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KATO TYPE OPERATORS AND WEYL'S THEOREM

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ABSTRACT. A Banach space operator T satisfies Weyl's theorem if and only if T or T^* has SVEP at all complex numbers λ in the complement of the Weyl spectrum of T and T is Kato type at all λ which are isolated eigenvalues of T of finite algebraic multiplicity. If T^* (resp., T) has SVEP and T is Kato type at all λ which are isolated eigenvalues of T of finite algebraic multiplicity (resp., T is Kato type at all $\lambda \in \text{iso}\sigma(T)$), then T satisfies a-Weyl's theorem (resp., T^* satisfies a-Weyl's theorem).

1. INTRODUCTION

Let $B(X) = B(X, X)$ denote the algebra of operators (equivalently, bounded linear transformations) on a Banach space X. Let $\sigma_w(T)$ denote the Weyl spectrum of T, and let $\pi_{00}(T)$ denote the set of isolated eigenvalues μ of T for which $\dim((T-\mu I)^{-1}(0)) < \infty$. An operator $T \in B(X)$ is said to satisfy Weyl's theorem if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. Sufficient conditions for an operator $T \in B(X)$ to satisfy Weyl's theorem have been considered by a number of authors in the recent past ([1], [4], [5], [8], [11] and [14]). Let $H_0(T - \lambda I) = \{x \in X : \lim_{n\longrightarrow\infty} ||(T - \lambda I)^n(x)||^{\frac{1}{n}} = 0\}$ denote the *quasi-nilpotent part* of $T \in B(X)$. One such condition which has attracted the attention of a number of authors is the property

$$\mathbf{H}(p) \qquad H_0(T - \lambda I) = (T - \lambda I)^{-p}(0)$$

for some integer $p \geq 1$ and all complex numbers λ (see [2], [4] and [14]). It is known that property $\mathbf{H}(p)$ is satisfied by a number of the commonly considered classes of operators (see [2] and [14]). Operators T satisfying property $\mathbf{H}(p)$ have finite ascent and are isoloid (i.e., isolated points μ of the spectrum are eigenvalues of T). For operators T satisfying property $\mathbf{H}(p)$, both T and T^* satisfy Weyl's theorem, and T^* satisfies a-Weyl's theorem. (See [2] and [14], where it is shown that property $\mathbf{H}(p)$ is equivalent to a number of other properties.)

An operator $T \in B(X)$ is said to be *semi-regular* if $T(X)$ is closed and $T^{-1}(0) \subset T^\infty(X) = \cap_{n\in\mathbf{N}} T^n(X)$; T admits a *generalized Kato decomposition*, GKD for short, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. We say that T is of *Kato type* at a point λ if $(T - \lambda I)|_M$ is nilpotent in the GKD for $(T - \lambda I)$. Fredholm operators are Kato type [10, Theorem 4], and operators $T \in B(X)$ satisfying property $\mathbf{H}(p)$ are Kato type at isolated points of $\sigma(T)$ (but not every Kato type operator T satisfies property $\mathbf{H}(p)$).

It is obvious that for an operator $T \in B(X)$, $(T - \lambda I)$ is Kato type for all $\lambda \in \sigma(T) \setminus \sigma_w(T)$, which implies that if T satisfies Weyl's theorem then T is Kato

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type at each $\lambda \in \pi_{00}(T)$. The role, if any, played by the Kato type of T at points $\lambda \in \pi_{00}(T)$ in determining T satisfies Weyl's theorem seems not to have attracted attention. This paper fills this gap by considering operators $T \in B(X)$ which are Kato type at points $\lambda \in \pi_{00}(T)$ or at points $\lambda \in \text{iso}\sigma(T)$. It is proved that a necessary and sufficient condition for $T \in B(X)$ to satisfy Weyl's theorem is that: (a) T or T^* has *the single valued extension property*, SVEP, at all points $\lambda \in \sigma(T) \setminus \sigma_w(T)$; (b) T is Kato type at each $\lambda \in \pi_{00}(T)$. If T satisfies (a) and $(b)'$ T is Kato type at points $\lambda \in \text{iso}\sigma(T)$, then T^* satisfies Weyl's theorem. Furthermore, if T^* has SVEP and hypothesis (b) is satisfied (resp., T has SVEP and hypothesis $(b)'$ is satisfied), then T satisfies a-Weyl's theorem (resp., T^* satisfies a-Weyl's theorem).

The plan of this note is as follows. We introduce our notation and terminology in Section 2 and prove our main result, along with some of its consequences, in Sections 3 and 4. We remark here that there exist operators such that they satisfy conditions (a) and (b), but do not satisfy property $\mathbf{H}(p)$: paranormal operators and the weighted unilateral shift $T = \text{shift}(\{\frac{1}{k+1}\}_{k=1}^{\infty})$ are but two examples of such operators.

2. Notation and terminology.

An operator $T \in B(X)$ is said to be Fredholm, $T \in \Phi(X)$, if $T(X)$ is closed and both the *deficiency indices* $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(X/T(X))$ are finite, and then the *index* of T, ind(T), is defined to be $\text{ind}(T) = \alpha(T) - \beta(T)$. The ascent of T, asc(T), is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the descent of T, dsc(T), is the least non-negative integer n such that $T^n(X) = T^{n+1}(X)$. (We shall, henceforth, shorten $(T - \lambda I)$ to $(T - \lambda)$.) The operator T is *Weyl* if it is Fredholm of index zero, and T is said to be *Browder* if it is Fredholm "of finite ascent and descent". Let $\mathbf{C}$ denote the set of complex numbers. The (Fredholm) essential spectrum $\sigma_e(T)$, the Browder spectrum $\sigma_b(T)$ and the Weyl spectrum $\sigma_w(T)$ of T of are the sets

$$\sigma_e(T) = \{\lambda \in \mathbf{C} : T - \lambda \quad \text{is not Fredholm}\};$$
$$\sigma_b(T) = \{\lambda \in \mathbf{C} : T - \lambda \quad \text{is not Browder}\}$$

and

$$\sigma_w(T) = \{\lambda \in \mathbf{C} : T - \lambda \quad \text{is not Weyl}\}.$$

If we let $\rho(T)$ denote the resolvent set of the operator T, $\sigma(T)$ denote the usual spectrum of T and acc $\sigma(T)$ denote the set of accumulation points of $\sigma(T)$, then:

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup \text{acc}\ \sigma(T).$$

Let $\pi_0(T)$ denote the set of *Riesz points* of T, and let $\pi_{00}(T)$ denote the set of eigenvalues of T of finite geometric multiplicity. Also, let $\pi_{a0}(T)$ be the set of $\lambda \in \mathbf{C}$ such that λ is an isolated point of $\sigma_a(T)$ and $0 < \dim(T - \lambda)^{-1}(0) < \infty$, where $\sigma_a(T)$ denote the approximate point spectrum of the operator $T \in B(X)$. Clearly, $\pi_{00}(T) \subseteq \pi_{a0}(T)$. We say that *Browder's theorem holds for* $T \in B(X)$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T),$$

Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and *a-Weyl's theorem holds for* T

$$\sigma_{wa}(T) = \sigma_a(T) \setminus \pi_{a0}(T),$$

where $\sigma_{wa}(T)$ denote *the essential approximate point spectrum* (i.e., $\sigma_{wa}(T) = \cap\{\sigma_a(T+K) : K \in K(X)\}$ with $K(X)$ denoting the ideal of compact operators on X). If we let $\Phi_+(X) = \{T \in B(X) : \alpha(T) < \infty$ and $T(X)$ is closed$\}$ denote the semi-group of *upper semi-Fredholm operators* in $B(X)$, then $\sigma_{wa}(T)$ is the complement in $\mathbf{C}$ of all those λ for which $(T-\lambda) \in \Phi_+(X)$ and $\text{ind}(T-\lambda) \leq 0$. The concept of a-Weyl's theorem was introduced by Rakočević: a-Weyl's theorem for $T \Longrightarrow$ Weyl's theorem for T, but the converse is generally false [15].

An operator $T \in B(X)$ has *the single-valued extension property* at $\lambda_0 \in \mathbf{C}$, SVEP at $\lambda_0 \in \mathbf{C}$ for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to X$ which satisfies

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\mathbf{C} \setminus \sigma(T)$; also T has SVEP at $\lambda \in \text{iso}\sigma(T)$. We say that T has SVEP if it has SVEP at every $\lambda \in \mathbf{C}$. The *analytic core* $K(T-\lambda)$ of $(T-\lambda)$ is defined by

$$\begin{aligned} K(T-\lambda) = \{x \in X : &\text{there exists a sequence } \{x_n\} \subset X \quad \text{and} \quad \delta > 0 \\ &\text{for which } x = x_0, T(x_{n+1}) = x_n \quad \text{and} \quad \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, ...\}. \end{aligned}$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T-\lambda)$ such that $(T-\lambda)^{-p}(0) \subseteq H_0(T-\lambda)$ for all $p = 0, 1, 2, ...$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$ [13].

3. Main results.

We start by collecting together some results, which will be used in the sequel without further specific reference to the source. Every semi-Fredholm operator $T \in B(X)$ is of Kato type [10, Theorem 4], and a Kato type operator $T-\lambda$ such that T has SVEP at λ (resp., T^* has SVEP at λ) satisfies the property that $\text{asc}(T-\lambda) < \infty$ (resp., $\text{dsc}(T-\lambda) < \infty$) [1, Theorems 2.6 and 2.9]. It is easily seen that if $(T-\lambda)$ is Kato type, then the adjoint operator $(T^* - \lambda I^*)$ is also Kato type. If $\text{asc}(T-\lambda) < \infty$ (resp., $\text{dsc}(T-\lambda) < \infty$) for an operator $T \in B(X)$ and $\lambda \in \mathbf{C}$, then $\text{ind}(T-\lambda) \leq 0$ (resp., $\text{ind}(T-\lambda) \geq 0$) [9, Proposition 38.5].

Recall that a point $\lambda \in \sigma(T)$ is in $\pi_0(T)$ if and only if $(T-\lambda)$ is Fredholm of finite ascent and descent [3]. Obviously, $\pi_0(T) \subseteq \pi_{00}(T)$.

Proposition 3.1. *If $T \in B(X)$, then $(T-\lambda)$ is Kato type for all $\lambda \in \pi_{00}(T)$ if and only if $\pi_{00}(T) = \pi_0(T)$.*

Proof. If $\pi_{00}(T) = \pi_0(T)$, then to each $\lambda \in \pi_{00}(T)$ there corresponds an integer $p \geq 1$ such that $X = (T-\lambda)^{-p}(0) \oplus (T-\lambda)^p(X) \Longrightarrow (T-\lambda)$ is Kato type.

Conversely, let $\lambda \in \pi_{00}(T)$. Then $(T-\lambda)$ is Kato type implies $X = M \oplus N$, where $(T-\lambda)|_M$ is p-nilpotent for some integer $p \geq 1$ and $(T-\lambda)|_N$ is semi-regular. Since $\lambda \in \text{iso}\sigma(T)$, T has SVEP at λ, and, also, $(T-\lambda)|_N$ has SVEP at 0. Hence, $(T-\lambda)|_N$ is a semi-regular operator with SVEP in 0 and by [14, Lemma 2.1 (i)] is injective. Now we have

$$(T-\lambda)^{-n}(0) = ((T-\lambda)|_N)^{-n}(0) \oplus ((T-\lambda)|_M)^{-n}(0) = 0 \oplus M = M$$

for all $n \geq p$, i.e., $\text{asc}(T-\lambda) \leq p$. Clearly, $\text{asc}(T-\lambda) < \infty \Longrightarrow \text{ind}(T-\lambda) \leq 0$. Since $(T^* - \lambda I^*)$ is also Kato type, and since $\lambda \in \text{iso}\sigma(T^*)$ implies T^* has SVEP at

λ, $\operatorname{ind}(T^* - \lambda I^*) \leq 0$. We have already seen that $\operatorname{ind}(T - \lambda) = -\operatorname{ind}(T^* - \lambda I^*) \leq 0$; hence $\operatorname{ind}(T - \lambda) = 0$. Since $\lambda \in \pi_{00}(T) \Longrightarrow 0 < \alpha(T - \lambda) < \infty$, it follows that $\alpha(T - \lambda) = \beta(T - \lambda) < \infty$. Taken together with $\operatorname{asc}(T - \lambda) = p$, this implies that $\operatorname{asc}(T - \lambda) = \operatorname{dsc}(T - \lambda) = p < \infty$ [9, Proposition 38.6], and hence that $\lambda \in \pi_0(T)$. Thus $\pi_{00}(T) \subseteq \pi_0(T)$. Since $\pi_0(T) \subseteq \pi_{00}(T)$ always, $\pi_{00}(T) = \pi_0(T)$. □

Let $\mathcal{N}(T)$ denote the *null space* of T, and let $\gamma(T)$ denote the *minimal modulus function* of T (see [9, pp. 155]), i.e.

$$\gamma(T) = \inf\{\frac{\|Tx\|}{d(x, \mathcal{N}(T))} : x \in X, x \notin \mathcal{N}(T)\}.$$

Corollary 3.2. *The following conditions are equivalent for $T \in B(X)$.*
(i) $\lambda \in \pi_{00}(T)$ *and* $(T - \lambda)$ *is Kato type.*
(ii) $\lambda \in \pi_0(T)$.
(iii) $\lambda \in \pi_{00}(T)$ *and* $(T - \lambda)(X)$ *is closed.*
(iv) $\lambda \in iso\sigma(T)$ *and* $\dim(H_0(T - \lambda)) < \infty$.
(v) $\lambda \in iso\sigma(T)$ *and co-dim*$(K(T - \lambda)) < \infty$.
(vi) $\lambda \in \pi_{00}(T)$ *and* $H_0(T - \lambda) = (T - \lambda)^{-p}(0)$ *for some integer* $p \geq 1$.
(vii) $\lambda \in \pi_{00}(T)$ *and* $K(T - \lambda) = (T - \lambda)^p(X)$ *for some integer* $p \geq 1$.
(viii) $\lambda \in \pi_{00}(T)$ *and* $dsc(T - \lambda) < \infty$.
(ix) $\lambda \in \pi_{00}(T)$ *and* λ *is a pole of the resolvent of* T.
(x) $\lambda \in \pi_{00}(T)$ *and* $\gamma(T)$ *is discontinuous at* λ.

Proof. Equivalence of conditions (ii) to (x) is proved in [14, Proposition 2.12] (see also [2] and [4]). The equivalence of (i) and (ii) follows from Proposition 3.1. □

If T has SVEP, then T satisfies Browder's theorem [4]. Thus if T is Kato type for $\lambda \in \pi_{00}(T)$ (or satisfies any of the equivalent conditions in Corollary 3.2), then T satisfies Weyl's theorem. The following theorem shows that the SVEP hypothesis on T can be weakened to SVEP at all points in the complement of $\sigma_w(T)$.

Theorem 3.3. *$T \in B(X)$ satisfies Weyl's theorem if and only if*
(a) *T or T^* has SVEP at all $\lambda \in \sigma(T) \setminus \sigma_w(T)$;*
(b) *T is Kato type at all $\lambda \in \pi_{00}(T)$.*

Proof. If T satisfies Weyl's theorem, then $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, and so, since $\lambda \in \sigma(T) \setminus \sigma_w(T) \Longrightarrow \lambda \in \operatorname{iso}\sigma(T)$, both T and T^* have SVEP at points $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Again, if $\lambda \in \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, then $\lambda \in \Phi(X)$ and $\operatorname{ind}(T - \lambda) = 0$ imply T is Kato type.

Conversely, let $\lambda \in \pi_{00}(T)$. Then, by hypothesis (b), $\lambda \in \pi_0(T)$ which implies that $(T - \lambda) \in \Phi(X)$ and $\operatorname{ind}(T - \lambda) = 0$. Hence $\lambda \in \sigma(T) \setminus \sigma_w(T)$, so that $\pi_{00}(T) \subseteq \sigma(T) \setminus \sigma_w(T)$. For the reverse inclusion, let $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $(T - \lambda) \in \Phi(X)$ and $\operatorname{ind}(T - \lambda) = 0$. If T or T^* has SVEP at λ, then both $\operatorname{asc}(T - \lambda)$ and $\operatorname{dsc}(T - \lambda))$ are finite [1, Corollary 2.10], which implies that $\lambda \in \pi_0(T) \subseteq \pi_{00}(T)$. Hence $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, and T satisfies Weyl's theorem. □

Theorem 3.4. *If T or T^* has SVEP at all $\lambda \in \sigma(T) \setminus \sigma_w(T)$ and T is Kato type at all $\lambda \in iso\sigma(T)$, then T^* satisfies Weyl's theorem.*

Proof. The hypotheses imply that T satisfies Weyl's theorem, and $\sigma(T)\setminus\sigma_w(T)=\pi_{00}(T)=\pi_0(T)$. Since

$$\begin{aligned}\lambda\notin\sigma_w(T) &\iff (T-\lambda)\in\Phi(X)\ \text{ and }\ \mathrm{ind}(T-\lambda)=0\\ &\iff (T^*-\lambda I^*)\in\Phi(X)\ \text{ and }\ \mathrm{ind}(T^*-\lambda I^*)=0\\ &\iff \lambda\notin\sigma_w(T^*),\end{aligned}$$

$\sigma_w(T)=\sigma_w(T^*)$. Hence, since $\sigma(T)=\sigma(T^*)$,

$$\sigma(T^*)\setminus\sigma_w(T^*)=\sigma(T)\setminus\sigma_w(T)=\pi_{00}(T)=\pi_0(T)=\pi_0(T^*)\subseteq\pi_{00}(T^*).$$

For the reverse inclusion, let $\lambda\in\pi_{00}(T^*)$. Then $\alpha(T^*-\lambda I^*)<\infty$. Since $\lambda\in\mathrm{iso}\sigma(T^*)\Longrightarrow\lambda\in\mathrm{iso}\sigma(T)$, both T and T^* have SVEP at λ. Thus, since $(T-\lambda)$ is Kato type, $\mathrm{asc}(T-\lambda)=\mathrm{dsc}(T-\lambda)<\infty$ (which implies that $\mathrm{asc}(T^*-\lambda I^*)=\mathrm{dsc}(T^*-\lambda I^*)<\infty$). Consequently, $0<\alpha(T^*-\lambda I^*)=\beta(T^*-\lambda I^*)<\infty\Longrightarrow 0<\alpha(T-\lambda)=\beta(T-\lambda)<\infty$. Hence $\lambda\in\pi_0(T)=\pi_{00}(T)$, which implies that $\sigma(T^*)\setminus\sigma_w(T^*)=\pi_{00}(T^*)$. □

Remark 3.5. Theorem 3.4 fails if T is not Kato type at all $\lambda\in\mathrm{iso}\sigma(T)$. Let $T\in B(\ell^2)$ be the weighted shift $T(x_1,x_2,...)=(0,\frac{x_1}{2},\frac{x_2}{3},...)$; then $T^*(x_1,x_2,x_3,...)=(\frac{x_2}{2},\frac{x_3}{3},...)$, both T and T^* have SVEP, $\pi_{00}(T)=\emptyset$, $\sigma(T)=\sigma(T^*)=\{0\}$, $\sigma_w(T)=\sigma_w(T^*)=\{0\}$ and $\pi_{00}(T^*)=\{0\}$. Trivially, T is Kato type at points $\lambda\in\sigma(T)\setminus\sigma_w(T)$, but T is not Kato type at $0\in\mathrm{iso}\sigma(T)$: T satisfies Weyl's theorem, but T^* does not satisfy Weyl's theorem.

Strengthening the hypothesis on SVEP, a result stronger than that of Theorems 3.3 and 3.4 holds.

Theorem 3.6. (i) *If T^* has SVEP and T is Kato type at each $\lambda\in\pi_{00}(T)$, then T satisfies a-Weyl's theorem.*

(ii) *If T has SVEP and is of Kato type at each $\lambda\in iso\sigma(T)$, then T^* satisfies a-Weyl's theorem.*

Proof. (i) The hypothesis T^* has SVEP implies $\sigma(T)=\sigma_a(T)$ [12, pp. 35], and hence $\pi_{a0}(T)=\pi_{00}(T)$. We prove that $\sigma_{wa}(T)=\sigma_w(T)$: since T satisfies Weyl's theorem (by Theorem 3.3), this would then imply that $\sigma_a(T)\setminus\sigma_{wa}(T)=\pi_{a0}(T)$. It being clear that $\lambda\notin\sigma_w(T)\Longrightarrow\lambda\notin\sigma_{wa}(T)$, we prove that $\lambda\notin\sigma_{wa}(T)\Longrightarrow\lambda\notin\sigma_w(T)$. Since $\lambda\notin\sigma_{wa}(T)\iff(T-\lambda)\in\Phi_+(X)$ and $\mathrm{ind}(T-\lambda)\le 0$, the hypothesis T^* has SVEP implies that $\mathrm{dsc}(T-\lambda)<\infty$, $0<\alpha(T-\lambda)<\infty$ and $\mathrm{ind}(T-\lambda)\le 0$. Again, since $\mathrm{dsc}(T-\lambda)<\infty$ implies $\mathrm{ind}(T-\lambda)\ge 0$, we have:

$$\begin{aligned}\lambda\notin\sigma_{wa}(T) &\iff \mathrm{dsc}(T-\lambda)<\infty\ \text{ and }\ 0<\alpha(T-\lambda)=\beta(T-\lambda)<\infty\\ &\Longrightarrow \lambda\notin\sigma_w(T).\end{aligned}$$

(ii) If T has SVEP, then $\sigma(T^*)=\sigma_a(T^*)$ [12, pp. 35] and $\pi_{a0}(T^*)=\pi_{00}(T^*)$. We prove that $\sigma_{wa}(T^*)=\sigma_w(T^*)$: since T^* satisfies Weyl's theorem (by Theorem 3.4), this would then imply that $\sigma_a(T^*)\setminus\sigma_{wa}(T^*)=\pi_{a0}(T^*)$. It being clear that $\sigma_{wa}(T^*)\subseteq\sigma_w(T^*)$, we prove the reverse inclusion. Since

$$\lambda\notin\sigma_{wa}(T^*)\iff(T^*-\lambda I^*)\in\Phi_+(X^*)\ \text{ and }\ \mathrm{ind}(T^*-\lambda I^*)\le 0,$$

the hypothesis T has SVEP implies that

$$\begin{aligned}&\mathrm{dsc}(T^*-\lambda I^*)<\infty, 0<\alpha(T^*-\lambda I^*)<\infty\ \text{ and }\ \mathrm{ind}(T^*-\lambda I^*)\le 0\\ \Longrightarrow\ &\mathrm{dsc}(T^*-\lambda I^*)<\infty, 0<\alpha(T^*-\lambda I^*)=\beta(T^*-\lambda I^*)<\infty\\ \Longrightarrow\ &\lambda\notin\sigma_w(T^*).\end{aligned}$$

□

Let $\mathcal{H}(\sigma(T))$ (resp., $\mathcal{H}_1(\sigma(T))$) denote the set of analytic functions which are defined on an open neighborhood $\mathcal{U}$ of $\sigma(T)$ (resp., the set of $f \in \mathcal{H}(\sigma(T))$ which are non-constant on each of the connected components of the open neighborhood $\mathcal{U}$ of $\sigma(T)$ on which f is defined). Recall that the operator T is said to be *isoloid* if each $\lambda \in \text{iso}\sigma(T)$ is an eigenvalue of T.

Lemma 3.7. *If T is Kato type at $\lambda \in iso\sigma(T)$, then $\lambda \in \sigma_p(T)$.*

Proof. If $\lambda \in \text{iso}\sigma(T)$, then T and T^* have SVEP at λ. Thus if $(T-\lambda)$ is Kato type, then $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) = p$ for some integer $p \geq 1$ and $X = (T-\lambda)^{-p}(0) \oplus (T-\lambda)^p(X)$ ([1, Theorems 2.6 and 2.9] and [12, Proposition 4.10.6]), which implies that λ is an eigenvalue of T [9, Proposition 50.2]. □

Theorem 3.8. (i) *If T or T^* has SVEP, and if T is Kato type at each $\lambda \in iso\sigma(T)$, then $f(T)$ satisfies Weyl's theorem for each $f \in \mathcal{H}(\sigma(T))$.*

(ii) *If T^* has SVEP, and if T is Kato type at each $\lambda \in iso\sigma(T)$, then $f(T)$ satisfies a-Weyl's theorem for each $f \in \mathcal{H}_1(\sigma(T))$.*

Proof. (i) T being isoloid (by Lemma 3.7), $\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T))$ [11, Lemma]. If T or T^* has SVEP, then $\sigma_w(f(T)) = f(\sigma_w(T))$ for every $f \in \mathcal{H}(\sigma(T))$ [4, Corollary 2.6]. We already know from Theorem 3.3 that T satisfies Weyl's theorem. Hence

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma_w(T)) = \sigma_w(f(T)),$$

i.e., $f(T)$ satisfies Weyl's theorem.

(ii) If T^* has SVEP, and $f \in \mathcal{H}_1(\sigma(T))$, then $f(T^*) = f(T)^*$ has SVEP [12, Theorem 3.3.9], which implies that $\sigma(f(T)) = \sigma_a(f(T))$. Arguing as in the proof of Theorem 3.6 it is seen that $\sigma_w(f(T)) = \sigma_{wa}(f(T))$. Since $f(T)$ satisfies Weyl's theorem by part (i), $f(T)$ satisfies a-Weyl's theorem. □

4. Applications

An operator $T \in B(X)$ is said to be *transaloid* if $(T-\lambda)$ is *normaloid* for every complex number λ (i.e., if the spectral radius $r(T-\lambda)$ equals $||T-\lambda||$ for all $\lambda \in \mathbf{C}$). The transaloid property implies $H_0(T-\lambda) = (T-\lambda)^{-1}(0)$ for every $\lambda \in \mathbf{C}$ [4, Lemma 2.3]. If we let $\mathbf{H}(p)$, $1 \leq p$ some integer, denote the class of $T \in B(X)$ for which $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for all $\lambda \in \mathbf{C}$, then the transaloid operators belong to the subclass $\mathbf{H}(1)$ of $\mathbf{H}(p)$. The class $\mathbf{H}(p)$ is large; it contains, in particular, the classes consisting of *generalized scalar*, *subscalar* and *totally paranormal* operators on a Banach space, *multipliers of semi-simple Banach algebras*, *hyponormal*, *p-hyponormal* $(0 < p < 1)$ and *M-hyponormal* operators on a Hilbert space (see [2], [4], [12] and [14] for further information). It is obvious that operators $T \in \mathbf{H}(p)$ have finite ascent (and hence, SVEP). If $\lambda \in \text{iso}\sigma(T)$, $T \in \mathbf{H}(p)$, then

$$\begin{aligned} & X = H_0(T-\lambda) \oplus K(T-\lambda) = (T-\lambda)^{-p}(0) \oplus K(T-\lambda) \\ \Longrightarrow\ & (T-\lambda)^{-p}(0) \quad \text{is complemented by the closed subspace} \\ & K(T-\lambda) \subseteq (T-\lambda)(X) \\ \Longrightarrow\ & K(T-\lambda) = (T-\lambda)^p(X). \end{aligned}$$

Hence operators $T \in \mathbf{H}(p)$ are Kato type at each $\lambda \in \text{iso}\sigma(T)$.

Corollary 4.1. *Let $T \in B(X) \cap \mathbf{H}(p)$. Then:*
(i) *$f(T)$ satisfies Weyl's theorem for each $f \in \mathcal{H}(\sigma(T))$.*
(ii) *If T^* has SVEP, then $f(T)$ satisfies a-Weyl's theorem for each $f \in \mathcal{H}_1(\sigma(T))$.*

Proof. Apply Theorem 3.8. □

Corollary 4.1 had earlier been proved in [14] and [2] using a different argument. The corollary shows that the hypotheses T has SVEP in [4, Theorem 2.5] and T is isoloid in [4, Theorem 3.5] are redundant.

An operator $T \in B(X)$ is *paranormal* if $||Tx||^2 \leq ||T^2x||$ for all unit vectors $x \in X$ [9, pp. 229]. Paranormal operators do not belong to $\mathbf{H}(p)$ (see [5], Remark following Lemma 2.3). However, isolated points of the spectrum of a paranormal operator T are simple poles of the resolvent of T [5, Lemma 2.1]. The definition of paranormality of T implies that $asc(T) \leq 1$; hence T has SVEP at 0.

Corollary 4.2. *If $T \in B(X)$ is paranormal, then both T and T^* satisfy Weyl's theorem.*

Proof. We have already seen that T is Kato type at $\lambda \in \text{iso}\sigma(T)$: if we prove that T has SVEP at $\lambda \in \sigma(T) \setminus \sigma_w(T)$ then Theorems 3.3 and 3.4 would imply that both T and T^* satisfy Weyl's theorem. Let, as before, $\mathcal{N}(T)$ and $\gamma(T)$ denote the null space and the minimal modulus function of T, and let $d(x, \mathcal{N}(T)) = \inf_{y \in \mathcal{N}(T)} ||x - y||$ denote the distance of $x \in X$ from $\mathcal{N}(T)$. Let $(0 \neq)\lambda \in \sigma(T) \setminus \sigma_w(T)$; then $(T - \lambda) \in \Phi(X)$ and $\text{ind}(T - \lambda) = 0$. We claim that $\lambda \in \text{iso}\sigma(T)$. For if $\lambda \notin \text{iso}\sigma(T)$, then, λ being a non-isolated eigenvalue of T, there exists a sequence of non-zero eigenvalues of T converging to λ. Recall from [5, Lemma 2.2] that eigenspaces corresponding to distinct non-zero eigenvalues of a paranormal operator are *orthogonal* (in the sense of G. Birkhoff [7, pp. 93]). Hence $d(x_n, \mathcal{N}(T - \lambda)) \geq 1$ for all $x_n \in \mathcal{N}(T - \lambda_n)$ such that $||x_n|| = 1$. We have:

$$\delta(\lambda_n, \lambda) = \sup\{d(x_n, \mathcal{N}(T - \lambda)) : x_n \in \mathcal{N}(T - \lambda_n), ||x_n|| = 1\} \geq 1$$

for all n, which implies that

$$|\lambda_n - \lambda| / \delta(\lambda_n, \lambda) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

But then

$$\gamma(T - \lambda) = |\lambda_n - \lambda| / \delta(\lambda_n, \lambda) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since $(T - \lambda)(X)$ is closed, this is a contradiction [9, Proposition 36.1]. Consequently, points $(0 \neq)\lambda \in \sigma(T) \setminus \sigma_w(T)$ are isolated in $\sigma(T)$ and T has SVEP at all such points. As remarked upon above, $\text{asc}(T) \leq 1$. Hence T has SVEP at all points $\lambda \in \sigma(T) \setminus \sigma_w(T)$. □

As a final application of the results of the previous section, we consider the *elementary operator* $d_{AB} \in B(B(H))$, where H is a Hilbert space, A and $B^* \in B(H)$ are *hyponormal* operators (i.e., $|A^*|^2 \leq |A|^2$ and $|B|^2 \leq |B^*|^2$), and d_{AB} is either the *generalized derivation* $\delta_{AB}(X) = AX - XB$ or the *elementary operator* $\triangle_{AB}(X) = AXB - X$. It is then known that $asc(d_{AB} - \lambda) \leq 1$ for all complex numbers λ [6, Corollary 2.4] and $H_0(d_{AB} - \lambda) = (d_{AB} - \lambda)^{-1}(0)$ for $\lambda \in iso\sigma(d_{AB})$ (this is the content of the proof of [6, Theorem 2.7]). Thus d_{AB} has SVEP, and is of Kato type at all $\lambda \in \text{iso}\sigma(d_{AB})$. Applying the results of the previous section, we have next corollary that extends [6, Theorem 3.1].

Corollary 4.3. (i) d_{AB} *(the conjugate operator)* d^*_{AB} *and* $f(d_{AB})$ $(f \in \mathcal{H}(\sigma(d_{AB})))$ *satisfy Weyl's theorem.*

(ii) d^*_{AB} *satisfies a-Weyl's theorem.*

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