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THREE DECADES OF THE LOMONOSOV INVARIANT SUBSPACE THEOREM

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Abstract. In 1973 Lomonosov presented a remarkable breakthrough on the invariant subspace problem for compact operators: *If a nonscalar operator commutes with a nonzero compact operator, then it has a nontrivial hyperinvariant subspace.* To commemorate three decades of the Lomonosov Theorem, we give a brief review on it, where the Lomonosov technique is discussed in a step-by-step fashion. An extension of the Lomonosov Theorem, including further results on invariant subspaces for hyponormal operators acting on a Hilbert space, closes this expository paper.

1. Introduction

By an operator we mean a *bounded* (i.e., continuous) linear transformation of normed space into itself, and by a subspace of normed space we mean a *closed* linear manifold of it. Let $\mathcal{B}[\mathcal{X}]$ denote the algebra of all operators on a normed space $\mathcal{X}$. A subspace $\mathcal{M}$ of $\mathcal{X}$ is invariant for $T \in \mathcal{B}[\mathcal{X}]$ (or T-invariant) if $T(\mathcal{M}) \subseteq \mathcal{M}$. It is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{X}$. The *invariant subspace problem* is: Does every operator have a nontrivial invariant subspace? A subspace is hyperinvariant for T if it is invariant for every operator that commutes with T; that is, for every operator in the commutant $\{T\}' = \{L \in \mathcal{B}[\mathcal{X}] \colon LT = TL\}$ of T. A related open question is the *hyperinvariant subspace problem*: Does every nonscalar operator have a nontrivial hyperinvariant subspace? (An operator is scalar if it is a scalar multiple of the identity.)

On a finite-dimensional *complex* normed space every operator has an eigenvalue, and eigenspaces of nonscalar operators are nontrivial and hyperinvariant, so that every operator on a complex finite-dimensional normed space of dimension greater than 1 has a nontrivial invariant subspace (hyperinvariant, actually, if it is nonscalar). On the other hand, on a nonseparable normed space every operator also has a nontrivial invariant subspace. In fact, if T is an operator on a nonseparable normed space $\mathcal{X}$, then the (closed) span of the orbit of any nonzero vector x in $\mathcal{X}$ under T (i.e., $\bigvee\{T^n x\}_{n\geq 0}$) is a nonzero separable (spanned by a countable set) invariant subspace for T, and hence a nontrivial invariant subspace for T. The invariant subspace problem trivially has a negative answer in a real space. For instance, the operator $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $\mathbb{R}^2$ has no nontrivial invariant subspace (when acting on the Euclidean *real* space but, of course, it has a nontrivial invariant subspace when acting on the *complex* space $\mathbb{C}^2$). Thus the invariant subspace problem refers

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to *infinite-dimensional, complex and separable* spaces and, from now on, all spaces are supposed to be complex.

It is not clear exactly when the problem was originally (or formally) posed but, according to [1], von Neumann had shown that compact operators on a Hilbert space have nontrivial invariant subspaces back in the early thirties. In 1954, Aronszajan and Smith [1] proved that compact operators on a Banach space have nontrivial invariant subspaces. It is worth noticing that, according to the Fredholm Alternative (if K is a compact operator on a Banach space, then $\sigma(K)\backslash\{0\} = \sigma_P(K)\backslash\{0\}$, where $\sigma(K)$ and $\sigma_P(K)$ denote spectrum and point spectrum, respectively), the significant result contained in the above paper refers to quasinilpotent compact operators without eigenvalues — if a compact operator is not quasinilpotent, then it has an eigenvalue. (An operator T is quasinilpotent if $\sigma(T) = \{0\}$ or, equivalently, if $r(T) = 0$, where $r(T)$ stands for the spectral radius of T.) A typical example of a compact quasinilpotent operator without eigenvalues is the Volterra operator $(Kx)(s) = \int_0^s x(t)\,dt$ on $L^2([0,1])$.

The question of the existence of nontrivial invariant subspace for polynomially compact operators (i.e., operators T such that $p(T)$ is compact for some nonzero polynomial p) was publicized by Halmos in [11]. This was proved by Bernstein and Robinson [5] in 1966 who used nonstandard analysis and, subsequently, the proof was translated into standard analysis by Halmos in [12].

The next significant step was the remarkable breakthrough of Lomonosov [19] in 1973, which will be the subject of the next section. Before moving on to the Lomonosov Theorem, let us comment on two pertinent points.

The first one refers to the space upon which the operators act. The invariant subspace problem has a negative answer in a Banach space. This remained as an open question for a long period, but constructions of Banach space operators without a nontrivial invariant subspace have been published between 1984 and 1987 by Enflo [10] and Reed [25] (an operator on ℓ^1 free of nontrivial invariant subspaces was exhibited in [26]). However, such constructions are all on nonreflexive Banach spaces; on a Hilbert space (infinite-dimensional, complex and separable), the invariant subspace problem remains a recalcitrant open question.

The second point refers to classes of Hilbert space operators. Recall that an operator T on a Hilbert space $\mathcal{H}$ is quasinormal if it commutes with T^*T, subnormal if it is a part a normal operator (i.e., restriction of a normal operator to an invariant subspace), hyponormal if $TT^* \leq T^*T$, and normaloid if $r(T) = \|T\|$. These classes are related by proper inclusion (Normal $\subset$ Quasinormal $\subset$ Subnormal $\subset$ Hyponormal $\subset$ Normaloid) if $\mathcal{H}$ is infinite-dimensional (otherwise hyponormal operators are normal). Normal operators (on a complex Hilbert space of dimension greater than 1) have a nontrivial invariant subspace (after the Spectral Theorem), and so does the quasinormal operators. That every subnormal operator has a nontrivial invariant subspace is a deep result proved by S. Brown [6] in 1978. However, it is still unknown whether every hyponormal operator has a nontrivial invariant subspace. Since a hyponormal operator is normaloid, the invariant subspace is, obviously, left open for normaloid operators as well. In fact, as pointed out in [2], p.339, every Banach space operator without a nontrivial invariant subspace that had been exhibited up to then, and for which the spectral radius had been computed, was nonnormaloid. We can think of quasinilpotent operators as a class lying opposite to the normaloid ones. It turns out that the invariant subspace also remains unanswered for quasinilpotent operators on a Hilbert space (but not for quasinilpotent operators on a nonreflexive Banach space [27]).

A classical reference for the invariant subspace problem is [23] (also see [2], [14] and [21]). For a select collection of expository and survey papers dealing with many aspects of the problem see e.g., [13], [22], [24], [28] and [30].

2. The Lomonosov Theorem

Compact operators (on a complex Banach space of dimension greater than 1) have a nontrivial invariant subspace; a nontrivial hyperinvariant subspace, actually, if it is nonscalar. The definitive result in this line is due to Lomonosov [19]: *An operator has a nontrivial invariant subspace if it commutes with a nonscalar operator that commutes with a nonzero compact operator.* In fact, *every nonscalar operator that commutes with a nonscalar compact operator* (itself, in particular) *has a nontrivial hyperinvariant subspace.* Recall that on an infinite-dimensional normed space the only scalar compact operator is the null operator; on a finite-dimensional normed space every operator is compact.

The Lomonosov Theorem is a remarkable breakthrough on the invariant subspace problem. The full version of it, mentioned above, can be split into two parts:

(i) *If an operator commutes with a nonzero compact operator, then it has a nontrivial invariant subspace;*

(ii) *if it is nonscalar, then it has a nontrivial hyperinvariant subspace.*

Part (i) is a slightly weaker version of the full statement, whose proof is known as Hilden's Proof of Lomonosov's Theorem [20]. We shall sketch the proof of part one.

A Sketch of Hilden's Proof. Take a nonzero compact operator K on a complex Banach space $\mathcal{X}$ of dimension greater than 1. Let T be an operator on $\mathcal{X}$ that commutes with K. If T has no nontrivial invariant subspace, then the following assertions hold.

(a) K is quasinilpotent, and hence αK is uniformly stable for all $\alpha \in \mathbb{C}$.

(b) For each nonzero vector x in $\mathcal{X}$ and each nonempty open subset U of $\mathcal{X}$, there exists a nonzero polynomial p such that $p(T)x \in U$.

Assertion (a) is a consequence of the Fredholm alternative for compact operators and the Beurling–Gelfand formula for the spectral radius ($r(T) = \lim_n \|T^n\|^{\frac{1}{n}}$ for every operator T). Assertion (b) is readily verified by recalling that if an operator T has no nontrivial invariant subspace, then every $x \neq 0$ is a cyclic vector (i.e., $\bigvee\{T^n x\}_{n\geq 0} = \{p(T)x \in \mathcal{X}\colon\ p \text{ is a nonzero polynomial}\}^- = \mathcal{X}$). Both (a) and (b) rely on the hypothesis that T has no nontrivial invariant subspace. The program is to show that this assumption, through assertions (a) and (b), leads to a contradiction. Take any vector x_0 in $\mathcal{X}$ such that the origin is not in the closed ball $B_1[x_0] = \{x \in \mathcal{X}\colon\ \|x - x_0\| \leq 1\}$ nor in the closure of its image under K;

$$0 \notin B_1[x_0] \quad \text{and} \quad 0 \notin K(B_1[x_0])^-.$$

Since $K \neq O$, this happens for any vector $x_0 \in \mathcal{X}$ such that $\|K\| < \|Kx_0\|$. Indeed, if 0 lies in $B_1[x_0]$, then $\|x_0\| \leq 1$, and so $\|Kx_0\| \leq \|K\|$. If 0 lies in the closure $K(B_1[x_0])^-$ of $K(B_1[x_0])$ (i.e., if 0 is a point of adherence of $K(B_1[x_0])$), then there is a sequence $\{x_n\}$ in $B_1[x_0]$ such that $Kx_n \to 0$, and $\|Kx_0\| = \lim_n \|K(x_n - x_0)\| \leq$

$\|K\| \limsup_n \|x_n - x_0\| \le \|K\|$. Thus take any x_0 in $\mathcal{X}$ that satisfies the above properties and, for each nonzero polynomial p, set

$$U_p(x_0) = \big\{x \in \mathcal{X}\colon\; \|p(T)x - x_0\| < 1\big\}.$$

Assertion (b) ensures that

(c) $U_p(x_0)$ is open in $\mathcal{X}$ and every nonzero vector of $\mathcal{X}$ lies in $U_p(x_0)$ for some p.

Since K is compact and $0 \notin K(B_1[x_0])^-$, assertion (c) implies the next one.

(d) There exists a finite set $\mathcal{P}$ of polynomials such that, if $x \in K(B_1[x_0])$, then $p(T)x$ lies in $B_1(x_0)$ for some $p \in \mathcal{P}$.

Finally, by using (d) and recalling that T commutes with K, we can show that assertion (e) below holds true.

(e) There is a sequence $\{p_k\}$ of polynomials in $\mathcal{P}$ such that, for each $n \ge 1$,

$$p_1(T) \dots p_n(T) K^n x_0 \;\in\; B_1[x_0].$$

Summing up: Assertion (b) ensures the existence of a sequence $\{x_n\}$ of vectors in $B_1[x_0]$ such that $x_n = p_1(T) \dots p_n(T) K^n x_0$ for every $n \ge 1$, with each p_k in $\mathcal{P}$, where $\mathcal{P}$ is a finite set of polynomials. Put $\alpha = \max_{p \in \mathcal{P}} \|p(T)\|$. On the other hand, assertion (a) ensures that $\|(\alpha K)^n\| \to 0$. Therefore,

$$\|x_n\| = \|p_1(T) \dots p_n(T) K^n x_0\| \le \alpha^n \|K^n x_0\| \le \|(\alpha K)^n\| \|x_0\| \to 0,$$

and the $B_1[x_0]$-valued sequence $\{x_n\}$ converges to 0. Since $B_1[x_0]$ is closed in $\mathcal{X}$, the Closed Set Theorem ensures that $0 \in B_1[x_0]$. But this contradicts the fact that $0 \notin B_1[x_0]$. Conclusion: *If an operator T commutes with a nonzero compact K, then T has a nontrivial invariant subspace,* which completes the proof of part (i) of the Lomonosov Theorem. □

Besides very basic functional analysis, Hilden's Proof of Lomonosov's Theorem uses only two elementary results from operator theory, viz., Fredholm alternative and Beurling–Gelfand formula. A proof of the full version of the Lomonosov Theorem requires more than that.

First recall that the convex hull of a subset G of a linear space $\mathcal{X}$, denoted by $\mathrm{co}(G)$, is the intersection of all convex sets containing G; that is, $\mathrm{co}(G)$ is the smallest (in the inclusion ordering) convex set that contains G. Also recall that $\mathrm{co}(G)$ coincides with the set of all convex linear combinations of vectors in G; that is, $x \in \mathrm{co}(G)$ if and only if $x = \sum_{i=1}^n \alpha_i x_i$ for some finite set $\{x_i\}_{i=1}^n$ of vectors in G and some finite set of positive scalars $\{\alpha_i\}_{i=1}^n$ such that $\sum_{i=1}^n \alpha_i = 1$. Clearly, $\mathrm{co}(G)^- \subseteq \mathrm{co}(G^-)^-$ for every subset G of a normed space $\mathcal{X}$. A classical result on the geometry of Banach spaces is the Mazur Theorem.

Mazur Theorem. *The closure of the convex hull of every compact subset of a Banach space is compact.*

That is, if C is a compact set in a Banach space $\mathcal{X}$, then so is $\mathrm{co}(C)^-$. In fact, this can be readily verified by showing that $\mathrm{co}(C)$ is totally bounded whenever C is (see e.g., [7], p.180).

Now we borrow the notion of *compact mapping* from nonlinear functional analysis. Let D be a nonempty subset of a normed space $\mathcal{X}$. A mapping $F\colon D \to \mathcal{X}$ is compact if it is continuous and $F(B)^-$ is a compact set in $\mathcal{X}$ whenever B is a bounded subset of D. Recall that a continuous image of any compact set is a compact set. Thus, if D is a compact subset of $\mathcal{X}$, then every continuous mapping $F\colon D \to \mathcal{X}$ is compact. We shall, however, be concerned with the case where D (the domain of F) is not compact but is bounded. In this case, if F is continuous and $F(D)^-$ is a compact set, then F is a compact mapping (because $F(B)^- \subseteq F(D)^-$ whenever $B \subseteq D$). The central result required for proving the full version of the Lomonosov Theorem is the Schauder Fixed Point Theorem (see e.g., [7], p.150), which reads as follows.

Schauder Fixed Point Theorem. *Let D be a closed, bounded and convex subset of a normed space $\mathcal{X}$, and let $F\colon D \to \mathcal{X}$ be a compact mapping. If D is F-invariant, then F has a fixed point* (i.e., *if $F(D) \subseteq D$, then there exists $x \in D$ such that $F(x) = x$*).

Next take the algebra $\mathcal{B}[\mathcal{X}]$ of all operators on a normed space $\mathcal{X}$ and let $\mathcal{A}$ be a unital subalgebra of $\mathcal{B}[\mathcal{X}]$, which means that $\mathcal{A}$ is a linear manifold of $\mathcal{B}[\mathcal{X}]$ that contains the identity and is such that AB lies in $\mathcal{A}$ whenever A and B lie in $\mathcal{A}$. A subspace $\mathcal{M}$ of $\mathcal{X}$ is invariant for $\mathcal{A}$ (or is $\mathcal{A}$-invariant) if it is invariant for every operator in $\mathcal{A}$ (i.e., if $A(\mathcal{M}) \subseteq \mathcal{M}$ for every $A \in \mathcal{A}$). We say that $\mathcal{A}$ has no nontrivial invariant subspace if there is no nontrivial subspace of $\mathcal{X}$ that is invariant for every operator in $\mathcal{A}$. An intermediate stage towards a proof of the full version of the Lomonosov Theorem is the so-called Lomonosov Lemma.

Lomonosov Lemma. *Let K be a nonzero compact operator on a complex Banach space $\mathcal{X}$ and let $\mathcal{A}$ be a unital subalgebra of $\mathcal{B}[\mathcal{X}]$. If there is no nontrivial invariant subspace of $\mathcal{X}$ that is invariant for every operator in $\mathcal{A}$, then there exists L in $\mathcal{A}$ such that 1 is an eigenvalue of LK* (i.e., *such that* $\ker(I - LK) \neq \{0\}$).

Below we sketch a step by step proof for the Lomonosov Lemma. The first steps actually prepare the ground for an application of the Schauder Fixed Point Theorem.

Proof of the Lomonosov Lemma. Let $\mathcal{X}$ be a complex Banach space and let $\mathcal{A}$ be a unital subalgebra of $\mathcal{B}[\mathcal{X}]$ that has no nontrivial invariant subspace. In this case we can show that the following assertion holds.

(a) For each nonzero vector x in $\mathcal{X}$ and each nonempty open subset U of $\mathcal{X}$ there exists an operator A in $\mathcal{A}$ such that $Ax \in U$.

Let K be a nonzero compact operator on $\mathcal{X}$. Take $x_0 \in \mathcal{X}$ as in the previous proof so that $0 \notin B_1[x_0]$ and $0 \notin K(B_1[x_0])^-$. For each operator A in $\mathcal{A}$ set

$$U_A(x_0) = \big\{x \in \mathcal{X}\colon\ \|Ax - x_0\| < 1\big\}.$$

Assertion (a) ensures that

(b) $U_A(x_0)$ is open in $\mathcal{X}$ and every nonzero vector of $\mathcal{X}$ lies in $U_A(x_0)$ for some operator A in $\mathcal{A}$.

Since K is compact and $0 \notin K(B_1[x_0])^-$, assertion (b) implies the next one.

(c) There exists a finite subset $\mathcal{F}$ of $\mathcal{A}$ such that, if $x \in K(B_1[x_0])$, then Ax lies in $B_1(x_0)$ for some $A \in \mathcal{F}$.

For each A in $\mathcal{F}$ consider the function $\alpha_A\colon K(B_1[x_0]) \to \mathbb{R}$ given by

$$\alpha_A(x) = \max\{0, 1 - \|Ax - x_0\|\} \quad \text{for every} \quad x \in K(B_1[x_0]).$$

Take an arbitrary $x \in K(B_1[x_0])$. Assertion (c) ensures that there exists A in $\mathcal{F}$ such that $\|Ax - x_0\| < 1$, and hence $0 < \alpha_A(x)$ for some A in $\mathcal{F}$. Therefore, since $0 \le \alpha_A(x)$ for every $x \in K(B_1[x_0])$ and each $A \in \mathcal{F}$, we get $0 < \sum_{A\in\mathcal{F}} \alpha_A(x) < \infty$ for every $x \in K(B_1[x_0])$. Put

$$\beta_A(x) = \frac{\alpha_A(x)}{\sum_{A\in\mathcal{F}} \alpha_A(x)} \quad \text{for each} \quad x \in K(B_1[x_0]),$$

which defines a function $\beta_A\colon K(B_1[x_0]) \to \mathbb{R}$. Now let $F\colon B_1[x_0] \to \mathcal{X}$ be a mapping defined by

$$F(x) = \sum_{A\in\mathcal{F}} \beta_A(Kx)AKx \quad \text{for every} \quad x \in B_1[x_0].$$

It is readily verified that F is continuous. Using the Mazur Theorem we can show that $F(B_1[x_0])^-$ is compact. Therefore,

(d) F is a compact mapping.

Moreover, it is also easy to show that $F(B_1[x_0]) \subseteq B_1[x_0]$. That is,

(e) F is $B_1[x_0]$-invariant.

Recall that $0 \notin B_1[x_0]$ and apply the Schauder Fixed Point Theorem to conclude that there exists x in $B_1[x_0]$ and $L = \sum_{A\in\mathcal{F}} \beta_A(Kx)A$ in $\mathcal{A}$ such that

(f) $LKx = F(x) = x \neq 0$ (i.e., $\ker(I - LK) \neq \{0\}$). □

The proof of part (i) of the Lomonosov Theorem uses only elementary results of operator theory. In order to prove part (ii), thus completing the full version of it, the Lomonosov Lemma is called forth. The full version of the Lomonosov Theorem can be rephrased as follows.

Lomonosov Theorem. *If a nonscalar operator commutes with a nonzero compact operator, then it has a nontrivial hyperinvariant subspace.*

Take an operator T on a complex Banach space $\mathcal{X}$. Let $\{T\}'$ be the commutant of T, which is the unital subalgebra of $\mathcal{B}[\mathcal{X}]$ consisting of all operators in $\mathcal{B}[\mathcal{X}]$ that commute with T. Recall that a nontrivial hyperinvariant subspace for T is a nontrivial subspace of $\mathcal{X}$ that is invariant for every operator in $\{T\}'$.

Proof of the Lomonosov Theorem. Let T be an operator acting on a complex Banach space $\mathcal{X}$. Suppose there exists a nonzero compact operator K in $\{T\}'$, and suppose T has no nontrivial hyperinvariant subspace. Setting $\mathcal{A} = \{T\}'$, the following assertion holds as a consequence of the Lomonosov Lemma.

(a) There exists an operator L in $\{T\}'$ such that $\ker(I - LK)$ is nonzero and T-invariant.

Using the above result it can be shown that

(b) T has an eigenvalue (i.e., there is a $\lambda \in \mathbb{C}$ such that $\ker(\lambda I - T) \neq \{0\}$).

But $\ker(\lambda I - T)$ is a hyperinvariant subspace for T. Therefore, if T has no nontrivial hyperinvariant subspace, then $\ker(\lambda I - T) = \mathcal{X}$. Equivalently, $T = \lambda I$; that is, T is scalar. Summing up: *If an operator T has no nontrivial hyperinvariant subspace and commutes with a nonzero compact K, then T must be scalar.* □

For detailed proofs of each assertion in this section the reader is referred to [7] (Section VI.4), [18] (Chapter 12), [21] (Chapter 7), or [22].

3. An Extension of the Lomonosov Theorem

Recall that rank means dimension of range. Thus T commutes with K if and only if $\operatorname{rank}(KT - TK) = 0$. For simplicity, throughout this section all operators are assumed to act on a Hilbert space.

Proposition 1. *Let T, S and K be operators acting on a complex Hilbert space.*

(a) *If* $0 < \dim(\ker S) < \infty$, $0 < \dim(\ker S^*) < \infty$ *and* $\operatorname{rank}(TS - ST) = 1$, *then* $\sigma_P(T) \cup \sigma_P(T^*) \neq \varnothing$, *and therefore*

(b) *if K is compact and* $\operatorname{rank}(KT - TK) = 1$, *then T has a nontrivial hyperinvariant subspace.*

If $\operatorname{rank}(KT - TK) = 1$), then K is nonzero and T is nonscalar. Thus, combining the Lomonosov Theorem with the above proposition yields the following useful extension of the Lomonosov Theorem.

Extension of the Lomonosov Theorem. *If a nonscalar operator T is such that* $\operatorname{rank}(KT - TK) \leq 1$ *for some nonzero compact operator K, then T has a nontrivial hyperinvariant subspace.*

The above results were stated in a Hilbert space setting — for a proof see [18] (Chapter 12). Let us just mention that they hold in a Banach space setting as well. Their proofs in a Banach space $\mathcal{X}$ are based on the quotient space $\mathcal{X}/\operatorname{ran}(S)^-$ rather than on the orthogonal complement $\mathcal{H} \ominus \operatorname{ran}(S)^- = \operatorname{ran}(S)^\perp = \ker(S^*)$ in a Hilbert space $\mathcal{H}$ (see [9], [15] and [16]).

It is worth noticing that if one could improve the Extension of the Lomonosov Theorem by replacing $\operatorname{rank}(KT - TK) \leq 1$ with $\operatorname{rank}(KT - TK) \leq 2$, then one would have solved affirmatively the hyperinvariant subspace problem. Indeed, for every operator T there exists a nonzero compact K such that $\operatorname{rank}(KT - TK) \leq 2$ (reason: this always holds whenever K is a rank-1 operator).

Let $\mathcal{H}$ be a complex Hilbert space of dimension greater than 1. Recall that an operator T on $\mathcal{H}$ essentially normal if it has a compact self-commutator (i.e., if $D_T = T^*T - TT^*$ is compact). It is called quasireducible if there exists a nonscalar operator L in $\{T\}'$ such that $\operatorname{rank}(D_T L - L D_T) \leq 1$. Every quasinormal operator is quasireducible, but there exist subnormal (thus hyponormal) operators that are not quasireducible. The next result is a corollary of the above extension of the Lomonosov Theorem [17].

Proposition 2. *Every essentially normal quasireducible operator has a nontrivial invariant subspace.*

An important result on hyponormal operators is the Berger–Shaw Theorem [3], [4] (also see [8], p.152). A consequence of it is that if a hyponormal operator has no nontrivial invariant subspace, then its self-commutator D_T is compact (trace-class, actually). That is, if there exists a hyponormal operator without a nontrivial invariant subspace, then it is essentially normal. Combining this with the above proposition, it follows that the invariant subspace problem for hyponormal operators is restricted to the class of nonquasireducible hyponormal operators [17].

Proposition 3. *Quasireducible hyponormal operators have a nontrivial invariant subspace.*

Remark. The fact that *a nonquasireducible subnormal operator has a nontrivial invariant subspace* is a trivial corollary of the S. Brown Theorem (every subnormal operator has a nontrivial invariant subspace — recall that there exist nonquasireducible subnormal operators). However, an independent proof of the above italicized result would lead to a new proof for the S. Brown Theorem (via Proposition 3), which would be a consequence of the Lomonosov and Berger–Shaw Theorems.

We have touched in one type of extension of the Lomonosov Theorem (based on [9], [15] and [16]). For further results considering extensions in various directions, the reader is referred to [21], [22], [29], [31] and the references therein.

References

1. N. Aronszajan and K.T. Smith, *Invariant subspaces of completely continuous operators*, Ann. Math. **60** (1954), 345–350.
2. B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam, 1988.
3. C.A. Berger and B.I. Shaw, *Selfcommutators of multicyclic hyponormal operators are always trace class*, Bull. Amer. Math. Soc. **79** (1973), 1193–1199.
4. C.A. Berger and B.I. Shaw, *Intertwining, analytic structures, and the trace norm estimate*, Proceedings of a Conference on Operator Theory, Halifax, 1973, Lecture Notes in Math., Vol. 345, Springer, Berlin, 1973, 1–6.
5. A.R. Bernstein and A. Robinson, *Solution of the invariant subspace problem of K.T. Smith and P.R. Halmos*, Pacific J. Math. **16** (1966), 421–431.
6. S.W. Brown, *Some invariant subspaces for subnormal operators*, Integral Equations Operator Theory **1** (1978), 310–333.
7. J.B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer, New York, 1990.
8. J.B. Conway, *The Theory of Subnormal Operators*, Mathematical Surveys and Monographs, Vol. 36, Amer. Math. Soc., Providence, 1991.
9. J. Daughtry, *An invariant subspace theorem*, Proc. Amer. Math. Soc. **49** (1975), 267–268.
10. P. Enflo, *On the invariant subspace problem for Banach spaces*, Acta Math. **158** (1987), 213–313.
11. P.R. Halmos, *A glimpse into Hilbert space*, Lectures on Modern Mathematics I, Wiley, New York, 1963, 1–22.
12. P.R. Halmos, *Invariant subspaces of polynomially compact operators*, Pacific J. Math. **16** (1966) 433–437.
13. P.R. Halmos, *Invariant subspaces 1969*, 7º Colóquio Brasileiro de Matemática, Poços de Caldas, 1969, (also see P.R. Halmos, *Selecta Research Contributions*, Springer, New York, 1983, 249–273).
14. H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.
15. H.W. Kim, C. Pearcy and A.L. Shields, *Rank-one commutators and hyperinvariant subspaces*, Michigan Math. J. **22** (1975), 193–194.
16. H.W. Kim, C. Pearcy and A.L. Shields, *Sufficient conditions for rank-one commutators and hyperinvariant subspaces*, Michigan Math. J. **23** (1976), 235–243.
17. C.S. Kubrusly, *Quasireducible operators*, Internat. J. Math. Math. Sci. **2003:31** (2003), 1993–2002.
18. C.S. Kubrusly, *Hilbert Space Operators*, Birkhäuser, Boston, 2003.
19. V.I. Lomonosov, *Invariant subspaces for the family of operators which commute with a completely continuous operator*, Functional Anal. Appl. **7** (1973), 213–214.
20. A.J. Michaels, *Hilden's simple proof of Lomonosov's invariant subspace theorem*, Adv. Math. **25** (1977), 56–58.
21. C.M. Pearcy, *Some Recent Developments in Operator Theory*, CBMS Regional Conference Series in Mathematics no. 36, Amer. Math. Soc., Providence, 1978.
22. C. Pearcy and A. Shields, *A survey of the Lomonosov technique in the theory of invariant subspaces*, Topics in Operator Theory, Mathematical Surveys no. 13, Amer. Math. Soc., Providence, 2nd pr., 1979, 219-230.
23. H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer, Berlin, 1973.

24. H. Radjavi and P. Rosenthal, *The invariant subspace problem*, Math. Intelligencer **4** (1982) 33–37.
25. C.J. Read, *A solution to the invariant subspace problem*, Bull. London Math. Soc. **16** (1984), 337–401.
26. C.J. Read, *A solution to the invariant subspace problem on the space* ℓ_1, Bull. London Math. Soc. **17** (1985), 305–307.
27. C.J. Read, *Quasinilpotent operators and the invariant subspace problem*, J. London Math, Soc. **56** (1997), 595–606.
28. P. Rosenthal, *Equivalents of the invariant subspace problem*, Paul Halmos: Celebrating 50 Years of Mathematics, Springer, New York, 1991, 179–188.
29. P. Rosenthal, (Featured Review) *Quasinilpotent operators and the invariant subspace problem* by C.J. Read, Mathematical Reviews (1998), 98m:47004.
30. D. Sarason, *Invariant subspaces*, Topics in Operator Theory, Mathematical Surveys no. 13, Amer. Math. Soc., Providence, 2nd pr., 1979, 1–47.
31. A. Simonič, *An extension of the Lomonosov techniques to non-compact operators*, Trans. Amer. Math. Soc. **348** (1996), 975–995.

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