's theorem.) Hence

$$\sigma(d^*_{AB}) \setminus \sigma_w(d^*_{AB}) = \pi_0(d^*_{AB}) \subseteq \pi_{00}(d^*_{AB}).$$

For the reverse inclusion, we let $\lambda \in \pi_{00}(d^*_{AB})$. Then $\lambda \in \sigma_{\text{iso}}(d_{AB})$ and the following implications hold:

$$\begin{aligned}
\lambda \in \sigma_{\text{iso}}(d_{AB}) &\implies B(H) = (d_{AB}-\lambda)^{-1}(0) \oplus (d_{AB}-\lambda)(B(H)) \\
&\implies B(H)^* = (d^*_{AB}-\lambda I^*)^{-1}(0) \oplus (d^*_{AB}-\lambda I^*)(B(H)^*) \\
&\implies \text{asc}(d^*_{AB}-\lambda I^*) = \text{dsc}(d^*_{AB}-\lambda I^*) \leq 1.
\end{aligned}$$

Since $0 < \alpha(d^*_{AB}-\lambda I^*) < \infty$, it follows that $\alpha(d^*_{AB}-\lambda I^*) = \beta(d^*_{AB}-\lambda I^*) < \infty \Longrightarrow \lambda \in \sigma(d^*_{AB}) \setminus \sigma_w(d^*_{AB})$. Hence $\pi_0(d^*_{AB}) = \pi_{00}(d^*_{AB})$, and d^*_{AB} satisfies Weyl's theorem. $\square$

Theorem 4.1 has an A, $B^* \in THN$ counterpart. More precisely one has:

Theorem 4.2. Let A, $B^* \in THN$. If $B^{*-1}(0) \subseteq B^{-1}(0)$ and d_{AB} has SVEP at all points $\lambda \in \sigma(d_{AB}) \setminus \sigma_w(d_{AB})$, then d_{AB} and d^*_{AB} satisfy Weyl's theorem.

The hypotheses $B^{*-1}(0) \subseteq B^{-1}(0)$ and d_{AB} has SVEP at all points $\lambda \in \sigma(d_{AB}) \setminus \sigma_w(d_{AB})$ are satisfied by many a choice of $A, B^* \in THN$. Thus, for example, if A, B^* are hyponormal operators, then $(B^*-\overline{\lambda})^{-1}(0) \subseteq (B-\lambda)^{-1}(0)$ and $d_{AB}-\lambda$

has SVEP for all complex λ (see [7, Corollary 2.4]). A stronger version of Theorem 4.1 holds for operators d_{AB} with SVEP.

We say that the Fredholm operator $T \in B(\mathcal{X})$ has *stable index* if $\text{ind}\,(T-\lambda)$ $\text{ind}\,(T-\mu) \geq 0$ for every λ, μ in the Fredholm region of T. Let $\pi_{a0}(T) = \{\lambda \in \mathbf{C}\colon \lambda \in \sigma_{a_{\text{iso}}}(T) \text{ and } 0 < \alpha(T-\lambda) < \infty\}$. We say that *a-Weyl's theorem holds for* T if

$$\sigma_{wa}(T) = \sigma_a(T) \setminus \pi_{a0}(T),$$

where $\sigma_{wa}(T)$ denotes *the essential approximate point spectrum* (i.e., $\sigma_{wa}(T) = \bigcap\{\sigma_a(T+K)\colon K \in K(X)\}$ with $K(X)$ denoting the ideal of compact operators on X). If we let $\Phi_+(X) = \{T \in B(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$ denote the semi-group of *upper semi-Fredholm operators* in $B(X)$, then $\sigma_{wa}(T)$ is the complement in $\mathbf{C}$ of all those λ for which $(T-\lambda) \in \Phi_+(X)$ and $\text{ind}\,(T-\lambda) \leq 0$. The concept of a-Weyl's theorem was introduced by Rakočvić: a-Weyl's theorem for $T \Longrightarrow$ Weyl's theorem for T, but the converse is generally false [22].

Let $\mathcal{H}(\sigma(T))$ denote the set of analytic functions f which are defined on an open neighborhood $\mathcal{U}$ of $\sigma(T)$.

Theorem 4.3. Let A and B be THN operators such that $B^{-1}(0) \subseteq B^{*-1}(0)$. If d_{AB} has SVEP, then:

(i) $f(d_{AB})$ and $f(d_{AB}^*)$ satisfy Weyl's theorem for every $f \in \mathcal{H}(\sigma(d_{AB}))$.
(ii) d_{AB} satisfies a-Weyl's theorem.

Proof. (i) Recall from Schmoeger [24, Theorem 1] that if an isoloid Banach space operator satisfying Weyl's theorem has stable index, then $f(T)$ satisfies Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$. As we have already seen, the operators d_{AB} and d_{AB}^* are isoloid and satisfy Weyl's theorem. Furthermore, if d_{AB} has SVEP at λ and $d_{AB} - \lambda$ is Fredholm, then $\text{asc}(d_{AB}-\lambda) = \text{dsc}(d_{AB}-\lambda) < \infty$ and $\text{ind}\,(d_{AB}-\lambda) = 0$ (combine [2, Theorem 2.6] with [16, Proposition 38.6]). Hence $\text{ind}\,(d_{AB}-\lambda) = \text{ind}\,(d_{AB}^* - \lambda I^*) = 0$, so that d_{AB} and d_{AB}^* have stable index.

(ii) If d_{AB} has SVEP, then $\sigma(d_{AB}) = \sigma_a(d_{AB}^*)$ [19, page 35], $\pi_{a0}(d_{AB}) = \pi_{00}(d_{AB}^*)$ and $\sigma_{wa}(d_{AB}) = \sigma_w(d_{AB}^*)$. Since d_{AB}^* satisfies Weyl's theorem, the proof follows. $\square$

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8 Redwood Grove, Northfields Avenue, Ealing, London W5 4SZ, England, U.K.
E-mail address: bpduggal@yahoo.co.uk

Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil
E-mail address: carlos@ele.puc-rio.br