WEYL'S THEOREMS FOR POSINORMAL OPERATORS

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ABSTRACT. An operator T belonging to the algebra $B(H)$ of bounded linear transformations on a Hilbert H into itself is said to be *posinormal* if there exists a positive operator $P \in B(H)$ such that $TT^* = T^*PT$. A posinormal operator T is said to be *conditionally totally posinormal* (resp., *totally posinormal*), shortened to $T \in CTP$ (resp., $T \in TP$), if to each complex number λ there corresponds a positive operator P_λ such that $|(T-\lambda I)^*|^2 = |P_\lambda^{1/2}(T-\lambda I)|^2$ (resp., if there exists a positive operator P such that $|(T-\lambda I)^*|^2 = |P^{1/2}(T-\lambda I)|^2$ for all λ). This paper proves Weyl's theorem type results for TP and CTP operators. If $A \in TP$, if $B^* \in CTP$ is *isoloid* and if $d_{AB} \in B(B(H))$ denotes either of the *elementary operators* $\delta_{AB}(X) = AX - XB$ and $\triangle_{AB}(X) = AXB - X$, then it is proved that d_{AB} satisfies Weyl's theorem and d_{AB}^* satisfies a-Weyl's theorem.

1. INTRODUCTION

Denoting the algebra of operators (equivalently, bounded linear transformations) on an infinite dimensional complex Hilbert space H into itself by $B(H)$, an operator $T \in B(H)$ is said to be *posinormal* (short for *positive-normal*) if there exists a $P \geq 0$ in $B(H)$ such that $TT^* = T^*PT$. Equivalently, $T \in B(H)$ is posinormal if there exists a co-isometry $V^* \in B(H)$ and a positive operator $P \in B(H)$ such that $T = T^*PV^*$. The class of posinormal operators is large: it contains in particular the classes consisting of *hyponormal* $(T \in B(H) : TT^* \leq T^*T)$, *M-hyponormal* $(T \in B(H) : |(T - \lambda I)^*|^2 \leq M|(T - \lambda I)|^2$ for some real number $M > 0$ and all complex numbers $\lambda)$ and *dominant* operators $(T \in B(H) : |(T - \lambda I)^*|^2 \leq M_\lambda|(T - \lambda I)|^2$ for some real number $M_\lambda > 0$ and all complex numbers $\lambda)$. The class of posinormal operators was introduced by Rhaly [22], and has since by considered by Jeon *et al* [14]. It is elementary to see that the restriction of a posinormal operator to an invariant subspace is again posinormal.

It is immediate from the definition of posinormality that a posinormal operator T satisfies $T^{-1}(0) \subseteq T^{*-1}(0)$, which implies that a posinormal operator has ascent ≤ 1. A posinormal operator T is said to be *conditionally totally posinormal* (resp., *totally posinormal*), shortened to $T \in CTP$ (resp., $T \in TP$), if to each complex number λ there corresponds a positive operator P_λ such that $|(T - \lambda I)^*|^2 = |P_\lambda^{1/2}(T - \lambda)|^2$ (resp., if there exists a positive operator P such that $|(T - \lambda I)^*|^2 = |P^{1/2}(T - \lambda)|^2$ for all λ). CTP operators have been considered by Jeon *et.al.* [14] (where they have been called totally posinormal). Obviously, if $T \in CTP$, then $(T - \lambda I)$ has ascent ≤ 1. Furthermore, $T \in CTP$ if and only if T is dominant [22, Proposition 3.5]. Restricting themselves to only those $T \in CTP$ for which the spectrum $\sigma((T - \lambda I)|_{\mathcal{M}}) = \{0\} \implies (T - \lambda I)|_{\mathcal{M}} = 0$ for every

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$\mathcal{M} \in Lat(T)$, Jeon *et al* [14, Theorem 13] have shown that T satisfies Weyl's theorem. In this note we prove that posinormal operators satisfy Weyl's theorems under conditions which are visibly weaker than those considered in [14]. The plan of this note is as follows: we explain our notation and terminology in Section 2, with Section 3 devoted to proving our main results. In addition to proving Weyl's theorem type results for TP and CTP operators, we prove that if $A \in TP$, if $B^* \in CTP$ is *isoloid* and if $d_{AB} \in B(B(H))$ denotes either of the *elementary operators* $\delta_{AB}(X) = AX - XB$ and $\triangle_{AB}(X) = AXB - X$, then d_{AB} satisfies Weyl's theorem and d_{AB}^* satisfies a-Weyl's theorem.

2. Notation and Terminology

A Banach space operator T, $T \in B(\mathcal{X})$, is said to be Fredholm, $T \in \Phi(\mathcal{X})$, if $T(\mathcal{X})$ is closed and both the *deficiency indices* $\alpha(T) = dim(T^{-1}(0))$ and $\beta(T) = dim(\mathcal{X}/T(\mathcal{X}))$ are finite, and then the *index* of T, $ind(T)$, is defined to be $ind(T) = \alpha(T) - \beta(T)$. The ascent of T, $asc(T)$, is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the descent of T, $dsc(T)$, is the least non-negative integer n such that $T^n((\mathcal{X})) = T^{n+1}((\mathcal{X}))$. We shall, henceforth, shorten $(T - \lambda I)$ to $(T - \lambda)$. The operator T is *Weyl* if it is Fredholm of index zero, and T is said to be *Browder* if it is Fredholm "of finite ascent and descent". Let $\mathbf{C}$ denote the set of complex numbers. The (Fredholm) essential spectrum $\sigma_e(T)$, the Browder spectrum $\sigma_b(T)$ and the Weyl spectrum $\sigma_w(T)$ of T of are the sets

$$\sigma_e(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not Fredholm}\};$$
$$\sigma_b(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not Browder}\}$$

and

$$\sigma_w(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not Weyl}\}.$$

If we let $\rho(T)$ denote the resolvent set of the operator T, $\sigma(T)$ denote the usual spectrum of T and $acc\, \sigma(T)$ denote the set of accumulation points of $\sigma(T)$, then:

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup acc\, \sigma(T).$$

Let $\pi_0(T)$ denote the set of *Riesz points* of T (i.e., the set of $\lambda \in \mathbf{C}$ such that $T - \lambda$ is Fredholm of finite ascent and descent), and let $\pi_{00}(T)$ denote the set of eigenvalues of T of finite geometric multiplicity. Also, let $\pi_{a0}(T)$ be the set of $\lambda \in \mathbf{C}$ such that λ is an isolated point of $\sigma_a(T)$ and $0 < \dim ker(T - \lambda) < \infty$, where $\sigma_a(T)$ denotes the approximate point spectrum of the operator $T \in B(X)$. Clearly, $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \pi_{a0}(T)$. We say that *Browder's theorem holds for $T \in B(X)$* if

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T),$$

Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and *a-Weyl's theorem holds for T* if

$$\sigma_{wa}(T) = \sigma_a(T) \setminus \pi_{a0}(T),$$

where $\sigma_{wa}(T)$ denote *the essential approximate point spectrum* (i.e., $\sigma_{wa}(T) = \bigcap\{\sigma_a(T + K) : K \in K(\mathcal{X})\}$ with $K(\mathcal{X})$ denoting the ideal of compact operators on $\mathcal{X}$). If we let $\Phi_+(\mathcal{X}) = \{T \in B(\mathcal{X}) : \alpha(T) < \infty \text{ and } T(\mathcal{X}) \text{ is closed}\}$ denote the semi-group of *upper semi-Fredholm operators* in $B(\mathcal{X})$, then $\sigma_{wa}(T)$ is the complement in $\mathbf{C}$ of all those λ for which $(T - \lambda) \in \Phi_+(\mathcal{X})$ and $ind(T - \lambda) \leq 0$. The

concept of a-Weyl's theorem was introduced by Rakočvić: a-Weyl's theorem for $T \implies$ Weyl's theorem for T, but the converse is generally false [21].

An operator $T \in B(\mathcal{X})$ has *the single-valued extension property* at $\lambda_0 \in \mathbf{C}$, SVEP at $\lambda_0 \in \mathbf{C}$ for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(T - \lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\mathbf{C} \setminus \sigma(T)$; also T has SVEP at $\lambda \in iso\sigma(T)$. We say that T *has SVEP* if it has SVEP at every $\lambda \in \mathbf{C}$. It is known that a Banach space operator T with SVEP satisfies Browder's theorem [1, Corollary 2.12] and that Browder's theorem holds for $T \iff$ Browder's theorem holds for T^* [12].

The *analytic core* $K(T - \lambda)$ of $(T - \lambda)$ is defined by

$$K(T - \lambda) = \{x \in X : \text{ there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which}$$
$$x = x_0, \ (T - \lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, ... \}.$$

We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T - \lambda)$ such that $(T - \lambda)^{-p}(0) \subseteq H_0(T - \lambda)$ for all $p = 0, 1, 2, ...$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ [18].

The operator $T \in B(\mathcal{X})$ is said to be *semi-regular* if $T(\mathcal{X})$ is closed and $T^{-1}(0) \subset T^\infty(\mathcal{X}) = \bigcap_{n \in \mathbf{N}} T^n(\mathcal{X})$; T admits a *generalized Kato decomposition*, GKD for short, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. An operator $T \in B(\mathcal{X})$ has a GKD at every $\lambda \in iso\sigma(T)$, namely $\mathcal{X} = H_0(T - \lambda) \oplus K(T - \lambda)$. We say that T is of *Kato type* at a point λ if $(T - \lambda)|_M$ is nilpotent in the GKD for $(T - \lambda)$. Fredholm operators are Kato type [15, Theorem 4], and operators $T \in B(\mathcal{X})$ satisfying property $\mathbf{H}(p)$,

$$\mathbf{H}(p) \qquad\qquad H_0(T - \lambda) = (T - \lambda)^{-p}(0)$$

for some integer $p \geq 1$, are Kato type at isolated points of $\sigma(T)$ (but not every Kato type operator T satisfies property $\mathbf{H}(p)$). Let $\sigma_{kt}(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not Kato type}\}$. The set $\sigma_{kt}(T)$ is known to be a closed subset of $\mathbf{C}$ such that $\sigma_{kt}(T) \subseteq \sigma_a(T)$ [3].

3. Main Results

We say in the following that an operator $T \in B(H)$ is *conditionally totally posinormal*, CTP for short, if to each $\lambda \in \mathbf{C}$ there corresponds an operator $P_\lambda \geq 0$ such that $|(T - \lambda)^*|^2 \leq |P_\lambda^{1/2}(T - \lambda)|^2$; T will be said to be *totally posinormal*, TP for short, if T is CTP and the positive operator P_λ can be chosen independent of λ. It is easy to see that T is CTP if and only if it is *dominant*, and that T is TP if and only if it is M-*hyponormal*. Clearly, TP operator satisfy Bishop's condition (β) [11] and hence are *subscalar*. If we let $\hat{T}$ denote the *generalized scalar* extension of T to some Hilbert space $K \supset H$, then there exists an integer $p \geq 1$ such that $H_0(\hat{T} - \lambda) = (\hat{T} - \lambda)^{-p}(0)$ for each $\lambda \in \mathbf{C}$ [5, Theorem 4.5, Chapter4]. The Hilbert space H being invariant for $\hat{T}$, $H_0(\hat{T}|_H - \lambda) = H_0(\hat{T} - \lambda) \cap H = (\hat{T} - \lambda)^{-p}(0) \cap H = (\hat{T}|_H - \lambda)^{-p}(0)$, i.e., T satisfies the property $\mathbf{H}(p)$ for all $\lambda \in \mathbf{C}$. Finally, since $asc(T - \lambda) \leq 1$, T satisfies property $\mathbf{H}(1)$.

Let $\mathcal{H}(\sigma(T))$ (resp., $\mathcal{H}_1(\sigma(T))$) denote the set of analytic functions which are defined on an open neighbourhood $\mathcal{U}$ of $\sigma(T)$ (resp., the set of $f \in \mathcal{H}(\sigma(T))$ which are non-constant on each of the connected components of the open neighbourhood $\mathcal{U}$ of $\sigma(T)$ on which f is defined). Recall that an operator T is said to be *isoloid* if each $\lambda \in iso\sigma(T)$ is an eigen-value of T.

Theorem 3.1. *Let $T \in TP$. Then:*

(i) *$f(T)$ and $f(T^*)$ satisfy Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$.*
(ii) *T^* satisfies a-Weyl's theorem.*
(iii) *If T^* has SVEP, then T satisfies a-Weyl's theorem.*
(iv) *If T is the quasi-affine transform of an operator $S \in B(H)$, then $f(S)$ and $f(S^*)$ satisfy Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. The proof of (i) follows from [19, Corollary 3.6] (see also [2]). We note that if T is the quasi-affine transform of S and T satisfies property $\mathbf{H}(p)$, then S satisfies property $\mathbf{H}(p)$ [19, Lemma 3.2], and this implies (iv) [19, Corollary 3.7]. To prove (ii), we notice that T satisfies property $\mathbf{H}(1) \Longrightarrow T$ has finite ascent (and hence SVEP), which implies that $\sigma(T^*) = \sigma_a(T^*)$ [17, pp. 35] and $\pi_{a0}(T^*) = \pi_{00}(T^*)$. We prove that $\sigma_{wa}(T^*) = \sigma_w(T^*)$: since T^* satisfies Weyl's theorem by (i), this would then imply that $\sigma_a(T^*) \backslash \sigma_{wa}(T^*) = \pi_{a0}(T^*)$. It being clear that $\sigma_{wa}(T^*) \subseteq \sigma_w(T^*)$, we prove the reverse inclusion. Since

$$\lambda \notin \sigma_{wa}(T^*) \iff (T-\lambda)^* \in \Phi_+(H) \ \text{ and } \ ind(T-\lambda)^* \leq 0,$$

the hypothesis T has SVEP implies that

$$dsc(T-\lambda)^* < \infty \ ([1, \text{Theorem 2.9}]), \ \alpha(T-\lambda)^* < \infty \ \text{ and } \ ind(T-\lambda)^* \leq 0.$$

Since $dsc(T-\lambda)^* < \infty \Longrightarrow ind(T-\lambda)^* \geq 0$ [13, Proposition 38.5], it follows that

$$dsc(T-\lambda)^* < \infty, \alpha(T-\lambda)^* = \beta(T-\lambda)^* < \infty \Longrightarrow \lambda \notin \sigma_w(T^*).$$

This leaves us with the proof of (iii).

The hypothesis T^* has SVEP implies $\sigma(T) = \sigma_a(T)$ [17, pp. 35], and hence $\pi_{a0}(T) = \pi_{00}(T)$. We prove that $\sigma_{wa}(T) = \sigma_w(T)$: since T satisfies Weyl's theorem (by part (i)), this would then imply that $\sigma_a(T) \setminus \sigma_{wa}(T) = \pi_{a0}(T)$. It being clear that $\lambda \notin \sigma_w(T) \Longrightarrow \lambda \notin \sigma_{wa}(T)$, we prove that $\lambda \notin \sigma_{wa}(T) \Longrightarrow \lambda \notin \sigma_w(T)$. Since $\lambda \notin \sigma_{wa}(T) \iff (T-\lambda) \in \Phi_+(X)$ and $ind(T-\lambda) \leq 0$, the hypothesis T^* has SVEP implies that $dsc(T-\lambda) < \infty$ [1, Theorem 2.9], $\alpha(T-\lambda) < \infty$ and $ind(T-\lambda) \leq 0$. Again, since $dsc(T-\lambda) < \infty$ implies $ind(T-\lambda) \geq 0$ [13, Proposition 38.5], we get:

$$\lambda \notin \sigma_{wa}(T) \iff dsc(T-\lambda) < \infty \ \text{ and } \ \alpha(T-\lambda) = \beta(T-\lambda) < \infty,$$

which implies that $\lambda \notin \sigma_w(T)$. $\qquad\square$

The example of a quasi-nilpotent CTP operator shows that CPT operators do not satisfy property $\mathbf{H}(p)$. (Such operators exist: see [14, Example 8].) Since $(T-\lambda)^{-1}(0) \subseteq (T-\lambda)^{*-1}(0)$ for CTP operators T, CTP operators have *ascent* ≤ 1. In particular, CTP operators have SVEP.

Lemma 3.2. *A necessary and sufficient condition for the isolated points of the spectrum of a Banach space operator T, $T \in B(\mathcal{X})$, to be poles of the resolvent of T is that $iso\sigma(T) \cap \sigma_{kt}(T) = \emptyset$.*

Proof. If $iso\sigma(T) \cap \sigma_{kt}(T) = \emptyset$, then $\lambda \in iso\sigma(T) \implies T - \lambda$ is Kato type. Since both $(T - \lambda)$ and $(T - \lambda)^*$ have SVEP at 0, it follows (from [1, Theorems 2.6 and 2.9] and [17, Proposition 4.10.6]) that $asc(T - \lambda) = dsc(T - \lambda) < \infty \implies \lambda$ is a pole of the resolvent of T [13, Proposition 50.2]. Conversely, if each $\lambda \in iso\sigma(T)$ is a pole (of some finite order p) of the resolvent of T, then $\mathcal{X} = (T - \lambda)^{-p}(0) \oplus (T - \lambda)^p(\mathcal{X}) \implies \lambda \notin \sigma_{kt}(\mathcal{X}) \implies iso\sigma(T) \cap \sigma_{kt}(T) = \emptyset$. $\square$

Theorem 3.3. *If $T \in CTP$ is such that $iso\sigma(T) \cap \sigma_{kt}(T) = \emptyset$, then:*

(i) *$f(T)$ satisfies Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$ and T^* satisfies a-Weyl's theorem.*

If also T^* has SVEP, then:

(ii) *$f(T)$ satisfies a-Weyl's theorem for every $f \in \mathcal{H}_1(\sigma(T))$.*

Proof. (i) We start by proving that T satisfies Weyl's theorem. Since $T \in CTP \implies (T - \lambda)^{-1}(0) \subseteq (T - \lambda)^{*-1}(0)$, $asc(T - \lambda) \le 1$ for all $\lambda \in \mathbf{C} \implies T$ has SVEP $\implies T$ satisfies Browder's theorem $\implies \sigma(T) \setminus \sigma_w(T) = \pi_0(T) \subseteq \pi_{00}(T)$. For the reverse inclusion, let $\lambda \in \pi_{00}(T)$. Then, since $iso\sigma(T) \cap \sigma_{kt}(T) = \emptyset$, $\lambda \in \pi_0(T)$ (by Lemma 3.2) $= \sigma(T) \setminus \sigma_w(T)$. Thus $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, and T satisfies Weyl's theorem. It is clear from Lemma 3.2 that T is isoloid $\implies \sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(f(T)))$ [16, Lemma]. Since T has SVEP, $\sigma_w(f(T)) = f(\sigma_w(T))$ for every $f \in \mathcal{H}(\sigma(T))$ [6, Corollary 2.6]. We already know that T satisfies Weyl's theorem. Hence

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma_w(T)) = \sigma_w(f(T)),$$

i.e., $f(T)$ satisfies Weyl's theorem.

The proof that T^* satisfies a-Weyl's theorem is similar to that of Theorem 3.1(ii), and is left to the reader.

(ii) Let $f \in \mathcal{H}_1(\sigma(T))$. Then T^* has SVEP $\implies f(T^*) = f(T)^*$ has SVEP [17, Theorem 3.3.9] $\implies \sigma(f(T)) = \sigma_a(f(T))$ [17, pp. 35]. Arguing as before it is seen that $\sigma_w(f(T)) = \sigma_{wa}(f(T))$. Since $f(T)$ satisfies Weyl's theorem by part (i),

$$\sigma_a(f(T)) \setminus \sigma_{wa}(f(T)) = \sigma(f(T)) \setminus \sigma_w(f(T)) = \pi_{00}(f(T)) = \pi_{0a}(f(T)),$$

i.e. $f(T)$ satisfies a-Weyl's theorem. $\square$

We remark here that Theorem 3.3 has a more general Banach space version; see [9]. A CTP operator T such that $\sigma(T - \lambda) = \{0\} \implies T = \lambda I$ satisfies $iso\sigma(T) \cap \sigma_{kt}(T) = \emptyset$: this is seen as follows. If $\lambda \in iso\sigma(T)$, then $H = H_0(T - \lambda) \oplus K(T - \lambda)$. The operator $(T - \lambda)|_{H_0(T-\lambda)}$ being CTP with $\sigma((T - \lambda)|_{H_0(T-\lambda)}) = \{0\}$, $(T - \lambda)|_{H_0(T-\lambda)} = 0 \implies T$ is Kato type. Obviously, Theorem 3.3 contains [14, Theorems 13 and 16].

An elementary operator. Let $A \in TP$ and let $B^* \in CTP$. Define the *elementary operator* $\triangle_{AB} \in B(B(H))$ and the *generalized derivation* $\delta_{AB} \in B(B(H))$ by $\triangle_{AB}(X) = AXB - X$ and $\delta_{AB}(X) = AX - XB$. Let $d_{AB} \in B(B(H))$ denote either of the operators $\triangle_{AB}$ and δ_{AB}. We prove in the following that if B^* has the isoloid property (i.e., if the isolated points of $\sigma(B^*)$ are eigenvalues of B^*), then $f(d_{AB})$ satisfies Weyl's theorem for each $f \in \mathcal{H}(\sigma(d_{AB}))$, thereby generalizing [8, Theorem 3.1]. We start with the following lemma on the ascent of d_{AB}.

Lemma 3.4. $asc(d_{AB} - \lambda) \le 1$ for all $\lambda \in \mathbf{C}$.

Proof. Recall that $A \in TP$ if and only if A is M-hyponormal and $B^* \in TCP$ if and only if B^* is dominant. Recall also that if $A \in B(H)$ is M-hyponormal and $B^* \in B(H)$ is dominant, then $\delta_{AB}(X) = 0 \Longrightarrow \delta_{A^*B^*}(X) = 0$ [20]. Thus $(d_{AB})^{-1}(0) \subseteq (d_{A^*B^*})^{-1}(0)$ [7, Theorem 2]. Evidently, if $B^* \in CTP$, then the operators αB^* and $\frac{1}{\beta}B^*$ are CTP for all α and non-zero β in $\mathbf{C}$. Hence $(d_{AB} - \lambda)^{-1}(0) \subseteq (d_{A^*B^*} - \lambda)^{-1}(0)$ for all $\lambda \in \mathbf{C}$. This, by [8, Proposition 2.3], implies that $asc(d_{AB} - \lambda) \leq 1$. $\qquad\square$

Lemma 3.5. *If B^* is isoloid, then $iso\sigma(d_{AB}) \cap \sigma_{kt}(d_{AB}) = \emptyset$.*

Proof. We prove that $H_0(d_{AB} - \lambda) = (d_{AB} - \lambda)^{-1}(0)$ for all $\lambda \in iso\sigma(d_{AB})$. Since $\lambda \in iso\sigma(d_{AB}) \Longrightarrow B(H) = H_0(d_{AB} - \lambda) \oplus K(d_{AB} - \lambda)$, this would then imply that $B(H) = (d_{AB} - \lambda)^{-1}(0) \oplus (d_{AB} - \lambda)(B(H))$, i.e. $d_{AB} - \lambda$ is Kato type and hence $\lambda \notin \sigma_{kt}(d_{AB})$. Recall that an M-hyponormal (equivalently, TP) operator is isoloid (indeed, the isolated points of the spectrum of such an operator are simple poles of the resolvent of the operator), and that the eigenvalues of a dominant (equivalently, CTP) operator are normal eigenvalues of the operator.

The case $d_{AB} = \triangle_{AB}$. Let $\lambda \in iso\sigma(\triangle_{AB})$. We divide the proof into the cases $\lambda = -1$ and $\lambda \neq -1$. Let $\Phi_{AB} = L_A R_B$, where L_A and $R_B \in B(B(H))$ are operators of "left multiplication by A" and "right multiplication by B" (respectively). If $\lambda = -1$, then $0 \in iso\sigma(\Phi_{AB})$. Since $\sigma(\Phi_{AB}) = \cup\{\sigma(zA) : z \in \sigma(B)\}$ [10, Theorem 3.2], we must have either $0 \in iso\sigma(B)$ or $0 \in iso\sigma(A)$. Suppose that $0 \in iso\sigma(B)$. (The other case is similarly dealt with.) Then 0 can not be a limit point of $\sigma(A)$. For if 0 is a limit point of $\sigma(A)$, then there exists a sequence $\{\alpha_n\} \in \sigma(A)$ such that $\alpha_n \to 0 \in \sigma(A)$. Choosing a non-zero $z \in \sigma(B)$ we then have a sequence $\{z\alpha_n\} \in \sigma(\Phi_{AB})$ such that $z\alpha_n \to 0$ which contradicts the fact that $0 \in iso\sigma(\Phi_{AB})$. (We remark here that such a choice of z is always possible, for if not then $\sigma(B) = \{0\}$ and B is the zero operator.) The conclusion that 0 can not be a limit point of $\sigma(A)$ implies that either $0 \notin \sigma(A)$ or $0 \in iso\sigma(A)$. If $0 \notin \sigma(A)$, then A is invertible and $H_0(\Phi_{AB}) = H_0(\Phi_{IB})$. Notice that $0 \in iso\sigma(B) \Longrightarrow 0 \in iso\sigma(B^*)$. Since $B^* \in CTP$ is isoloid, $ker(B^*)$ reduces B and $B = 0 \oplus B_2$ with respect to the decomposition $H = ker(B^*) \oplus ker^\perp(B^*) = H_1 \oplus H_2$, say, of H. Clearly, the operator $B_2 = B|_{H_2}$ is invertible. Let $X : H_1 \oplus H_2 \to H_1 \oplus H_2$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then

$$\lim_{n\to\infty} ||\Phi_{IB}^n(X)||^{\frac{1}{n}} = \lim_{n\to\infty} \left|\left|\left[\begin{array}{cc} 0 & X_{12}B_2^n \\ 0 & X_{22}B_2^n \end{array}\right]\right|\right|^{\frac{1}{n}} = 0$$

if and only if both $||X_{12}B_2^n||^{\frac{1}{n}}$ and $||X_{22}B_2^n||^{\frac{1}{n}} \to 0$ as $n \to \infty$. The operator B_2 being invertible, Φ_{IB_2} is invertible $\Longrightarrow X_{12} = X_{22} = 0$ and $H_0(\Phi_{AB}) = (\Phi_{AB})^{-1}(0)$. Now let $0 \in iso\sigma(A)$. Then $A = 0 \oplus A_2$ with respect to the decomposition $H = ker(A) \oplus ker^\perp(A) = H_1' \oplus H_2'$, say, of H, where the operator $A_2 = A|_{H_2'}$ is invertible. Let $X : H_1 \oplus H_2 \to H_1' \oplus H_2'$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then

$$\lim_{n\to\infty} ||\Phi_{AB}^n(X)||^{\frac{1}{n}} = \lim_{n\to\infty} \left|\left|\left[\begin{array}{cc} 0 & 0 \\ 0 & A_2^n X_{22}B_2^n \end{array}\right]\right|\right|^{\frac{1}{n}} = 0$$
$$\Longleftrightarrow \lim_{n\to\infty} ||A_2^n X_{22}B_2^n||^{\frac{1}{n}} = 0 \iff X_{22} = 0,$$

which implies $H_0(\Phi_{AB}) = (\Phi_{AB}^{-1}(0)$. This leaves us with the case $\lambda \neq -1$, which we consider next.

If $\lambda \neq -1$, then $(\triangle_{AB} - \lambda)(X) = AXB - (1 + \lambda)X$, and it follows from [10, Theorem 3.2] that

$$\sigma(\triangle_{AB} - \lambda) = \bigcup\{\sigma(-(1 + \lambda) + zA) : z \in \sigma(B)\}.$$

If $\lambda \in iso\sigma(\triangle_{AB})$, then $0 \in iso\sigma(\triangle_{AB} - \lambda)$. There exists a finite set $\{\beta_1, \beta_2, ..., \beta_n\}$ of distinct non-zero values of $z \in iso\sigma(B)$, and corresponding to these values of z a finite set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of distinct non-zero values $\alpha_i \in iso\sigma(A)$ such that $\alpha_i\beta_i = 1 + \lambda$ for all $1 \leq i \leq n$. Let

$$H_1 = \bigvee_{i=1}^{n} ker(B - \beta_i)^*, \ H_1' = \bigvee_{i=1}^{n} ker(A - \alpha_i), \ H_2 = H \ominus H_1 \text{ and } H_2' = H \ominus H_1'.$$

Then A and B have the direct sum decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where $A_1 = A|_{H_1'}$ and $B_1 = B|_{H_1}$ are normal operators with finite spectrum, B_1 is invertible, $A_2 = A|_{H_2'}$ $B_2 = B|_{H_2}$, and $\sigma(A_1) \cap \sigma(A_2) = \emptyset = \sigma(B_1) \cap \sigma(B_2)$. Let $X : H_1 \oplus H_2 \to H_1' \oplus H_2'$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then

$$\lim_{n \to \infty} ||(\triangle_{AB} - \lambda)^n(X)||^{\frac{1}{n}} = \lim_{n \to \infty} \left|\left|\begin{bmatrix} (\triangle_{A_1 B_1} - \lambda)^n(X_{11}) & (\triangle_{A_1 B_2} - \lambda)^n(X_{12}) \\ (\triangle_{A_2 B_1} - \lambda)^n(X_{21}) & (\triangle_{A_2 B_2} - \lambda)^n(X_{22}) \end{bmatrix}\right|\right|^{\frac{1}{n}} = 0$$

$$\iff \lim_{n \to \infty} ||(\triangle_{A_i B_j} - \lambda)^n(X_{ij})||^{\frac{1}{n}} = 0$$

for all $1 \leq i, j \leq 2$. Clearly, $0 \notin \sigma(\triangle_{A_i B_j} - \lambda)$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$; hence

$$\lim_{n \to \infty} ||(\triangle_{A_i B_j} - \lambda)^n(X_{ij})||^{\frac{1}{n}} = 0 \implies X_{ij} = 0$$

for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. The operators A_1 and B_1 are normal. Since B_1 is invertible implies $(\triangle_{A_1 B_1} - \lambda)(X_{11}) = (A_1 X_{11} - X_{11}((1+\lambda)B_1^{-1}))B_1 = \delta_{A_1((1+\lambda)B_1^{-1})}(X_{11}B_1)$, and since

$$\lim_{n \to \infty} ||\delta_{CD}^n(Y)||^{\frac{1}{n}} = 0 \iff \delta_{CD}(Y) = 0$$

for normal C and D [20, Lemma 2] (see also [5]),

$$H_0(\triangle_{A_1 B_1} - \lambda) = (\triangle_{A_1 B_1} - \lambda)^{-1}(0) \implies H_0(\triangle_{AB} - \lambda) = (\triangle_{AB} - \lambda)^{-1}(0).$$

The case $d_{AB} = \delta_{AB}$. Let $\lambda \in iso\sigma(\delta_{AB})$. Then $0 \in iso\sigma(\delta_{AB} - \lambda)$, where $\sigma(\delta_{AB} - \lambda) = \sigma(A) - \sigma(B + \lambda)$ [10]. Hence $\sigma(A) \cap \sigma(B + \lambda)$ consists of points which are isolated in both $\sigma(A)$ and $\sigma(B + \lambda)$. In particular, $\sigma(A) \cap \sigma(B + \lambda)$ does not contain any limit points of $\sigma(A) \cup \sigma(B + \lambda)$. There exist finite sets $S_1 = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $S_2 = (\beta_1, \beta_2, ..., \beta_n)$ of distinct values α_i and β_i such that each α_i is an isolated point of $\sigma(A)$, each β_i is an isolated point of $\sigma(B)$, and $\alpha_i - \beta_i = \lambda$ for all $1 \leq i \leq n$. Let

$$H_1 = \bigvee_{i=1}^{n} ker(B - \alpha_i)^*, \ H_1' = \bigvee_{i=1}^{n} ker(A - \alpha_i), \ H_2 = H \ominus H_1 \text{ and } H_2' = H \ominus H_1'.$$

Define the normal operators A_1 and B_1, and the operators A_2 and B_2, as before. Letting $X : H_1 \oplus H_2 \to H_1' \oplus H_2'$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$, it is then seen that

$$\lim_{n \to \infty} ||(\delta_{AB} - \lambda)^n(X)||^{\frac{1}{n}} = \lim_{n \to \infty} \left|\left|\begin{bmatrix} (\delta_{A_1 B_1} - \lambda)^n(X_{11}) & (\delta_{A_1 B_2} - \lambda)^n(X_{12}) \\ (\delta_{A_2 B_1} - \lambda)^n(X_{21}) & (\delta_{A_2 B_2} - \lambda)^n(X_{22}) \end{bmatrix}\right|\right|^{\frac{1}{n}} = 0$$

$$\iff \lim_{n \to \infty} ||(\delta_{A_i B_j} - \lambda)^n(X_{ij})||^{\frac{1}{n}} = 0$$

for all $1 \leq i, j \leq 2$. Since $\sigma(A_i) \cap \sigma(B_j + \lambda) = \emptyset$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$ (so that $0 \notin \sigma(\delta_{ab} - \lambda)$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$), $X_{ij} = 0$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. The operators A_1 and B_1 being normal

$$\lim_{n \to \infty} ||(\delta_{A_1 B_1} - \lambda)^n (X_{11})||^{\frac{1}{n}} = 0 \iff (\delta_{A_1 B_1} - \lambda)(X_{11}) = 0$$

[20, Lemma 2]. Hence $H_0(\delta_{A_1 B_1} - \lambda) = (\delta_{A_1 B_1} - \lambda)^{-1}(0)$, which implies that $H_0(d_{AB} - \lambda) = (\delta_{AB} - \lambda)^{-1}(0)$. □

Theorem 3.6. *Let $A \in TP$ and $B^* \in CTP$. If B^* is isoloid, then we have the following:*

(i) $f(d_{AB})$ *satisfies Weyl's theorem for each $f \in \mathcal{H}(\sigma(d_{AB}))$.*
(ii) d_{AB}^* *satisfies a-Weyl's theorem.*

If also d_{AB}^* has SVEP, then:

(iii) $f(d_{AB})$ *satisfies a-Weyl's theorem for each $f \in \mathcal{H}_1(\sigma(d_{AB}))$.*

Proof. (i) We start by proving that d_{AB} and d_{AB}^* satisfy Weyl's theorem. Since $asc(d_{AB} - \lambda) \leq 1$ (by Lemma 3.4), d_{AB} has SVEP $\implies d_{AB}$ and d_{AB}^* satisfy Browder's theorem (see [1, Corollary 2.12] and [12]). We prove that $\pi_{00}(d_{AB}) \subseteq \pi_0(d_{AB})$, which would then imply $\pi_{00}(d_{AB}) = \pi_0(d_{AB})$ and hence that d_{AB} satisfies Weyl's theorem. Let $\lambda \in \pi_{00}(d_{AB})$; then $\lambda \in iso\sigma(d_{AB})$ and $0 < \alpha(d_{AB} - \lambda) < \infty$. Since $iso\sigma(d_{AB}) \cap \sigma_{kt}(d_{AB}) = \emptyset$ (by Lemma 3.5), $d_{AB} - \lambda$ is Kato type and $B(H) = (d_{AB} - \lambda)^{-1}(0) \oplus (d_{AB} - \lambda)(B(H)) \implies \lambda$ is a simple pole of the resolvent of $d_{AB} \implies \pi_{00}(d_{AB}) \subseteq \pi_0(d_{AB})$. The conclusion d_{AB} satisfies Weyl's theorem implies that $\sigma(d_{AB}) \setminus \sigma_w(d_{AB}) = \pi_{00}(d_{AB}) = \pi_0(d_{AB})$. Since

$$\begin{aligned} \lambda \notin \sigma_w(d_{AB}) &\iff (d_{AB} - \lambda) \in \Phi(B(H)) \text{ and } ind(d_{AB} - \lambda) = 0 \\ &\iff (d_{AB}^* - \lambda I^*) \in \Phi(B(H)) \text{ and } ind(d_{AB}^* - \lambda I^*) = 0 \\ &\iff \lambda \notin \sigma_w(d_{AB}^*), \end{aligned}$$

$\sigma_w(d_{AB}) = \sigma_w(d_{AB}^*)$. Hence, since $\sigma(d_{AB}) = \sigma(d_{AB}^*)$,

$$\sigma(d_{AB}^*) \setminus \sigma_w(d_{AB}^*) = \sigma(d_{AB}) \setminus \sigma_w(d_{AB}) = \pi_{00}(d_{AB}) = \pi_0(d_{AB}) = \pi_0(d_{AB}^*) \subseteq \pi_{00}(d_{AB}^*).$$

For the reverse inclusion, let $\lambda \in \pi_{00}(d_{AB}^*)$. Then $\alpha(d_{AB}^* - \lambda I^*) < \infty \implies \beta(d_{AB} - \lambda) < \infty$. Since $\lambda \in iso\sigma(d_{AB}^*) \implies \lambda \in iso\sigma(d_{AB})$, both d_{AB} and d_{AB}^* have SVEP at λ. Thus, since $(T - \lambda)$ is Kato type, $asc(d_{AB} - \lambda) = dsc(d_{AB} - \lambda) < \infty$ ([1, Theorems 2.6 and 2.9]) and $0 < \alpha(d_{AB} - \lambda) = \beta(d_{AB} - \lambda) < \infty$ ([13, Proposition 38.6]). Hence $\lambda \in \pi_0(d_{AB}) = \pi_{00}(d_{AB})$, which implies that $\sigma(d_{AB}^*) \setminus \sigma_w(d_{AB}^*) = \pi_{00}(d_{AB}^*)$.

The isoloid property of d_{AB} (see the proof of Lemma 3.5) implies that $\sigma(f(d_{AB}) \setminus \pi_{00}(f(d_{AB})) = f(\sigma(d_{AB}) \setminus \pi_{00}(d_{AB}))$ [16, Lemma]. Since d_{AB} has SVEP, we get $\sigma_w(f(d_{AB})) = f(\sigma_w(d_{AB}))$ for every $f \in \mathcal{H}(\sigma(d_{AB}))$ [6, Corollary 2.6]. Hence, since $\sigma(d_{AB}) \setminus \sigma_w(d_{AB}) = \pi_{00}(d_{AB})$,

$$\sigma(f(d_{AB}) \setminus \pi_{00}(f(d_{AB}))) = f(\sigma_w(d_{AB})) = \sigma_w(fd_{AB})),$$

i.e., $f(d_{AB})$ satisfies Weyl's theorem.

To prove parts (ii) and (iii) one argues as in the proof of Theorem 3.3. □

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