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PARANORMAL CONTRACTIONS HAVE PROPERTY PF

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ABSTRACT. The assertion in the title is equivalent to saying that the completely nonunitary direct summand of a paranormal contraction is of class $\mathcal{C}_{\cdot 0}$. In this paper we give a new proof for this result, and extend it to k-quasihyponormal contractions and to k-paranormal contractions.

1. Notation and Terminology

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ stand for nonzero complex Hilbert spaces, and $\mathcal{B}[\mathcal{H},\mathcal{K}]$ stands for the Banach space of all bounded linear transformations of $\mathcal{H}$ into $\mathcal{K}$. If X lies in $\mathcal{B}[\mathcal{H},\mathcal{K}]$, then X^* in $\mathcal{B}[\mathcal{K},\mathcal{H}]$ denotes the adjoint of X. The range of $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ will be denoted by $\mathcal{R}(X)$ and its closure, which is a subspace (i.e., a closed linear manifold) of $\mathcal{K}$, by $\mathcal{R}(X)^-$. The null space (kernel) of $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, which is a subspace of $\mathcal{H}$, will be denoted by $\mathcal{N}(X)$. Set $\mathcal{B}[\mathcal{H}] = \mathcal{B}[\mathcal{H},\mathcal{H}]$ for short. If T lies in $\mathcal{B}[\mathcal{H}]$, then we say that T is an operator on $\mathcal{H}$.

By a contraction we mean an operator T such that $\|T\| \le 1$ (i.e., $\|Tx\| \le \|x\|$ for every $x \in \mathcal{H}$). An isometry is a contraction T such that $\|Tx\| = \|x\|$ for every $x \in \mathcal{H}$, and T is a coisometry if T^* is an isometry. If T is an isometry and a coisometry, then it is a unitary operator. A contraction is completely nonunitary if it has no unitary direct summand. For any contraction T the sequence of positive numbers $\{\|T^n x\|\}$ is decreasing (thus convergent) for every $x \in \mathcal{H}$. A contraction T is of class $\mathcal{C}_{0\cdot}$ if it is strongly stable; that is, if $\{\|T^n x\|\}$ converges to zero for every $x \in \mathcal{H}$, and of class $\mathcal{C}_{1\cdot}$ if $\{\|T^n x\|\}$ does not converge to zero for every nonzero $x \in \mathcal{H}$. It is of class $\mathcal{C}_{\cdot 0}$ or of class $\mathcal{C}_{\cdot 1}$ if its adjoint T^* is of class $\mathcal{C}_{0\cdot}$ or $\mathcal{C}_{1\cdot}$, respectively. All combinations are possible, leading to the Nagy–Foiaş classes of contractions $\mathcal{C}_{00}$, $\mathcal{C}_{01}$, $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$ [16, p.72].

We shall be dealing with the following well-known classes of operators. An operator T is *dominant* if, for each $\lambda \in \mathbb{C}$, $\|(\lambda I - T)^* x\| \le M_\lambda \|(\lambda I - T)x\|$ for every $x \in \mathcal{H}$ for some real number $M_\lambda \ge 0$ or, equivalently, $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda} I - T^*)$. As usual, put $|T| = (T^*T)^{\frac{1}{2}}$. An operator T is *p-hyponormal* if $|T^*|^{2p} \le |T|^{2p}$ for some real number $0 < p \le 1$, and *M-hyponormal* if, for all $\lambda \in \mathbb{C}$, $\|(\lambda I - T)^* x\| \le M\|(\lambda I - T)x\|$ for every $x \in \mathcal{H}$ for some real number $M \ge 1$. A *hyponormal* is precisely a 1-hyponormal operator (i.e., an operator T such that $TT^* \le T^*T$ or, equivalently, $\|(\lambda I - T)^* x\| \le \|(\lambda I - T)x\|$ for every $\lambda \in \mathbb{C}$ and every $x \in \mathcal{H}$). An operator T is *k-quasihyponormal* if $T^{*k}(T^*T - TT^*)T^k \ge O$ for some integer $k \ge 1$, and *quasi-p-hyponormal* (also called *p-quasihyponormal*) if $T^*(|T|^{2p} - |T^*|^{2p})T \ge O$ for some real $0 < p \le 1$. A *quasihyponormal* is a 1-quasihyponormal or a quasi-1-hyponormal operator or, equivalently, an operator T such that $|T|^4 \le |T^2|^2$; and so a *semi-quasihyponormal* is an operator T such that $|T|^2 \le |T^2|$ (also called *class*

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$\mathcal{A}$ or *class* $\mathcal{U}$). An operator T is *k-paranormal* if $\|Tx\|^{k+1} \leq \|T^{k+1}x\|\,\|x\|^k$ for some integer $k \geq 1$ and every $x \in \mathcal{H}$. A *paranormal* is simply a 1-paranormal operator.

See [2], [4], [8], [10], and [15] for properties of operators belonging to the above classes. Observe that M-hyponormal operators are dominant, and so are the hyponormal operators, which are paranormal too. Paranormal operators comprise a large class that will play a central role in this paper. Recall that quasi-p-hyponormal, as well as semi-quasihyponormal, are all paranormal operators. Since p-hyponormal operators are quasi-p-hyponormal, these are again paranormal. (In particular, hyponormal are quasihyponormal, which are semi-quasihyponormal, which in turn are paranormal.) Also recall that a paranormal operator is k-paranormal for every positive integer k, and therefore an operator is paranormal if and only if it is k-paranormal for every $k \geq 1$. The diagram below summarizes the relationship among these classes that will be required later in this paper.

		p-hyponormal	$\longrightarrow$	quasi-p-hyponormal				
	$\nearrow$		$\searrow_1$		$\searrow_2$			
hyponormal	$\longrightarrow$	quasihyponormal	$\longrightarrow$	semi-quasihyponormal	$\longrightarrow$	paranormal	$\longrightarrow$	k-paranormal
	$\searrow$		$\searrow$					
		M-hyponormal		k-quasihyponormal				
			$\searrow$					
				dominant				

For the nontrivial implications in the central row (from hyponormal to k-paranormal) see e.g., [10, p.94]. Those in 1 and 2 can be found in [8, pp.162,166] and [1], respectively. The remaining implications are either readily verified or trivial.

2. Introduction

Every contraction $T \in \mathcal{B}[\mathcal{H}]$ has a unique direct sum decomposition

$$T = U \oplus C,$$

where U is unitary and C is a completely nonunitary contraction. This is the celebrated Nagy–Foiaş–Langer decomposition for contractions [16, p.9]. Contractions T for which the completely nonunitary direct summand C is of class $\mathcal{C}_{\cdot 0}$ have been characterized in [4] as follows. A contraction $T \in \mathcal{B}[\mathcal{H}]$ has property PF (short for Putnam–Fuglede commutativity property) if, whenever the equation

$$TX = XJ^*$$

holds for some isometry $J \in \mathcal{B}[\mathcal{K}]$ and some $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, then

$$T^*X = XJ.$$

That is, a contraction T has property PF if either no coisometry is intertwined to T or, if $X \neq O$ intertwines a coisometry J^* to T, then the same X also intertwines the isometry J to T^*.

Lemma 1. *The completely nonunitary direct summand of a contraction T is of class $\mathcal{C}_{\cdot 0}$ if and only if T has property* PF.

Proof. [4] — see also [11]. □

This characterization works beautifully for contractions for which a Putnam–Fuglede type commutativity theorem is known to hold. (e.g., dominant operators, for which it is known that if $TX = XH^*$, then $T^*X = XH$ whenever T is dominant and H is hyponormal [2]), but is not easy to apply in many other cases (e.g., if T paranormal). The purpose of this paper is to extend Lemma 2.1 of [4] to take care of some of these cases.

3. Main Results

If $T \in \mathcal{B}[\mathcal{H}]$ is a contraction, then the sequence $\{T^nT^{*n}\}$ converges strongly to a nonnegative contraction A (i.e., $T^nT^{*n} \xrightarrow{s} A$, where $O \leq A \leq I$). The operators T and A are related by $TAT^* = A$ and $\mathcal{N}(A) = \{x \in \mathcal{H}\colon\; T^{*n}x \to 0\}$ — the kernel of A is the subspace of strong stability for T^*. Let $A^{\frac{1}{2}}$ be the square root of A, which is again a nonnegative contraction with $\mathcal{N}(A^{\frac{1}{2}}) = \mathcal{N}(A)$ and $\mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^-$. Moreover, there exists an isometry $V\colon \mathcal{R}(A)^- \to \mathcal{R}(A)^-$ such that $A^{\frac{1}{2}}T^* = VA^{\frac{1}{2}}$, and so $TA^{\frac{1}{2}} = A^{\frac{1}{2}}V^*$ on $\mathcal{R}(A)^-$, and $\|A^{\frac{1}{2}}V^nx\| \to \|x\|$ for every $x \in \mathcal{R}(A)^-$ (see e.g., [7] or [9, Chapter 3], where the notation A_* is used instead of our A). We shall be particularly concerned with contractions T for which A is a projection (see e.g., [3], [12] and [11]).

Theorem 1. *Let $T \in \mathcal{B}[\mathcal{H}]$ be a contraction. Take any vector x in $\mathcal{R}(A)^-$ and put $x_n = A^{\frac{1}{2}}V^nx$ for each integer $n \geq 0$. Consider the assertions*

(a) *the completely nonunitary direct summand of T is of class $\mathcal{C}_{\cdot 0}$,*

(b) *T has property* PF*,*

and also

(c) *$\{||x_n||\}$ is a convex sequence,*

(d) *$\{||x_n||\}$ is a constant sequence,*

(e) *A is a projection*

(f) *the completely nonunitary direct summand of T is either a $\mathcal{C}_{\cdot 0}$-contraction, a backward unilateral shift, or a direct sum of a $\mathcal{C}_{\cdot 0}$-contraction and a backward unilateral shift.*

Claim: (a) *and* (b) *are equivalent,* (b) *implies* (c)*, and* (c)*,* (d)*,* (e) *and* (f) *are equivalent.*

Proof. According to Lemma 1 (a) and (b) are equivalent [4]. If (b) holds, then $TA^{\frac{1}{2}} = A^{\frac{1}{2}}V^*$ implies $T^*A^{\frac{1}{2}} = A^{\frac{1}{2}}V$, so that $TA^{\frac{1}{2}}V^{n+1} = A^{\frac{1}{2}}V^*V^{n+1} = A^{\frac{1}{2}}V^n = A^{\frac{1}{2}}V\,V^{n-1} = T^*A^{\frac{1}{2}}V^{n-1}$ on $\mathcal{R}(A)^-$ (V is an isometry). Hence, with $x \in \mathcal{R}(A)^-$,

$$\begin{aligned}\|x_n\|^2 \;&=\; \|A^{\frac{1}{2}}V^nx\|^2 = \|T^*A^{\frac{1}{2}}V^{n-1}x\|\,\|TA^{\frac{1}{2}}V^{n+1}x\| \\ &\leq\; \|x_{n-1}\|\,\|x_{n+1}\| \leq \tfrac{1}{4}\big(\|x_{n-1}\| + \|x_{n+1}\|\big)^2\end{aligned}$$

(T is a contraction), and $\{\|x_n\|\}$ is convex. Thus (b) implies (c). Since $x_n = Tx_{n+1}$, the sequence $\{\|x_n\|\}$ is increasing and bounded ($A^{\frac{1}{2}}V^n = A^{\frac{1}{2}}V^*V^{n+1} = TA^{\frac{1}{2}}V^{n+1}$ so that $\|A^{\frac{1}{2}}V^nx\| \leq \|A^{\frac{1}{2}}V^{n+1}x\| \leq \|x\|$ for every $x \in \mathcal{R}(A)^-$). Hence (c) implies (d) because a bounded increasing convex sequence must be constant. Since the converse is trivial, it follows that (c) and (d) are equivalent. Now, if (d) holds, then $\|A^{\frac{1}{2}}x\| = \|A^{\frac{1}{2}}V^nx\| \to \|x\|$ so that $\|A^{\frac{1}{2}}x\| = \|x\|$ for every $x \in \mathcal{R}(A)^-$. Thus $A^{\frac{1}{2}}$ is an isometry on $\mathcal{R}(A)^-$. Since $A^{\frac{1}{2}} \geq O$, it follows that $\mathcal{H}$ admits the decomposition $\mathcal{R}(A)^- \oplus \mathcal{N}(A)$ into A-invariant subspaces and $A^{\frac{1}{2}}|_{\mathcal{R}(A)^-} = I$ (because it is a nonnegative isometry) so that $A^{\frac{1}{2}} = I \oplus O$ on $\mathcal{H} = \mathcal{R}(A)^- \oplus \mathcal{N}(A)$. Then A is a projection so that (d) implies (e). Clearly, (e) implies (d). Indeed, If A is a projection, then $A^{\frac{1}{2}} = A = I$ on $\mathcal{R}(A) = \mathcal{R}(A)^-$, and hence $\|x_n\| = \|V^nx\| = \|x\|$

for all $n \geq 0$, since $\mathcal{R}(V) \subseteq \mathcal{R}(A)^-$. Now, if A is a projection, then T^* admits the decomposition $T^* = U \oplus S \oplus G$ so that

$$T = U^* \oplus S^* \oplus G^*, \tag{1}$$

where U is unitary, S is a unilateral shift and G is a strongly stable contraction [12], which means that G^* is a $\mathcal{C}_{\cdot 0}$-contraction. Thus the completely nonunitary direct summand of T is $C = S^* \oplus G^*$ (either a backward unilateral shift, a $\mathcal{C}_{\cdot 0}$-contraction, or a direct sum of a backward unilateral shift and a $\mathcal{C}_{\cdot 0}$-contraction), and therefore (e) implies (f). Conversely, if (f) holds, then T admits a decomposition as in (1) so that $T^nT^{*n} = I \oplus I \oplus G^{*n}G^n$ for every $n \geq 0$, and hence $A = I \oplus I \oplus O$ because $G^{*n}G^n \xrightarrow{s} O$ (since G is a strongly stable contraction or, equivalently, G^* is a $\mathcal{C}_{\cdot 0}$-contraction). Thus A is a projection, and so (f) implies (e). □

Remark 1. We have relied on [4] for the equivalence between (a) and (b), and on [12] for the equivalence between (e) and (f). It was recently shown that (b) implies (e) — see [17] and [11] for a pair of distinct proofs. Note that the above argument, via assertions (c) and (d), yields still another proof that (b) implies (e).

Remark 2. It is well-known that dominant and paranormal contractions have property PF (see [4], [17], and the references therein), and so hyponormal contractions have property PF (every hyponormal operator is both dominant and paranormal). Thus, according to Lemma 1, dominant and paranormal (in particular, hyponormal) contractions have completely nonunitary direct summands of class $\mathcal{C}_{\cdot 0}$.

Remark 3. Perhaps a systematic investigation in this line has been initiated after Putnam's paper [14], which contains the first proof that a completely nonunitary cohyponormal contraction is strongly stable or, equivalently, that every hyponormal contraction has a completely nonunitary direct summand of class $\mathcal{C}_{\cdot 0}$. This was extended to paranormal contractions in [13] and to dominant contractions in [15].

Theorem 1 not only yields a different proof that the completely nonunitary direct summand of a paranormal contraction is of class $\mathcal{C}_{\cdot 0}$, but it goes beyond that: the result extends to k-quasihyponormal and to k-paranormal contractions.

Corollary 1. *If a contraction $T \in \mathcal{B}[\mathcal{H}]$ is k-quasihyponormal or k-paranormal, then the completely nonunitary direct summand of T is of class $\mathcal{C}_{\cdot 0}$.*

Proof. Take any contraction $T \in \mathcal{B}[\mathcal{H}]$ and let $\{x_n\}$ be a sequence defined as in Theorem 1. It is readily verified by induction on k that $x_n = T^kx_{n+k}$ for all $k \geq 0$, for each $n \geq 0$. Thus $x_{n-k} = T^kx_n$ whenever $k \leq n$. In particular, $T^{k-1}x_{n+k} = x_{n+1}$ for $k \geq 1$ and $n \geq 0$, and $T^{k+1}x_{n+k} = x_{n-1}$ for $k \geq 0$ and $n \geq 1$. We shall split the proof into four parts.

(i) Suppose T is k-quasihyponormal, which means that $\|T^*T^kx\| \leq \|T^{k+1}x\|$ for every $x \in \mathcal{H}$, for some $k \geq 1$. Recall that $\|Tx\|^2 \leq \|T^*Tx\|\|x\|$ for every $x \in \mathcal{H}$, for all $T \in \mathcal{B}[\mathcal{H}]$, by the Schwartz inequality. Therefore,

$$\begin{aligned}\|x_n\|^2 &= \|T^kx_{n+k}\|^2 = \|TT^{k-1}x_{n+k}\|^2 \leq \|T^*T^kx_{n+k}\|\|T^{k-1}x_{n+k}\| \\ &\leq \|T^{k+1}x_{n+k}\|\|T^{k-1}x_{n+k}\| = \|x_{n-1}\|\|x_{n+1}\| \leq \tfrac{1}{4}\big(\|x_{n-1}\| + \|x_{n+1}\|\big)^2\end{aligned}$$

for each $n \geq 1$ so that $\{\|x_n\|\}$ is a convex sequence.

(ii) Now suppose T is k-paranormal (i.e., $\|Tx\|^{k+1} \leq \|T^{k+1}x\|\|x\|^k$ for every $x \in \mathcal{H}$) for some integer $k \geq 1$ so that

$$\begin{aligned}\|x_n\| &= \big(\|Tx_{n+1}\|^{k+1}\big)^{\frac{1}{k+1}} \le \big(\|T^{k+1}x_{n+1}\|\,\|x_{n+1}\|^k\big)^{\frac{1}{k+1}}\\ &= \|T^{k+1}x_{n+1}\|^{\frac{1}{k+1}}\,\|x_{n+1}\|^{\frac{k}{k+1}} \le \tfrac{1}{k+1}\big(\|T^{k+1}x_{n+1}\| + k\|x_{n+1}\|\big) \qquad (2)\end{aligned}$$

for every $n \ge 0$ (the last inequality follows from the fact that $\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ for every pair of positive real numbers α and β whenever p and q are Hölder conjugates). Recalling that $T^k x_n = x_{n-k}$ whenever $k \le n$, we get

$$\|x_n\| \le \|x_{n-k}\|^{\frac{1}{k+1}}\,\|x_{n+1}\|^{\frac{k}{k+1}} \le \tfrac{1}{k+1}\big(\|x_{n-k}\| + k\|x_{n+1}\|\big) \qquad (3)$$

for every $n \ge k$. If $k = 1$ (i.e., if T is paranormal), then

$$\|x_n\| \le \|x_{n-1}\|^{\frac{1}{2}}\,\|x_{n+1}\|^{\frac{1}{2}} \le \tfrac{1}{2}\big(\|x_{n-1}\| + \|x_{n+1}\|\big)$$

for each $n \ge 1$ so that $\{\|x_n\|\}$ is a convex sequence.

(iii) Next we show that if T is k-paranormal for any k, then $\{\|x_n\|\}$ still is convex. Take an arbitrary integer $k \ge 1$. Since $\|T^{k+1}x_{n+1}\| = \|T^{k-1}x_{n-1}\| \le \|x_{n-1}\|$ for every $n \ge 1$ (T is a contraction), we get from (2) that

$$(k+1)\|x_n\| \le \|x_{n-1}\| + k\|x_{n+1}\|$$

for every integer $n \ge 1$. The above inequality says that if there exists an integer $n_0 \ge 1$ such that $\|x_{n_0}\| = \|x_{n_0+1}\|$, then $\|x_{n_0}\| \le \|x_{n_0-1}\|$, and hence (since $\{\|x_n\|\}$ is increasing) $\|x_{n_0-1}\| = \|x_{n_0}\|$. This implies that $\{\|x_n\|\}$ is constant for every $n \le n_0 + 1$, and therefore the increasing sequence $\{\|x_n\|\}$ is either constant or eventually strictly increasing with a constant initial segment. Suppose it is eventually strictly increasing so that there exits an integer $n_1 \ge 0$ for which $\{\|x_{n+n_1}\|\}$ is strictly increasing. Recall that $x_n = A^{\frac{1}{2}}V^n x$ for each $n \ge 0$, where x lies in $\mathcal{R}(A)^-$. Replace the arbitrary generating vector x with $V^{n_1}x$. Note that this again is an admissible generating vector once it also lies in $\mathcal{R}(A)^-$ (because $\mathcal{R}(V) \subseteq \mathcal{R}(A)^-$). Then each x_n is transformed into $x_{n+n_1} = A^{\frac{1}{2}}V^{n+n_1}x$, which means that the original sequence is shifted by n_1. This shifted sequence, also denoted by $\{x_n\}$, is such that $\{\|x_n\|\}$ is strictly increasing (and, clearly, satisfies the inequality in (3), is bounded, and converges to the same limit $\|x\|$). Now it follows from (3) that

$$(k+1)\|x_n\| \le \|x_{n-k}\| + k\|x_{n+1}\|$$

for every $n \ge k$. Adding up this inequality from k to an arbitrary $m \ge k$ we get

$$k\sum_{n=k}^{m}\|x_n\| + \sum_{n=k}^{m}\|x_n\| \le \sum_{n=k}^{m}\|x_{n-k}\| + k\sum_{n=k}^{m}\|x_{n+1}\|,$$

and hence

$$\begin{aligned}k\big(\|x_k\| - \|x_{m+1}\|\big) &= k\sum_{n=k}^{m}\|x_n\| - k\sum_{n=k}^{m}\|x_{n+1}\|\\ &\le \sum_{n=k}^{m}\|x_{n-k}\| - \sum_{n=k}^{m}\|x_n\| = \sum_{n=0}^{k-1}\|x_n\| - \sum_{n=m-k+1}^{m}\|x_n\|.\end{aligned}$$

Since $\|x_m\| \to \|x\|$ as $m \to \infty$, it follows that $k(\|x_k\| - \|x\|) \le \sum_{n=0}^{k-1}\|x_n\| - k\|x\|$ and so $k\|x_k\| \le \sum_{n=0}^{k-1}\|x_n\|$, which (since $\{\|x_n\|\}$ is increasing) implies that

$$\sum_{n=0}^{k-1}\|x_n\| = k\|x_k\|.$$

Thus, recalling again that $\{\|x_n\|\}$ is increasing,

$$\|x_0\| = \|x_n\| \quad \text{for all} \quad 0 \le n \le k;$$

in particular, $\|x_0\| = \|x_1\|$. But this contradicts the assumption that the shifted sequence is strictly increasing. Therefore, the original sequence must be constant, thus convex.

(iv) We have shown that, in all cases, $\{\|x_n\|\}$ is a convex sequence. Since a backward unilateral shift cannot be a direct summand of a k-quasihyponormal or k-paranormal contraction (because a backward unilateral shift does not belong to these classes), the desired result follows from Theorem 1. □

Remark 4. If T is a contraction and $k \ge 1$, then for each $0 \le j \le k+1$ we get $\|T^{k+1}x_{n+1}\| \le \|T^j x_{n+1}\| = \|x_{n+1-j}\|$ whenever $n+1-j \ge 0$. Thus, if T is a k-hyponormal contraction for some $k \ge 1$, then the inequality (2) says that

$$\|x_n\| \le \tfrac{1}{k+1}\big(\|x_{n+1-j}\| + k\|x_{n+1}\|\big)$$

for every $n \ge j-1$, for each $j = 2, \dots, k+1$,. In fact, this inequality holds for each $j = 0, \dots, k+1$, but it says nothing for $j = 0, 1$ except that $\{\|x_n\|\}$ is increasing. For $j = k+1$ we have the inequality (3), which played a crucial role in the proof of Corollary 1. It is worth noticing that for proving the assertion "$\{\|x_n\|\}$ is a constant sequence" (equivalently, "$\{\|x_n\|\}$ is a convex sequence"), it is not enough to assume that the above inequality holds for each $2 \le j \le k$; it is actually necessary that it also holds for $j = k+1$. For instance, put $k = 2$ and observe that the positive, strictly increasing and bounded sequence $\{\alpha_n\}$ given by $\alpha_n = \frac{1}{3}(7 - 2^{2-n})$ for every $n \ge 0$ satisfies the above inequality for $j = k = 2$, yielding the identity

$$\alpha_n = \tfrac{1}{3}\big(\alpha_{n-1} + 2\alpha_{n+1}\big)$$

for every $n \ge 1$, but $\{\alpha_n\}$ does not satisfy the above inequality for $j = k+1 = 3$ (otherwise it would be constant). Indeed, $\alpha_2 > \frac{1}{3}(\alpha_0 + 2\alpha_3)$.

It is well-known that dominant contractions have $\mathcal{C}_{\cdot 0}$ completely nonunitary direct summands (Remarks 2 and 3). Thus Corollary 1 ensures that every contraction in any of those classes appearing in the diagram of Section 1 has a $\mathcal{C}_{\cdot 0}$ completely nonunitary direct summand — all of them are included in the union of the dominant, k-quasihyponormal and k-paranormal (which includes the paranormal and, in particular, the quasi-p-hyponormal and semi-quasihyponormal) contractions.

4. A Final Remark

Recall that a part of an operator is a restriction of it to an invariant subspace, and also that an operator is *normaloid* if its spectral radius coincides with its norm. An operator in $\mathcal{B}[\mathcal{H}]$ is *hereditarily normaloid* if every part of it is normaloid, and *totally hereditarily normaloid* if, in addition, every invertible part of it has a normaloid inverse. These classes were introduced in [5] and are denoted by $\mathcal{HN}$ and $\mathcal{THN}$, respectively. See [6] for properties of operators belonging to them; in particular,

$$\text{paranormal} \;\rightarrow\; \mathcal{THN} \;\rightarrow\; \mathcal{HN} \;\rightarrow\; \text{normaloid}.$$

In fact, it is known that k-paranormal operators are normaloid and that a part of a k-paranormal operator is again k-paranormal. Thus

$$k\text{-paranormal} \;\to\; \mathcal{HN} \;\to\; \text{normaloid}.$$

This shows that these classes are quite large (see the diagram of Section 1). Recall that a $\mathcal{C}_{00}$-contraction T is of class $\mathcal{C}_0$ if there exists an inner function u such that $u(T) = 0$, and also that the defect operator of a contraction T is the nonnegative contraction $(I - T^*T)^{\frac{1}{2}}$. The following result is to be found in [6, Theorem 1].

Proposition 1. [6] *If T is a $\mathcal{THN}$ contraction with a Hilbert–Schmidt defect operator, and if normal subspaces of T reduce T, then*

$$T = U \oplus D \oplus S,$$

where U is unitary, D is a diagonal $\mathcal{C}_0$-contraction, and S is a $\mathcal{C}_{10}$-contraction with no invertible parts.

As we have commented above, contractions in any of those classes appearing in the diagram of Section 1 have $\mathcal{C}_{\cdot 0}$ completely nonunitary direct summands (see Remarks 2 and 3 and Corollary 1). Observe that $\mathcal{THN}$ contractions satisfying the hypothesis of Proposition 1 also have a $\mathcal{C}_{\cdot 0}$ completely nonunitary direct summand.

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