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CONTRACTIONS OF CLASS Q AND INVARIANT SUBSPACES

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Abstract. A Hilbert Space operator T is of class $\mathcal{Q}$ if $T^{2*}T^2 - 2T^*T + I$ is nonnegative. Every paranormal operator is of class $\mathcal{Q}$, but class-$\mathcal{Q}$ operators are not necessarily normaloid. It is shown that if a class-$\mathcal{Q}$ contraction T has no nontrivial invariant subspace, then it is a proper contraction. Moreover, the nonnegative operator $Q = T^{2*}T^2 - 2T^*T + I$ also is a proper contraction.

1. Introduction

Let $\mathcal{H}$ be a nonzero complex Hilbert space. By a subspace $\mathcal{M}$ of $\mathcal{H}$ we mean a closed linear manifold of $\mathcal{H}$, and by an operator T on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. A subspace $\mathcal{M}$ is invariant for T if $T(\mathcal{M}) \subseteq \mathcal{M}$, and nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$. Let $\mathcal{B}[\mathcal{H}]$ denote the algebra of all operators on $\mathcal{H}$. For an arbitrary operator T in $\mathcal{B}[\mathcal{H}]$ set, as usual, $|T| = (T^*T)^{\frac{1}{2}}$ (the absolute value of T) and $[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$ (the self-commutator of T), where T^* is the adjoint of T, and consider the following standard definitions: T is hyponormal if $[T^*, T]$ is nonnegative (i.e., $|T^*|^2 \leq |T|^2$; equivalently, $\|T^*x\| \leq \|Tx\|$ for every x in $\mathcal{H}$), T is of class $\mathcal{U}$ if $|T^2| - |T|^2$ is nonnegative (i.e., $|T|^2 \leq |T^2|$), paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every x in $\mathcal{H}$, and normaloid if $r(T) = \|T\|$ (where $r(T)$ denotes the spectral radius of T). These are related by proper inclusion:

$$\text{Hyponormal} \subset \text{Class } \mathcal{U} \subset \text{Paranormal} \subset \text{Normaloid}.$$

A contraction is an operator T such that $\|T\| \leq 1$ (i.e., $\|Tx\| \leq \|x\|$ for every x in $\mathcal{H}$; equivalently, $T^*T \leq I$). A proper contraction is an operator T such that $\|Tx\| < \|x\|$ for every nonzero x in $\mathcal{H}$ (equivalently, $T^*T < I$). A strict contraction is an operator T such that $\|T\| < 1$ (i.e., $\sup_{0\neq x}(\|Tx\|/\|x\|) < 1$ or, equivalently, $T^*T \prec I$, which means that $T^*T \leq \gamma I$ for some $\gamma \in (0, 1)$). Again, these are related by proper inclusion: Strict Contraction $\subset$ Proper Contraction $\subset$ Contraction.

It was recently proved in [10] that *if a hyponormal contraction T has no nontrivial invariant subspace, then T is a proper contraction and its self-commutator $[T^*, T]$ is a strict contraction.* This was extended in [5] to contractions of class $\mathcal{U}$ (*if a contraction T in $\mathcal{U}$ has no nontrivial invariant subspace, then both T and the nonnegative operator $|T^2| - |T|^2$ are proper contractions*), and to paranormal contractions in [6]: *If a paranormal contraction T has no nontrivial invariant subspace, then T is a proper contraction and so is the nonnegative operator $|T^2|^2 - 2|T|^2 + I$.* In the present paper we extend this result to contractions of class $\mathcal{Q}$. Operators of

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class $\mathcal{Q}$ are defined below. This is a class of operators that properly includes the paranormal operators.

2. Operators of Class Q

In this section we define operators of class $\mathcal{Q}$ and consider some basic properties, examples and counterexamples, in order to put this class in its due place. Recall that, for any real λ and any operator $T \in \mathcal{B}[\mathcal{H}]$,

$$\lambda\|T^2x\|\|x\| \;\le\; \tfrac{1}{2}\big(\|T^2x\|^2 + \lambda^2\|x\|^2\big)$$

and, in particular, for $\lambda = 1$,

$$\|T^2x\|\|x\| \;\le\; \tfrac{1}{2}\big(\|T^2x\|^2 + \|x\|^2\big),$$

for every $x \in \mathcal{H}$. An operator $T \in \mathcal{B}[\mathcal{H}]$ is paranormal if

$$\|Tx\|^2 \;\le\; \|T^2x\|\|x\|$$

for every $x \in \mathcal{H}$. Paranormal operators have been much investigated since [8] (see e.g., [7] and [9]). The following alternative definition is well-known. An operator $T \in \mathcal{B}[\mathcal{H}]$ is paranormal if and only if

$$O \;\le\; T^{2*}T^2 - 2\,\lambda\,T^*T + \lambda^2 I$$

for all $\lambda > 0$ (cf. [1], also see [12]). Equivalently, T is paranormal if and only if

$$\lambda\|Tx\|^2 \;\le\; \tfrac{1}{2}\big(\|T^2x\|^2 + \lambda^2\|x\|^2\big)$$

for every $x \in \mathcal{H}$, for all $\lambda > 0$. Note that the above inequalities hold trivially for every $\lambda \le 0$ for all operators $T \in \mathcal{B}[\mathcal{H}]$. Take any operator T in $\mathcal{B}[\mathcal{H}]$ and set

$$Q \;=\; T^{2*}T^2 - 2\,T^*T + I.$$

Definition 1. An operator T is of *class* $\mathcal{Q}$ if $O \le Q$. Equivalently, $T \in \mathcal{Q}$ if

$$\|Tx\|^2 \;\le\; \tfrac{1}{2}\big(\|T^2x\|^2 + \|x\|^2\big) \quad \text{for every } x.$$

Since $O \le T^{2*}T^2 - 2\,\lambda\,T^*T + \lambda^2 I$ if and only if $\lambda^{-\frac{1}{2}}T \in \mathcal{Q}$ for any $\lambda > 0$,

T is paranormal if and only if $\lambda\,T \in \mathcal{Q}$ for all $\lambda > 0$.

Every paranormal operator is a normaloid of class $\mathcal{Q}$. That is, with $\mathcal{N}$ and $\mathcal{P}$ standing for the classes of all normaloid and paranormal operators from $\mathcal{B}[\mathcal{H}]$, respectively, it is clear that

$$\mathcal{P} \;\subseteq\; \mathcal{Q}\cap\mathcal{N}.$$

However, $\mathcal{Q} \not\subseteq \mathcal{N}$ and $\mathcal{Q}\cap\mathcal{N} \not\subseteq \mathcal{P}$. Indeed, $S = \lambda\big(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\big) \in \mathcal{Q}$ for every $\lambda \in (0, 1/\sqrt{2}]$ but $S \notin \mathcal{N}$ (nonzero nilpotent) for all $\lambda \ne 0$. Moreover, $T = I \oplus S$ lies in $(\mathcal{Q}\cap\mathcal{N})\backslash\mathcal{P}$ for any $\lambda \in (0, 1/\sqrt{2}]$. In fact, S is not normaloid, and hence not paranormal, which implies that T is not paranormal (restriction of a paranormal to an invariant subspace is again paranormal), and $r(T) = \|T\| = 1$. Thus T is a normaloid contraction of class $\mathcal{Q}$ that is not paranormal.

Proposition 1. *Let $T \in \mathcal{B}[\mathcal{H}]$ be an operator of class $\mathcal{Q}$.*

(a) *The restriction of T to an invariant subspace is again a class-$\mathcal{Q}$ operator.*

(b) *If T is invertible, then T^{-1} is of class $\mathcal{Q}$.*

Proof. Let T be an operator of class $\mathcal{Q}$ and let $\mathcal{M}$ be a T-invariant subspace.

(a) If $u \in \mathcal{M}$, then $2\|T|_{\mathcal{M}}u\|^2 = 2\|Tu\|^2 \le \|T^2u\|^2 + \|u\|^2 = \|(T|_{\mathcal{M}})^2u\|^2 + \|u\|^2$, and so $T|_{\mathcal{M}}$ is of class $\mathcal{Q}$.

(b) If T is invertible, then $2\|x\|^2 = 2\|TT^{-1}x\|^2 \le \|T^2(T^{-1}x)\|^2 + \|T^{-1}x\|^2$ for every $x \in \mathcal{H}$. Take any y in $\mathcal{H} = \operatorname{ran}(T)$ so that $y = Tx$, $x = T^{-1}y$ and $T^{-1}x = T^{-2}y$ for some x in $\mathcal{H}$. Thus $2\|T^{-1}y\|^2 \le \|y\|^2 + \|T^{-2}y\|^2$ by the above inequality, and so T^{-1} is of class $\mathcal{Q}$. □

Some properties that the paranormal operators inherit from the hyponormals survive up to class $\mathcal{Q}$, as in the case of Proposition 1. However, many important properties shared by the hyponormals do not travel well up to class $\mathcal{Q}$. For instance, there exist nonzero quasinilpotent operators of class $\mathcal{Q}$ (a quasinilpotent normaloid is obviously null), compact operators of class $\mathcal{Q}$ that are not normal (every compact paranormal is normal [11]), and also operators of class $\mathcal{Q}$ for which isolated points of the spectrum are not eigenvalues (isolated points of the spectrum of a paranormal are eigenvalues [2]). Here is an example. The compact weighted unilateral shift $T = \operatorname{shift}(\{\frac{1}{k+1}\}_{k=1}^{\infty})$ is a quasinilpotent ($r(T) = 0$) contraction ($\|T\| = \frac{1}{2}$) with no eigenvalues (0 is in the residual spectrum of T). Clearly, since T is not normaloid, it is not paranormal. But it is of class $\mathcal{Q}$. Indeed,

$$O < \operatorname{diag}\big(\{1 - \tfrac{2}{(k+1)^2}\}_{k=1}^{\infty}\big) = I - 2\,T^*T < T^{2*}T^2 - 2\,T^*T + I.$$

Another common property of hyponormal and paranormal operators that does not apply to class $\mathcal{Q}$ is that a multiple of a class-$\mathcal{Q}$ operator may not be of class $\mathcal{Q}$. For example, $S = \lambda\big(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\big) \in \mathcal{Q}$ for every $\lambda \in (0, 1/\sqrt{2}]$, but $S \notin \mathcal{Q}$ for all $\lambda > 1/\sqrt{2}$. Actually, $\mathcal{Q}$ is not a cone in $\mathcal{B}[\mathcal{H}]$, although its intersection with the closed unit ball is balanced (a subset A of a linear space is balanced if $\alpha A \subseteq A$ whenever $|\alpha| \le 1$).

Proposition 2. *Let T be a Hilbert space operator.*

(a) *If $\|T\| \le 1/\sqrt{2}$, then $T \in \mathcal{Q}$.*

(b) *If $T^2 = O$, then $T \in \mathcal{Q}$ if and only if $\|T\| \le 1/\sqrt{2}$.*

(c) *If $T \in \mathcal{Q}$, $T^2 \ne O$ and $|\alpha| \le \min\{1, \|T^2\|^{-1}\}$, then $\alpha T \in \mathcal{Q}$.*

In particular, if $T \in \mathcal{Q}$ is a contraction, then $\alpha T \in \mathcal{Q}$ whenever $|\alpha| \le 1$.

(d) *A contraction T in $\mathcal{Q}$ is paranormal if and only if $O \le T^{2*}T^2 - 2\,\lambda\, T^*T + \lambda^2 I$ for all $\lambda \in (0, 1)$.*

Proof. Let T be any operator in $\mathcal{B}[\mathcal{H}]$.

(a) Since $O \le I - 2\,T^*T$ (that is, $2\,T^*T \le I$) if and only if αT for $|\alpha| = \sqrt{2}$ is a contraction, it follows that $\|\sqrt{2}\,T\| \le 1$ implies $T \in \mathcal{Q}$ because

$$I - 2\,T^*T \le T^{2*}T^2 - 2\,T^*T + I.$$

(b) If $T^2 = O$, then $T \in \mathcal{Q}$ if and only if $O \le I - 2\,T^*T$.

(c) If T lies in $\mathcal{Q}$, then

$$2|\alpha|^2 T^*T \le |\alpha|^2 T^{2*}T^2 + |\alpha|^2 I$$

and hence, for every scalar α,

$$2|\alpha|^2 T^*T - |\alpha|^4 T^{2*}T^2 - I \le \big(1 - |\alpha|^2\big)\big(|\alpha|^2 T^{2*}T^2 - I\big).$$

Suppose $T^2 \neq O$. Note: $|\alpha| \leq \|T^2\|^{-1}$ (i.e., $\alpha\, T^2$ is a contraction) if and only if $|\alpha|^2 T^{2*} T^2 \leq I$. If, in addition, $|\alpha| \leq 1$, then $(1 - |\alpha|^2)\,(|\alpha|^2 T^{2*} T^2 - I) \leq O$, and therefore $\alpha\, T \in \mathcal{Q}$.

(d) If $T \in \mathcal{Q}$ is a contraction, then αT lies in $\mathcal{Q}$ for all $\alpha \in (0,1]$ or, equivalently (with $\lambda = \alpha^{-1}$), $O \leq T^{2*}T^2 - 2\,\lambda\, T^*T + \lambda^2 I$ for all $\lambda \geq 1$. Thus, if $T \in \mathcal{Q}$ is a contraction, then the above inequality holds for all $\lambda > 0$ if and only if it holds for all $\lambda \in (0,1)$. Therefore, a contraction T of class $\mathcal{Q}$ is paranormal if and only if $O \leq T^{2*}T^2 - 2\,\lambda\, T^*T + \lambda^2 I$ for all $\lambda \in (0,1)$. □

Corollary 1. *If $T \in \mathcal{Q}$ is invertible, then $\alpha\, T \in \mathcal{Q}$ for every scalar α such that either $|\alpha| \leq \min\{1, \|T^2\|^{-1}\}$ or $|\alpha| \geq \max\{1, \|T^{-2}\|\}$.*

Proof. Take an invertible $T \in \mathcal{Q}$ and any scalar α. Proposition 2 ensures that

$$\alpha\, T \in \mathcal{Q} \quad \text{whenever} \quad |\alpha| \leq \min\big\{1, \|T^2\|^{-1}\big\},$$

and Proposition 1 says that $T^{-1} \in \mathcal{Q}$. Then $\beta\, T^{-1} \in \mathcal{Q}$ for every nonzero scalar β such that $|\beta| \leq \min\{1, \|T^{-2}\|^{-1}\}$ by Proposition 2. Put $\gamma = \beta^{-1}$ so that $(\gamma\, T)^{-1}$ lies in $\mathcal{Q}$ for each scalar γ such that $|\gamma|^{-1} \leq \min\{1, \|T^{-2}\|^{-1}\}$; equivalently, such that $|\gamma| \geq \max\{1, \|T^{-2}\|\}$. Therefore, applying Proposition 1 again, it follows that

$$\gamma\, T \in \mathcal{Q} \quad \text{whenever} \quad |\gamma| \geq \max\big\{1, \|T^{-2}\|\big\},$$

which completes the proof. □

If T is an invertible operator in $\mathcal{Q}$ and $\min\{1, \|T^2\|^{-1}\} = \max\{1, \|T^{-2}\|\}$, then the above corollary ensures that T is paranormal. In particular, if T is an invertible contraction in $\mathcal{Q}$ for which the above min and max coincide, then T is an invertible paranormal contraction; a unitary operator, actually, as we shall see in Proposition 3 below (*every invertible contraction for which the above* min *and* max *coincide is unitary*). Note that there exist invertible normaloid contractions in $\mathcal{Q}$ that are not unitary so that the above min and max do not coincide. For instance, a weighted bilateral shift with increasing positive weights in $(1/2, 1)$ is a nonunitary invertible hyponormal contraction, thus paranormal, and so a normaloid of class $\mathcal{Q}$.

Proposition 3. *If T is an invertible contraction and*

$$\min\{1, \|T^n\|^{-1}\} \;=\; \max\{1, \|T^{-n}\|\}$$

for some positive integer n, then T is unitary.

Proof. Take any positive integer n. If T is an invertible operator, then so is T^n. If $\|T\| \leq 1$, then $\|T^n\|^{-1} \geq 1$ and hence $\min\{1, \|T^n\|^{-1}\} = 1$. But $1 \leq \|T^{-n}\|\,\|T^n\|$, and so $\|T^{-n}\| \geq 1$, which implies that $\max\{1, \|T^{-n}\|\} = \|T^{-n}\|$. If min and max coincide, then $\|T^{-n}\| = 1$ and T^n is unitary (reason: $\|T^n\| \leq 1$, and an invertible operator U such that U and U^{-1} are both contractions must be unitary). But if T is a contraction and T^n is an isometry, then T is an isometry. Indeed, if T is a contraction, then so is $T^{(n-1)}$, which means that $T^{*(n-1)}T^{(n-1)} \leq I$, and therefore

$$I \;=\; T^{*n}T^n \;=\; T^*\big(T^{*(n-1)}T^{(n-1)}\big)T \;\leq\; T^*T \leq I$$

so that T is an isometry. Dually, if T is a contraction and T^n is a coisometry, then T is a coisometry. Thus, if T contraction and T^n unitary, then T unitary. □

Proposition 4. *Suppose T is an operator of class $\mathcal{Q}$.*

(a) *If T^2 is a contraction, then so is T.*

(b) *If T^2 is an isometry, then T is paranormal.*

Proof. Let $T \in \mathcal{B}[\mathcal{H}]$ be an operator of class $\mathcal{Q}$.

(a) Observe that T is of class $\mathcal{Q}$ if and only if

$$2(T^*T - I) \;\le\; T^{*2}T^2 - I.$$

Thus $T^{*2}T^2 \le I$ implies $T^*T \le I$; that is, T is a contraction whenever T^2 is.

(b) Take any x in $\mathcal{H}$ and note that T is of class $\mathcal{Q}$ if and only if

$$2\|Tx\|^2 \;\le\; \big(\|T^2x\| - \|x\|\big)^2 + 2\|T^2x\|\,\|x\|.$$

Hence $\|T^2x\| = \|x\|$ implies $\|Tx\|^2 \le \|T^2x\|\,\|x\|$, for every $x \in \mathcal{H}$. □

Therefore, *if T is an operator of class $\mathcal{Q}$ for which T^2 is an isometry, then T is a paranormal contraction.* Since $T^{*2}T^2 = I$ implies $Q = 2(I - T^*T)$, it follows that if T^2 is an isometry, then $T \in \mathcal{Q}$ if and only if T is a contraction and, in this case, T is paranormal. Note that the converses fail. For instance, the weighted unilateral shift $T = \text{shift}(2, \frac{1}{2}, 2, \frac{1}{2}, \cdots)$ is such that T^2 coincides with the square of the "unweighted" unilateral shift. Thus T^2 is an isometry, but T is not a contraction ($\|T\| = 2$), and hence $T \notin \mathcal{Q}$ by Proposition 4 (so that T is not paranormal — in fact, T is not even normaloid: $r(T) = 1$).

A part of an operator is a restriction of it to an invariant subspace. An operator T is *hereditarily normaloid* if every part of it is normaloid, and *totally hereditarily normaloid* if it is hereditarily normaloid and every invertible part of it has a normaloid inverse [3]. The class of all hereditarily normaloid operators from $\mathcal{B}[\mathcal{H}]$ is denoted by $\mathcal{HN}$, and the class of all totally hereditarily normaloid operators from $\mathcal{HN}$ is denoted by $\mathcal{THN}$. Recall that (see e.g., [4])

$$\mathcal{P} \;\subset\; \mathcal{THN} \;\subset\; \mathcal{HN} \;\subset\; \mathcal{N}.$$

Let $\mathcal{M}$ be any invariant subspace for T. Proposition 1 ensures that the following assertions hold true.

(a) *If $T \in \mathcal{Q} \cap \mathcal{HN}$, then $T|_{\mathcal{M}} \in \mathcal{Q} \cap \mathcal{HN}$.*

(b) *If $T \in \mathcal{Q} \cap \mathcal{THN}$ then $T|_{\mathcal{M}} \in \mathcal{Q} \cap \mathcal{THN}$ and, if $T|_{\mathcal{M}}$ is invertible, then $(T|_{\mathcal{M}})^{-1} \in \mathcal{Q} \cap \mathcal{N}$.*

Note that $T = I \oplus S$, with $S = \lambda\big(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\big)$ for any $\lambda \in (0, 1/\sqrt{2}]$, is a contraction in $(\mathcal{Q} \cap \mathcal{N}) \backslash \mathcal{HN}$. In fact, S is not normaloid so that T is not in $\mathcal{HN}$. There are two ways for an operator T to be in $\mathcal{THN}$: either $T \in \mathcal{HN}$ has no invertible part, or it has invertible parts and all of them have a normaloid inverse. The latter case prompts the question: are the invertible operators in $\mathcal{Q} \cap \mathcal{THN}$ paranormal? More generally, is it true that, if T is an invertible normaloid operator with a normaloid inverse, then $T \in \mathcal{Q}$ implies $T \in \mathcal{P}$? (i.e., $T \in \mathcal{Q}$ implies $\lambda T \in \mathcal{Q}$ for all $\lambda > 0$?)

3. An Invariant Subspace Theorem for Contractions of Class Q

Take any operator T in $\mathcal{B}[\mathcal{H}]$ and set $D = I - T^*T$. Recall that T is a contraction if and only if D is nonnegative. In this case, $D^{\frac{1}{2}}$ is the defect operator of T.

Proposition 5. *A contraction T lies in $\mathcal{Q}$ if and only if $\|D^{\frac{1}{2}}Tx\| \leq \|D^{\frac{1}{2}}x\|$ for every x in $\mathcal{H}$.*

Proof. For any $T \in \mathcal{B}[\mathcal{H}]$ put $Q = T^{2*}T^2 - 2\,T^*T + I$ and $D = I - T^*T$. Since

$$Q = D - T^*DT,$$

it follows that $O \leq Q$ if and only if $\langle T^*DTx\,;x\rangle \leq \langle Dx\,;x\rangle$ for every $x \in \mathcal{H}$ or, equivalently, $\|D^{\frac{1}{2}}Tx\|^2 \leq \|D^{\frac{1}{2}}x\|^2$ for every $x \in \mathcal{H}$ if T is a contraction. □

If a contraction T has no nontrivial invariant subspace, then D is a proper contraction. Indeed, if T is a contraction with no nontrivial invariant subspace, then $\ker(T) = \{0\}$ so that $\|D^{\frac{1}{2}}x\|^2 = \|x\|^2 - \|Tx\|^2 < \|x\|^2$ for every nonzero x in $\mathcal{H}$, which means that $D^{\frac{1}{2}}$ (and so D) is a proper contraction. If, in addition, T is of class $\mathcal{Q}$, then more is true.

Theorem 1. *If a contraction $T \in \mathcal{Q}$ has no nontrivial invariant subspace, then both T and Q are proper contractions.*

Proof. Let $T \neq O$ be a contraction of class $\mathcal{Q}$. Since $\ker(D) = \ker(D^{\frac{1}{2}})$, it follows by Proposition 5 that $\ker(D)$ is an invariant subspace for T. Suppose T has no nontrivial invariant subspace so that either $\ker(D) = \mathcal{H}$ or $\ker(D) = \{0\}$. In the former case $D = O$; that is, $T^*T = I$, and so T is an isometry, which is a contradiction: isometries have nontrivial invariant subspaces. In the latter case $D > O$; that is, $T^*T < I$, which means that T is a proper contraction. Moreover, If T is a contraction of class $\mathcal{Q}$, then the nonnegative operator Q is such that the power sequence $\{Q^n\}_{n\geq 1}$ converges strongly to P (i.e., $Q^n \xrightarrow{s} P$), where P is an orthogonal projection, and $TP = O$ so that $PT^* = O$ (P is self-adjoint) [6]. If T has no nontrivial invariant subspace, then T^* has no nontrivial invariant subspace as well. Since $\ker(P)$ is a nonzero invariant subspace for T^* whenever $PT^* = O$ and $T \neq O$, it follows that $\ker(P) = \mathcal{H}$. Hence $P = O$, and therefore $Q^n \xrightarrow{s} O$; that is, the nonnegative operator Q is strongly stable. But strong stability coincides with proper contractiveness for quasinormal operators [6]; in particular, for nonnegative operators. Thus Q also is a proper contraction. □

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