Acta Scientiarum Mathematicarum (Szeged) **71** (2005) 337–352

HEREDITARILY NORMALOID CONTRACTIONS

B.P. DUGGAL, S.V. DJORDJEVIĆ, AND C.S. KUBRUSLY

Abstract. A Hilbert space operator $T \in \mathcal{B}[\mathcal{H}]$ is said to be *totally hereditarily normaloid,* or $\mathcal{THN}$, if for every T-invariant subspace $\mathcal{M} \subseteq \mathcal{H}$ the restriction $T|_{\mathcal{M}}$ of T to $\mathcal{M}$ is normaloid and, whenever $T|_{\mathcal{M}} \in \mathcal{B}[\mathcal{M}]$ is invertible, the inverse $(T|_{\mathcal{M}})^{-1}$ is normaloid as well. In this paper we explore the structure of $\mathcal{THN}$ contractions, and conclude some properties which follow from such a structure, specially for $\mathcal{THN}$ contractions with either compact or Hilbert–Schmidt defect operators.

1. Introduction

Let $\mathcal{H}$ be a (complex nonzero) Hilbert space. By a subspace of $\mathcal{H}$ we mean a closed linear manifold of $\mathcal{H}$ (which is again a Hilbert space), and by an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. Let $\mathcal{B}[\mathcal{H}]$ be the algebra of all operators on $\mathcal{H}$. A subspace $\mathcal{M}$ of $\mathcal{H}$ is invariant for $T \in \mathcal{B}[\mathcal{H}]$ if $T(\mathcal{M}) \subseteq \mathcal{M}$. A part of an operator $T \in \mathcal{B}[\mathcal{H}]$ is a restriction $T|_{\mathcal{M}} \in \mathcal{B}[\mathcal{M}]$ of it to an invariant subspace $\mathcal{M}$. Recall that an operator T is a contraction if $\|T\| \leq 1$, a strict contraction if $\|T\| < 1$, power bounded if $\sup_n \|T^n\| < \infty$, normaloid if the spectral radius $r(T)$ coincides with the norm $\|T\|$, and invertible if it has a bounded inverse. Let $\mathbb{D}$ denote the open unit disc, $\mathbb{D}^-$ the closed unit disc, and $\partial\mathbb{D}$ the unit circle.

Definition 1. An operator is *hereditarily normaloid* if every part of it is normaloid. Let $\mathcal{HN}$ denote the class of all hereditarily normaloid operators from $\mathcal{B}[\mathcal{H}]$. An operator is *totally hereditarily normaloid* if every part of it is normaloid and every invertible part of it has a normaloid inverse. Let $\mathcal{THN}$ denote the class of all totally hereditarily normaloid operators from $\mathcal{B}[\mathcal{H}]$.

The classes $\mathcal{HN}$ and $\mathcal{THN}$ were introduced in [5] where it was proved that, if $T \in \mathcal{THN}$, then Weyl's theorem holds for both T and its adjoint T^*. (Recall that, according to usual terminology, Weyl's theorem is said to hold for an operator T if $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$, where $\sigma(T)$ is the spectrum of T, $\sigma_w(T)$ is the set of all $\lambda \in \mathbb{C}$ for which $(\lambda I - T)$ is not Fredholm of index zero, and $\pi_{00}(T)$ is the set of all isolated eigenvalues of T of finite multiplicity.) Observe that $\mathcal{THN}$ is closed under nonzero scaling ($\alpha T \in \mathcal{THN}$ for every $\alpha \neq 0$ whenever $T \in \mathcal{THN}$), and hence it is sufficient to investigate contractions in $\mathcal{THN}$. It is also worth noticing that if T is in $\mathcal{THN}$ then so is every part $T|_{\mathcal{M}}$ of it.

This paper considers contractions T in $\mathcal{THN}$. $\mathcal{C}_{11}$-contractions, and those with defect operator $D_T = (I - T^*T)^{\frac{1}{2}}$ either compact or Hilbert–Schmidt, are considered in Propositions 5 to 11. The main result shows that if D_T is Hilbert–Schmidt and normal subspaces of T (i.e., subspaces $\mathcal{M}$ such that $T|_{\mathcal{M}}$ is normal) reduce

Date: December 27, 2004.
2000 *Mathematics Subject Classification.* 47A45, 47B20.
Key words and phrases. Hereditarily normaloid, contractions, defect operator, decompositions.

T, then T is the direct sum of a unitary operator, a normal $\mathcal{C}_0$-contraction and a $\mathcal{C}_{10}$-contraction (with no invertible parts). Moreover, still under the assumptions that the defect operator D_T is Hilbert–Schmidt and normal subspaces of T reduce T, we also show that (a) if T has no normal direct summand, then T is reflexive, and (b) if $\dim(\ker(\lambda I - T^*)) \neq 0$ for every λ in $H \subset \mathbb{D}$ with $\sum_{\lambda\in H}(1-|\lambda|) = \infty$, then T has the bicommutant property.

$\mathcal{THN}$ is quite a large class. For instance, recall that an operator $T \in \mathcal{B}[\mathcal{H}]$ is quasinormal if it commutes with T^*T, subnormal if it is a part of a normal operator, hyponormal if $TT^* \leq T^*T$, and paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every $x \in \mathcal{H}$. These are related by proper inclusion (Normal $\subset$ Quasinormal $\subset$ Subnormal $\subset$ Hyponormal $\subset$ Paranormal) if $\mathcal{H}$ is infinite-dimensional (otherwise they all coincide with the class of normal operators). The classes $\mathcal{THN}$ and $\mathcal{HN}$ lie properly between the paranormal and normaloid operators:

$$\text{Paranormal} \subset \mathcal{THN} \subset \mathcal{HN} \subset \text{Normaloid}.$$

Indeed, paranormal operators are normaloid, a part of a paranormal is again paranormal, and so is the inverse of any invertible paranormal. (See [11] for a detailed discussion of theses classes.) In this introductory section we shall pose a few basic properties that will be required in the sequel and will help to situate the classes $\mathcal{THN}$ and $\mathcal{HN}$ in their due place.

Remark 1. The above inclusions are, in fact, proper inclusions. For instance, the direct sum $T = S \oplus Q$ of a normaloid nonstrict contraction S ($r(S) = \|S\| = 1$) with a quasinilpotent nonzero contraction Q ($r(Q) = 0 \neq \|Q\| \leq 1$) is a normaloid nonstrict contraction ($r(T) = \|T\| = 1$) but is not in $\mathcal{HN}$ (because Q is not normaloid). Sample: $T = 1 \oplus \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ *is a normaloid not in* $\mathcal{HN}$. Examples of operators in $\mathcal{HN}$ but not in $\mathcal{THN}$, and also of operators in $\mathcal{THN}$ that are not paranormal will come out below in Remarks 3 and 5 with the help of Propositions 1 and 4.

For any contraction $T \in \mathcal{B}[\mathcal{H}]$ the sequence of nonnegative numbers $\{\|T^nx\|\}$ is decreasing (thus convergent) for every $x \in \mathcal{H}$. A contraction T is of class $\mathcal{C}_{0\cdot}$ if it is strongly stable; that is, if $\{\|T^nx\|\}$ converges to zero for every $x \in \mathcal{H}$, and of class $\mathcal{C}_{1\cdot}$ if $\{\|T^nx\|\}$ does not converge to zero for every nonzero $x \in \mathcal{H}$. It is of class $\mathcal{C}_{\cdot 0}$ or of class $\mathcal{C}_{\cdot 1}$ if its adjoint T^* is of class $\mathcal{C}_{0\cdot}$ or $\mathcal{C}_{1\cdot}$, respectively. All combinations are possible, leading to classes $\mathcal{C}_{00}$, $\mathcal{C}_{01}$, $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$. An operator T is uniformly stable if $\{\|T^n\|\}$ converges to zero. Recall that T is uniformly stable if and only if $r(T) < 1$, and uniform stability for T trivially implies $T \in \mathcal{C}_{00}$.

Proposition 1. *Every $\mathcal{C}_{1\cdot}$-contraction is a nonstrict contraction in $\mathcal{HN}$.*

Proof. Every nonzero part of a $\mathcal{C}_{1\cdot}$-contraction is again a $\mathcal{C}_{1\cdot}$-contraction (reason: if $\mathcal{M} \neq \{0\}$ is T-invariant, then a trivial induction ensures that $(T|_{\mathcal{M}})^n u = T^n u$ for all $u \in \mathcal{M}$ and every $n \geq 1$). Now recall that *every contraction not in class $\mathcal{C}_{00}$ is normaloid and nonstrict.* Indeed, if a contraction T is not in $\mathcal{C}_{00}$, then it is not uniformly stable so that $r(T) \geq 1$. Therefore, $1 \leq r(T) \leq \|T\| \leq 1$. $\square$

Remark 2. Every isometry lies in $\mathcal{THN}$ since isometries are quasinormal. However, in light of the above proposition, it is worth noticing that an isometry is a particular case of $\mathcal{C}_{1\cdot}$-contraction in $\mathcal{THN}$; and this happens because completely nonunitary isometries have no invertible parts. Indeed, a part of an isometry is again an isometry, and an invertible isometry is precisely a unitary operator. Therefore, if an isometry is completely nonunitary (i.e., has no unitary direct summand), then it is a unilateral shift (after the von Neumann–Wold decomposition), which is not invertible. This exhibits a $\mathcal{C}_{10}$-contraction in $\mathcal{THN}$: a unilateral shift is a $\mathcal{C}_{10}$-contraction such that every part of it is again a unilateral shift (which are noninvertible isometries), and so it has no invertible part; thus lying in $\mathcal{THN}$.

Remark 3. Now consider the unilateral weighted shift $T = \mathrm{shift}(\{\omega_k\}_{k=1}^{\infty})$ on ℓ_+^2 with weights $\omega_k = 1$ for all k except for $k = 2$ where $\omega_2 = \omega$ for any $\omega \in (0,1)$. Since the nonnegative weight sequence is not increasing, T is not hyponormal, and hence not paranormal (recall that the concepts of hyponormality and paranormality coincide for unilateral weighted shifts; see e.g., [11, p.95]). It is readily verified that T is a normaloid nonstrict contraction (reason: $\|T^n\| = 1$ for all $n \geq 1$, which implies $r(T) = 1$ by the Gelfand–Beurling formula), and also that it is of class $\mathcal{C}_{10}$. Indeed, it is of class $\mathcal{C}_{1\cdot}$ because $\|T^n x\| = \|T^2 x\|$ for every $x \in \ell_+^2$ and $\ker T^n = \{0\}$ for all $n \geq 2$, and of class $\mathcal{C}_{\cdot 0}$ because $\|T^{*n}x\| \leq \|S^{*n}x\|$ for every $x \in \ell_+^2$ and each $n \geq 1$, where S is the "unweighted" unilateral shift. Thus $T \in \mathcal{HN}$ by Proposition 1. If there exists an invertible part of T, say $T|_{\mathcal{M}}$ for some T-invariant subspace $\mathcal{M} \neq \{0\}$, then $T^n(T|_{\mathcal{M}})^{-n}u = u$ for every $u \in \mathcal{M}$ (reason: $(T|_{\mathcal{M}})^{-n}u$ lies in $\mathcal{M}$ and $(T|_{\mathcal{M}})^n y = T^n y$ whenever $y \in \mathcal{M}$), and hence $\mathcal{M} \subseteq \mathrm{ran}\, T^n$ for every $n \geq 1$, which implies $\mathcal{M} = \{0\}$; a contradiction. Therefore, all parts of T are noninvertible. Consequently, *T is a nonparanormal $\mathcal{C}_{10}$-contraction in $\mathcal{THN}$.*

If $T \in \mathcal{C}_{\cdot 1}$, then $T^* \in \mathcal{C}_{1\cdot}$ and so $T^* \in \mathcal{HN}$ by Proposition 1. Thus, *if $T \in \mathcal{C}_{11}$, then both T and T^* lie in $\mathcal{HN}$.* Clearly, every unitary operator is a $\mathcal{C}_{11}$-contraction in $\mathcal{THN}$. Are they the only $\mathcal{C}_{11}$-contractions in $\mathcal{THN}$?

2. $\mathcal{THN}$ Contractions of Class $\mathcal{C}_{11}$

Let $\rho(T) = \big\{\lambda \in \mathbb{C}\colon \ker(\lambda I - T) = \{0\} \text{ and } \mathrm{ran}\,(\lambda I - T) = \mathcal{H}\big\}$ be the resolvent set of an operator T on $\mathcal{H}$. Consider the classical partition $\big\{\sigma_P(T), \sigma_R(T), \sigma_C(T)\big\}$ of the spectrum $\sigma(T) = \mathbb{C}\backslash\rho(T)$, where $\sigma_P(T) = \big\{\lambda \in \mathbb{C}\colon \ker(\lambda I - T) \neq \{0\}\big\}$ (the set of all eigenvalues of T) is the point spectrum, $\sigma_R(T) = \sigma_P(T^*)^*\backslash\sigma_P(T)$ is the residual spectrum, and $\sigma_C(T) = \sigma(T)\backslash(\sigma_P(T) \cup \sigma_P(T^*)^*)$ is the continuous spectrum. (We are using the standard notation $\Lambda^* := \{\lambda \in \mathbb{C}\colon \overline{\lambda} \in \Lambda\}$.) The joint point spectrum $\sigma_{JP}(T) = \big\{\lambda \in \mathbb{C}\colon \{0\} \neq \ker(\lambda I - T) \subseteq \ker(\overline{\lambda} I - T^*)\big\}$ is a subset of $\sigma_P(T)$. The elements of $\sigma_{JP}(T)$ are the normal eigenvalues of T. It is clear that if $\sigma_P(T) = \sigma_{JP}(T)$, then $\sigma_P(T) \subseteq \sigma_P(T^*)^*$, and hence $\sigma_R(T^*)$ is empty; this is the case whenever T is a hyponormal operator: (a) *every eigenvalue of a hyponormal operator is a normal eigenvalue*. Moreover, as is also well-known, (b) *isolated points in the spectrum of a hyponormal operator are eigenvalues.* However, the above italicized statements can be extended to operators in $\mathcal{THN}$ only partially, as we shall see in Proposition 3 below. To prove Proposition 3 we need the following result, which will be applied often throughout the text.

Proposition 2. (a) *If $T \in \mathcal{THN}$ is such that $\sigma(T) \subseteq \partial\mathbb{D}$, then T is unitary.* (b) *In particular, if $T \in \mathcal{THN}$ is similar to a unitary operator, then it is unitary.*

Proof. Take an invertible operator $S \in \mathcal{B}[\mathcal{H}]$. Recall that *$S$ is unitary if and only if $\|S\| = \|S^{-1}\| = 1$*. Indeed, if $\|S\| = \|S^{-1}\| = 1$, then $\|x\| = \|S^{-1}Sx\| \leq \|Sx\| \leq \|x\|$, and hence $\|Sx\| = \|x\|$ for every $x \in \mathcal{H}$ (thus an invertible isometry, which means a unitary operator). If $\sigma(T) \subseteq \partial\mathbb{D}$, then T is invertible and $\sigma(T^{-1}) \subseteq \partial\mathbb{D}$ so that

$r(T) = r(T^{-1}) = 1$. If an invertible T lies in $\mathcal{THN}$, then T and T^{-1} are normaloid. Thus, if $T \in \mathcal{THN}$ and $\sigma(T) \subseteq \partial\mathbb{D}$, then $\|T\| = \|T^{-1}\| = 1$, and T is unitary. This proves (a). If T is similar to a unitary operator U, then $\sigma(T) = \sigma(U) \subseteq \partial\mathbb{D}$ so that T is unitary by item (a) if, in addition, $T \in \mathcal{THN}$. □

Proposition 3. (a) *If $T \in \mathcal{THN}$, then isolated points of $\sigma(T)$ are eigenvalues of T.* (b) *Furthermore, if $T \in \mathcal{THN}$ and $\sigma(T)$ is finite, then T is a diagonal operator.*

Proof. If λ is an isolated point of $\sigma(T)$, then it follows by the Riesz Decomposition Theorem that $\mathcal{H}$ has a direct sum (not necessarily orthogonal) decomposition $\mathcal{H} = \mathcal{H}_1 \dotplus \mathcal{H}_2$ into T-invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)\backslash\{\lambda\}$, where $T_1 = T|_{\mathcal{H}_1}$ and $T_2 = T|_{\mathcal{H}_2}$. Since $T \in \mathcal{THN}$ we get $T_1 \in \mathcal{THN}$. Therefore, if $\lambda = 0$, then $T_1 = O$ and λ is an eigenvalue of T. If $\lambda \neq 0$, then put $U = \frac{1}{\lambda}T_1$ so that $U \in \mathcal{THN}$ and $\sigma(U) = \{1\}$. Proposition 2 ensures that U is unitary, and hence $U - I$ is quasinilpotent and normal so that $U = I$. Thus $T_1 = \lambda I$. (i.e., λ is a simple pole of the resolvent of T) and $\mathcal{H} = \ker(\lambda I - T) \dotplus \mathcal{H}_2$, which implies $\mathcal{H}_2 = \operatorname{ran}(\lambda I - T)$, and so $\mathcal{H} = \ker(\lambda I - T) \dotplus \operatorname{ran}(\lambda I - T)$. This proves (a): isolated points in $\sigma(T)$ are eigenvalues. To complete the proof, assume now that $\sigma(T)$ is finite (equivalently, $\sigma(T)$ consists of isolated points only). Then the points $\lambda \in \sigma(T)$ are simple poles of the resolvent of T. For any operator $S \in \mathcal{B}[\mathcal{H}]$ let $\sigma_\pi(S) = \{\lambda \in \sigma(S) \colon |\lambda| = r(S)\}$ denote the *peripheral spectrum* of S [6, p.225], which is nonempty. Let P_{λ_1} denote the Riesz projection corresponding to λ_1 in $\sigma_\pi(T)$. Then P_{λ_1} has norm 1, and $\ker(\lambda_1 I - T)$ is orthogonal (in the usual Hilbert space sense) to $\operatorname{ran}(\lambda_1 I - T)$ (see [6, Proposition 54.4]). (We remark here that the G. Birkhoff definition of orthogonality used in [6, Proposition 54.4] reduces to orthogonality in the usual sense for Hilbert spaces.) Thus λ_1 is a normal eigenvalue of T. Repeating this process a finite number of times, starting with $T|_{\operatorname{ran}(\lambda_1 I - T)}$ in $\mathcal{THN}$, it follows that T is a diagonal operator. □

Remark 4. An operator is *isoloid* if isolated points of the spectrum are eigenvalues. Proposition 3(a) says that *$\mathcal{THN}$ operators are isoloid*. However, it happens that

> even *nonzero isolated points of the spectrum of a $\mathcal{THN}$ operator* (which are eigenvalues) *are not necessarily normal eigenvalues.*

For instance, let S be the canonical unilateral shift on ℓ^2_+ and put $A = \frac{1}{2}(I + S)$, which is a subnormal operator. Hence $A \in \mathcal{THN}$. It is clear that $\sigma(A) = \frac{1}{2}(1 + \mathbb{D}^-)$ and A^* is not an isometry, and so $D_{A^*} = (I - AA^*)^{\frac{1}{2}} \neq O$. Take any $\lambda \in \mathbb{D}\backslash\sigma(A)$ and any $u \in \ell^2_+\backslash\ker D_{A^*}$ such that $2\|u\|^2 + |\lambda|^2 \leq 1$. Consider the operator $T = \left(\begin{smallmatrix} A & B \\ O & C \end{smallmatrix}\right)$ on the orthogonal direct sum $\mathcal{H} = \ell^2_+ \oplus \mathbb{C}$, where $B = v \otimes 1$ (i.e., $B\zeta = \zeta v$ for every $\zeta \in \mathbb{C}$) with $v = D^2_{A^*}u = (I - AA^*)u \neq 0$ and $C = \lambda$. We claim that

$$\lambda \text{ is a nonzero isolated nonnormal eigenvalue of } T.$$

Indeed, $\lambda \neq 0$ because $\lambda \in \mathbb{D}\backslash\sigma(A)$, and it is an eigenvalue of T since $T(x_0 \oplus 1) = \lambda(x_0 \oplus 1)$ if and only if $Ax_0 + v = \lambda x_0$, which happens for $x_0 = (\lambda I - A)^{-1}v$. Since $\mathcal{H} = \ell^2_+ \dotplus \mathbb{C}(x_0 \oplus 1)$, we have $\sigma(T) = \sigma(A) \cup \{\lambda\}$. Thus λ is an isolated point of $\sigma(T)$. Moreover, since $x_0 \neq 0$, it follows that $\ker(\lambda I - T) = \mathbb{C}(x_0 \oplus 1)$ is not orthogonal to $\ell^2_+ = \operatorname{ran}(\lambda I - T)$. Then $\ker(\overline{\lambda} I - T^*) \cap \ker(\lambda I - T) = \{0\}$, and so $\ker(\lambda I - T)$ does not reduce T. Now, using an argument from [2], we verify that

$$\|T\| = 1.$$

Actually, $\|T\| \geq \|A\| = r(A) = 1$. On the other hand, for any $x \oplus \zeta$ in $\ell^2_+ \oplus \mathbb{C}$,

$$\begin{aligned}\|T(x\oplus\zeta)\|^2 &= \|(Ax+\zeta v)\oplus\lambda\zeta\|^2 = \|Ax+\zeta v\|^2 + |\lambda|^2|\zeta|^2\\ &\leq \|Ax\|^2 + 2|\zeta|\,|\langle Ax\,;v\rangle| + |\zeta|^2\|v\|^2 + |\lambda|^2|\zeta|^2\\ &\leq \|Ax\|^2 + 2|\zeta|\,|\langle AD_A^2x\,;u\rangle| + |\zeta|^2\|u\|^2 + |\lambda|^2|\zeta|^2\\ &\leq \|Ax\|^2 + 2|\zeta|\,\|u\|\,\|D_Ax\| + |\zeta|^2\|u\|^2 + |\lambda|^2|\zeta|^2\\ &\leq \|Ax\|^2 + \|D_Ax\|^2 + 2|\zeta|^2\|u\|^2 + |\lambda|^2|\zeta|^2\\ &\leq \|Ax\|^2 + \|D_Ax\|^2 + |\zeta|^2 = \|x\|^2 + |\zeta|^2 = \|x\oplus\zeta\|^2.\end{aligned}$$

Finally, we show that $T \in \mathcal{THN}$. In fact, since $\sigma(T) = \sigma(A) \cup \{\lambda\}$ and $\|T\| = 1$, it follows that T is a noninvertible normaloid operator. Every nonzero T-invariant subspace is of the form $\mathcal{M} \dotplus \mathcal{N}$, where the subspaces $\mathcal{M}$ and $\mathcal{N}$ are invariant for $A = T|_{\mathcal{H}_2}$ and $T|_{\mathcal{H}_1}$, respectively, with $\mathcal{H}_2 = \ell^2_+$ and $\mathcal{H}_1 = \mathbb{C}(x_0 \oplus 1)$. Here we are applying the Riesz Decomposition Theorem again: $\mathcal{H}$ has a unique direct sum decomposition $\mathcal{H} = \mathcal{H}_2 \dotplus \mathcal{H}_1$ into T-invariant (not necessarily orthogonal) subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with $\sigma(T|_{\mathcal{H}_1}) = \{\lambda\}$ and $\sigma(T|_{\mathcal{H}_2}) = \sigma(T)\backslash\{\lambda\}$. Furthermore, the two projections determined by this decomposition are norm-limits of polynomials of T by Runge's Theorem. If $\mathcal{M} \neq \{0\}$, then $T|_{\mathcal{M}\dotplus\mathcal{N}} = A|_{\mathcal{M}} \dotplus \lambda I$ is noninvertible and normaloid, where I denotes the identity on $\mathcal{N}$. If $\mathcal{M} = \{0\}$, then $T|_{\mathcal{M}\dotplus\mathcal{N}} = \lambda I$ is normaloid with a normaloid inverse. Outcome: *T lies in $\mathcal{THN}$.*

We shall focus our attention on $\mathcal{C}_{1\cdot}$-contractions in $\mathcal{THN}$; in particular, on $\mathcal{C}_{11}$-contractions in $\mathcal{THN}$. These will be fully characterized in Propositions 5 and 6.

Proposition 4. *Let S be an invertible part of a $\mathcal{C}_{1\cdot}$-contraction in $\mathcal{THN}$. If S^{-1} is power bounded, then S is unitary.*

Proof. Take a $\mathcal{C}_{1\cdot}$-contraction T in $\mathcal{THN}$. Let S be a nonzero part of T. Thus S is a $\mathcal{C}_{1\cdot}$-contraction and so $\|S\| = 1$ (cf. proof of Proposition 1). Now suppose S is invertible. Since $T \in \mathcal{THN}$, it follows that S^{-1} is normaloid, and therefore $\|S\| = 1 = \|S^{-1}S\| \leq \|S^{-1}\| = r(S^{-1})$. If S^{-1} is power bounded, then $r(S^{-1}) \leq 1$ so that $\|S\| = \|S^{-1}\| = 1$, and S is unitary (cf. proof of Proposition 2). □

The above proposition shows which are the natural candidates to be in $\mathcal{HN}$ but not in $\mathcal{THN}$ (recall: although $\mathcal{C}_{11}$ is precisely the class of all contractions quasisimilar to a unitary operator [13, pp.71,75], a $\mathcal{C}_{11}$-contraction is not necessarily similar to a unitary operator).

Corollary 1. *If a nonunitary $\mathcal{C}_{11}$-contraction is similar to a unitary operator, then it lies in $\mathcal{HN}\backslash\mathcal{THN}$.*

Proof. If an operator is similar to a unitary operator, then it is invertible with a power bounded inverse, and the result follows by Propositions 1 and 4. □

A complete spectral characterization of $\mathcal{C}_{1\cdot}$-contractions is known. Let $\Gamma(\mathbb{D})$ be the collection of all nonempty compact subsets K of $\mathbb{D}^-$ such that every nonempty clopen (closed and open) subset C of K is such that $\mu(C \cap \partial\mathbb{D}) > 0$, where μ stands for the normalized Lebesgue measure on $\partial\mathbb{D}$. It was shown in [3] that the spectrum of a completely nonunitary $\mathcal{C}_{1\cdot}$-contraction lies in $\Gamma(\mathbb{D})$ and every set in $\Gamma(\mathbb{D})$ can be the spectrum of a $\mathcal{C}_{11}$-contraction, and in [8] that every set in $\Gamma(\mathbb{D})$ can also be the spectrum of a $\mathcal{C}_{10}$-contraction. Recall that a completely nonunitary $\mathcal{C}_{11}$-contraction is quasisimilar to an absolutely continuous unitary operator [13, pp.71,75,84,85], and an absolutely continuous unitary operator is similar to a completely nonunitary $\mathcal{C}_{11}$-contraction [7] (see also [2]). However, in accordance with the aforementioned results, the point zero may be in the spectrum of a $\mathcal{C}_{11}$-contraction (in fact, if it is there, then it is in the continuous spectrum). We give a concrete example.

Remark 5. Take an arbitrary integer $n \geq 1$ and let $T_n = \text{shift}(\{\omega_k\}_{k=-\infty}^{\infty})$ be a bilateral weighted shift on ℓ^2 with weights $\omega_k = 1$ for all k except for $k = 0$ where $\omega_0 = (n+1)^{-1}$. Each T_n is a nonunitary $\mathcal{C}_{11}$-contraction similar to a unitary operator, and $T = \bigoplus_{n=1}^{\infty} T_n$ is a $\mathcal{C}_{11}$-contraction not similar to any unitary operator [10, p.65] such that $0 \in \sigma(T)$ (T is injective but not bounded below). First note that T_n *lies in* $\mathcal{HN} \backslash \mathcal{THN}$ (by Corollary 1), and hence the $\mathcal{C}_{11}$-contraction T does not lie in $\mathcal{THN}$ (each direct summand T_n is invertible with a power bounded inverse, but not unitary — Proposition 4). This again prompts the question: If a $\mathcal{C}_{11}$-contraction T lies in $\mathcal{THN}$, then is it true that T must be unitary?

Proposition 5. *If* $T \in \mathcal{C}_{11} \cap \mathcal{THN}$, *then* T *is unitary.*

Proof. If T is a $\mathcal{C}_{11}$-contraction, then it is quasisimilar to a unitary operator. In this case, it follows from [1] that there exists an increasing sequence $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ of T-invariant subspaces that span $\mathcal{H}$ (i.e., $\bigvee_{n\in\mathbb{N}} \mathcal{M}_n = \mathcal{H}$) such that each part $T|_{\mathcal{M}_n}$ is similar to a unitary operator. If, in addition, $T \in \mathcal{THN}$, then each $T|_{\mathcal{M}_n}$ lies in $\mathcal{THN}$, and hence is unitary by Proposition 2. Therefore, T is unitary. □

Proposition 6. *If* $T \in \mathcal{THN}$ *is a* $\mathcal{C}_{1\cdot}$*-contraction, then* T *is the* (*orthogonal*) *direct sum of a unitary operator and a* $\mathcal{C}_{10}$*-contraction.*

Proof. Every $\mathcal{C}_{1\cdot}$-contraction T has a triangulation

$$T = \begin{pmatrix} T_{11} & * & * \\ O & T_{00} & * \\ O & O & T_{10} \end{pmatrix}$$

where $T_{11} \in \mathcal{C}_{11}$, $T_{00} \in \mathcal{C}_{00}$ and $T_{10} \in \mathcal{C}_{10}$ [13, p.75]. Since $T_{11} \in \mathcal{C}_{11}$, it follows that $T_{11} \in \mathcal{C}_{11} \cap \mathcal{THN}$ (it is a $\mathcal{C}_{11}$-part of a $\mathcal{THN}$ contraction), and hence T_{11} is unitary by Proposition 5. Recall that if the restriction of a contraction T to an invariant subspace is unitary, then the subspace reduces T. Therefore,

$$T = T_{11} \oplus \begin{pmatrix} T_{00} & * \\ O & T_{10} \end{pmatrix}$$

so that T_{00} is a part of T. Since $T \in \mathcal{C}_{1\cdot}$, it follows that T_{00} acts on $\{0\}$, and hence

$$T = T_{11} \oplus T_{10}.$$ □

3. $\mathcal{THN}$ Contractions with Compact Defect Operator

Recall that T is a contraction if and only if $I - T^*T$ is a nonnegative contraction. In this case, the nonnegative contraction $D_T = (I - T^*T)^{\frac{1}{2}}$ is called the defect operator of T. The characterization of $\mathcal{C}_{11}$-contractions in $\mathcal{THN}$ is complete by Proposition 5: *a contraction lies in* $\mathcal{C}_{11} \cap \mathcal{THN}$ *if and only if it is unitary.* In order to deal with the cases of $\mathcal{C}_{10}$ or $\mathcal{C}_{01}$-contractions in $\mathcal{THN}$ we shall assume that their defect operator is compact.

Proposition 7. *If a $\mathcal{C}_{1\cdot}$-contraction $T \in \mathcal{B}[\mathcal{H}]$ has a compact defect operator, then $\sigma(T) \cap \mathbb{D} \neq \varnothing$ if and only if $\mathbb{D} \subseteq \sigma_P(T^*)$.*

Proof. Let D_T be the defect operator of a contraction T. We claim that

$$\text{if } T \in \mathcal{C}_{1\cdot} \text{ and } D_T \text{ is compact, then } T \text{ is bounded below.}$$

If D_T is compact, then so is D_T^2 so that $I - T^*T$ is compact, and hence zero is the only possible accumulation point of $\sigma(I - T^*T)$, which implies that zero is not an accumulation point of $\sigma(T^*T)$. Therefore, if $0 \notin \sigma_P(T^*T)$ (i.e., if $0 \neq \|T^*Tx\|$ for every nonzero x), then T^*T is bounded below and so is T (since $\|T^*Tx\| \leq \|T^*\|\|Tx\|$ for every x). But if $T \in \mathcal{C}_{1\cdot}$, then $0 \notin \sigma_P(T^*T)$. (Indeed, if $0 \in \sigma_P(T^*T)$, then $\ker T = \ker(T^*T) \neq \{0\}$, which implies that $T^n x = 0$ for some $0 \neq x \in \ker T$ and every positive integer n, and hence $T \notin \mathcal{C}_{1\cdot}$.)

Now take any $\lambda \in \mathbb{D}$ and consider the Möbius transform $T_\lambda = (\lambda I - T)(\overline{\lambda} T - I)^{-1} = (\overline{\lambda} T - I)^{-1}(\lambda I - T)$, which is a $\mathcal{C}_{1\cdot}$-contraction with a compact defect operator [13, p.240], and hence bounded below by the above result. Thus $(\lambda I - T)$ is bounded below (since $\|T_\lambda x\| \leq \|(\overline{\lambda} T - I)^{-1}\|\|(\lambda I - T)x\|$ for every x), which means that λ is not in the approximate point spectrum; that is,

$$\mathbb{D} \cap \sigma_{AP}(T) = \varnothing.$$

Therefore, $\sigma_{AP}(T) = \partial\sigma(T) \subseteq \partial\mathbb{D}$ because $\sigma(T) \subseteq \mathbb{D}^-$ and $\partial\sigma(T) \subseteq \sigma_{AP}(T)$. Then $\sigma(T)\backslash\sigma_{AP}(T) = \mathbb{D}$ if $\sigma(T) \cap \mathbb{D} \neq \varnothing$. But $\sigma(T)\backslash\sigma_{AP}(T) \subseteq \sigma_R(T) \subseteq \sigma_P(T^*)^*$ so that

$$\sigma(T) \cap \mathbb{D} \neq \varnothing \quad \text{implies} \quad \mathbb{D} \subseteq \sigma_P(T^*).$$

The converse is trivial. □

A straightforward corollary reads as follows. *If a $\mathcal{C}_{1\cdot}$-contraction T has a compact defect operator and $\mathbb{D} \not\subseteq \sigma_P(T^*)$, then $\sigma(T) \subseteq \partial\mathbb{D}$.* By using Proposition 7 we can extend the result of Remark 2 on completely nonunitary isometries (i.e., on unilateral shifts, and hence on $\mathcal{C}_{10}$-contractions with null defect operator) to $\mathcal{C}_{10}$-contractions with compact defect operators.

Proposition 8. *If T is a $\mathcal{C}_{10}$-contraction with a compact defect operator, then it lies in $\mathcal{THN}$ if and only if it has no invertible part.*

Proof. We shall show that the following assertions are pairwise equivalent.

(a) Every nonzero part of T is not invertible.

(b) $\mathbb{D} \subseteq \sigma_P((T|_\mathcal{M})^*)$ whenever $\mathcal{M}$ is a nonzero T-invariant subspace.

(c) $T \in \mathcal{THN}$.

Suppose T is a $\mathcal{C}_{10}$-contraction with a compact defect operator, let $\mathcal{M}$ be an arbitrary nonzero T-invariant subspace, and consider the part $T|_\mathcal{M}$. First observe that $T|_\mathcal{M}$ is a $\mathcal{C}_{1\cdot}$-contraction with a compact defect operator. Thus, if $T|_\mathcal{M}$ is not invertible, then $0 \in \sigma(T|_\mathcal{M})$ so that $\sigma(T|_\mathcal{M}) \cap \mathbb{D} \neq \varnothing$, and hence Proposition 7 ensures that $\mathbb{D} \subseteq \sigma_P((T|_\mathcal{M})^*)$; that is, (a) implies (b). Now if $\mathbb{D} \subseteq \sigma_P((T|_\mathcal{M})^*)$, then $0 \in \sigma_P((T|_\mathcal{M})^*)$ so that $0 \in \sigma(T|_\mathcal{M})$, which ensures that $T|_\mathcal{M}$ is not invertible so that (b) implies (a), and (a) implies (c) trivially (because $T \in \mathcal{HN}$ according to Proposition 1). Finally, if $T|_\mathcal{M}$ is invertible, then $0 \notin \sigma(T|_\mathcal{M})$ so that $0 \notin \sigma_P((T|_\mathcal{M})^*)$, and hence $\mathbb{D} \not\subseteq \sigma_P((T|_\mathcal{M})^*)$, which implies that $\sigma(T|_\mathcal{M}) \subseteq \partial\mathbb{D}$ by Proposition 7 (because $T|_\mathcal{M}$ is a $\mathcal{C}_{1\cdot}$-contraction with a compact defect operator), and therefore $T|_{\mathcal{M}}$ is unitary according to Proposition 2 whenever $T|_{\mathcal{M}} \in \mathcal{THN}$, that is, whenever $T \in \mathcal{THN}$. Summing up: $T|_{\mathcal{M}}$ invertible and $T \in \mathcal{THN}$ leads to $T|_{\mathcal{M}}$ is unitary, which is a contradiction. In fact, if $T|_{\mathcal{M}}$ is unitary, then $\mathcal{M}$ reduces T (if a part of a contraction is unitary, then it is a direct summand) so that $T^*|_{\mathcal{M}}$ is unitary, which contradicts the fact that $T \in \mathcal{C}_{10}$. Thus (c) implies (a). □

An immediate consequence of Propositions 6 and 8 reads as follows. *A $\mathcal{C}_{1\cdot}$-contraction in $\mathcal{THN}$ with a compact defect operator is the* (*orthogonal*) *direct sum of a unitary operator and a $\mathcal{C}_{10}$-contraction with no invertible parts.*

On the other hand, there is no way for a $\mathcal{C}_{01}$-contraction T (acting on a nonzero Hilbert space $\mathcal{H}$) to be in $\mathcal{THN}$, provided it has a compact defect operator D_T and $\ker T \subseteq \ker T^*$. (Observe that $\ker T \subseteq \ker T^*$ for a $\mathcal{C}_{1\cdot}$-contraction.)

Proposition 9. *If T is a $\mathcal{C}_{01}$-contraction with a compact defect operator such that* $\ker T \subseteq \ker T^*$, *then it is not in $\mathcal{THN}$.*

Proof. Since $TD_T = D_{T^*}T$ [13, p.7] and $(\operatorname{ran} T)^- = \mathcal{H} \neq \{0\}$ (for $T \in \mathcal{C}_{01}$ so that $\ker T^* = \{0\}$), it can be shown by using the polar decomposition of T^* that D_{T^*} is compact whenever D_T is. Moreover, $T^* \in \mathcal{C}_{10}$ implies $\sigma_P(T^*) \cap \mathbb{D} = \varnothing$. The hypothesis $\ker T \subseteq \ker T^*$ implies that either T is injective or 0 is a normal eigenvalue of T (which cannot occur once $T \in \mathcal{C}_{01}$), and hence $\mathbb{D} \not\subseteq \sigma_P(T)$. Thus, the fact that T^* is a $\mathcal{C}_{1\cdot}$-contraction with a compact defect operator implies, by Proposition 7, that $\sigma(T^*) \subseteq \partial\mathbb{D}$, and so $\sigma(T) \subseteq \partial\mathbb{D}$. Therefore, T is unitary by Proposition 2 whenever $T \in \mathcal{THN}$, which contradicts the hypothesis that $T \in \mathcal{C}_{01}$. □

If T is a $\mathcal{C}_{00}$-contraction with compact defect operator, then T is a semi-Fredholm operator with a finite-dimensional kernel. The operator T may or may not have Fredholm index 0. If, however $\operatorname{ind} T = 0$, then T is a compact perturbation of a unitary operator, which implies that its essential spectrum $\sigma_e(T)$ is a subset of $\partial\mathbb{D}$. It follows that $\sigma(T) \cap \mathbb{D} = \sigma_P(T) \cap \mathbb{D}$.

Proposition 10. *Let T be a $\mathcal{C}_{00}$-contraction with a compact defect operator such that* $\operatorname{ind} T = 0$. *If $T \in \mathcal{THN}$ and normal subspaces of T reduce T, then T is a diagonal operator, the eigenvalues of T are of finite multiplicity, and $\sigma_P(T)$ has no accumulation point in $\mathbb{D}$.*

Proof. Let T be a $\mathcal{C}_{00}$-contraction in $\mathcal{THN}$ with a compact defect operator and with $\operatorname{ind} T = 0$. Then $\sigma(T) \cap \mathbb{D} = \sigma_P(T) \cap \mathbb{D}$. For any $\lambda \in \sigma_P(T) \subseteq \mathbb{D}$ it follows that $\mathcal{N}_\lambda = \ker(\lambda I - T)$ is a normal subspace of T. Hence $\mathcal{N}_\lambda$ reduces T, and so do the subspaces $\mathcal{H}_0 = \bigoplus_{\lambda \in \sigma_P(T)} \mathcal{N}_\lambda$ and $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$. Since $\sigma(T|_{\mathcal{H}_1}) \subseteq \partial\mathbb{D}$, it follows that the $\mathcal{C}_{00}$-contraction $T|_{\mathcal{H}_1}$ is unitary, thus acting on $\mathcal{H}_1 = \{0\}$. Hence T is a diagonal operator. Since the defect operator D_T is compact, the eigenvalues of T are of finite multiplicity and $\sigma_P(T)$ has no accumulation point in $\mathbb{D}$. □

Remark 6. There exist $\mathcal{C}_{00}$-contractions T in $\mathcal{THN}$ such that D_T is compact but $\operatorname{ind} T \neq 0$. For example, consider a unilateral weighted shift $T = \operatorname{shift}(\{\omega_k\}_{k=1}^\infty)$ on ℓ^2_+ with increasing weights $\omega_k = \frac{k}{k+1}$. Then $0 < \omega_k \to 1$, $\prod_k \omega_k = 0$, $\operatorname{ind} T \neq 0$ and D_T belongs to the Schatten p-class for all $2 < p < \infty$. Furthermore, T is a $\mathcal{C}_{00}$-contraction, which being hyponormal is in $\mathcal{THN}$. If, however, the defect operator D_T of a $\mathcal{C}_{00}$-contraction in $\mathcal{THN}$ is of Hilbert–Schmidt class (i.e., of Schatten 2-class), then $\operatorname{ind} T = 0$ whenever normal subspaces of T are reducing, as we shall see in the next section.

4. $\mathcal{THN}$ Contractions with Hilbert–Schmidt Defect Operator

An operator $D \in \mathcal{B}[\mathcal{H}]$ is Hilbert–Schmidt if $\{\|De\|^2\}_{e\in B}$ is a summable family for any orthonormal basis B for $\mathcal{H}$. This sum does not depend on the choice of the orthonormal basis. For simplicity we shall assume from now on that $\mathcal{H}$ is separable. Let $B = \{e_n\}$ be any orthonormal basis for $\mathcal{H}$. Thus the defect operator D_T of a contraction $T \in \mathcal{B}[\mathcal{H}]$ is Hilbert–Schmidt if $\sum_n \|D_T e_n\|^2 < \infty$ or, equivalently, if $\sum_n (1 - \|Te_n\|^2) < \infty$. This is the trace of D_T^2; that is, $\operatorname{tr}(D_T^2) = \sum_n \langle D_T^2 e_n\,;e_n\rangle = \sum_n \|D_T e_n\|^2 < \infty$ (D_T is Hilbert–Schmidt if and only if D_T^2 is trace class).

Recall that a $\mathcal{C}_{00}$-contraction T is of class $\mathcal{C}_0$ if there exists an inner function u such that $u(T) = 0$. A contraction T is a weak contraction if $\sigma(T) \neq \mathbb{D}^-$ and D_T^2 is trace class [13, p.323]. Equivalently, $\sigma(T) \neq \mathbb{D}^-$ and D_T is Hilbert–Schmidt.

Proposition 11. *If $T \in \mathcal{THN}$ is a $\mathcal{C}_{0\cdot}$-contraction with a Hilbert–Schmidt defect operator D_T such that normal subspaces of T reduce T, then T is a diagonal $\mathcal{C}_0$-contraction.*

Proof. Every $\mathcal{C}_{0\cdot}$-contraction T has a triangulation

$$T = \begin{pmatrix} T_{01} & * \\ O & T_{00} \end{pmatrix},$$

where $T_{01} \in \mathcal{C}_{01}$ and $T_{00} \in \mathcal{C}_{00}$ [13, p.75]. The hypothesis D_T is Hilbert–Schmidt implies $D_{T_{01}}$ is Hilbert–Schmidt. In particular, $D_{T_{01}}$ is compact. Since $T \in \mathcal{THN}$ implies $T_{01} \in \mathcal{THN}$, and $\ker T_{01} \subseteq \ker T_{01}^*$ (because normal subspaces of T are reducing), it follows from an application of Proposition 9 that T_{01} acts on the zero space $\{0\}$. Hence $T = T_{00}$ is a $\mathcal{C}_{00}$-contraction with a Hilbert–Schmidt defect operator, and so a $\mathcal{C}_0$-contraction [16]. The normal subspace $\mathcal{H}_0 = \ker T$ reduces T. Put $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$. Since T is a $\mathcal{C}_0$-contraction, we infer that $\mathcal{H}_0$ is finite-dimensional and that $T|_{\mathcal{H}_1}$ is invertible. It follows that T is Fredholm with $\operatorname{ind} T = 0$, and so Proposition 10 applies. □

We next prove our main result, which says that a $\mathcal{THN}$ contraction T with Hilbert–Schmidt defect operator, such that normal subspaces of T reduce T, is the direct sum of a unitary, a normal $\mathcal{C}_0$-contraction and a $\mathcal{C}_{10}$-contraction.

Theorem 1. *Let $T \in \mathcal{THN}$ be a contraction such that D_T is Hilbert–Schmidt. If normal subspaces of T reduce T, then*

$$T = T_u \oplus T_n \oplus T_{10},$$

where T_u is unitary, T_n is a normal $\mathcal{C}_0$-contraction and T_{10} is a $\mathcal{C}_{10}$-contraction with no invertible parts.

Proof. Since T is contraction, it has a triangulation (cf. [13, p.73])

$$T = \begin{pmatrix} T_{0\cdot} & * \\ O & T_{1\cdot} \end{pmatrix},$$

where $T_{0\cdot} \in \mathcal{C}_{0\cdot}$ and $T_{1\cdot} \in \mathcal{C}_{1\cdot}$. If D_T is Hilbert–Schmidt, then so is $D_{T_{0\cdot}}$, and hence $T_{0\cdot}$ is a normal (indeed, diagonal) $\mathcal{C}_0$-contraction by Proposition 11. Since normal subspaces of T are reducing, we get $T = T_{0\cdot} \oplus T_{1\cdot}$. Therefore, by Proposition 6,

$$T = T_{0\cdot} \oplus T_u \oplus T_{10},$$

where T_u is unitary and T_{10} is a $\mathcal{C}_{10}$-contraction with no invertible parts according to Proposition 8. □

Recall that an operator $T \in \mathcal{B}[\mathcal{H}]$ is said to have the bicommutant property if $\{T\}'' = \operatorname{Alg} T$, where $\{T\}''$ denotes the double commutant of T and $\operatorname{Alg} T$ denotes the weakly closed algebra generated by T and the identity. An operator T is said to be reflexive if $\operatorname{Alg} T = \operatorname{Alg} \operatorname{Lat} T$, where $\operatorname{Lat} T$ denotes the lattice of all T-invariant subspaces and $\operatorname{Alg} \operatorname{Lat} T$ is the algebra of all operators S for which $\operatorname{Lat} T \subseteq \operatorname{Lat} S$. The proof of the following lemma can be found in [14, Theorem 4] and [15, Theorem 5] — also see [17] for some earlier results. As usual, "cnu" is a short for "completely nonunitary".

Proposition 12. (a) *A cnu $\mathcal{C}_{1\cdot}$-contraction with a Hilbert–Schmidt defect operator is reflexive.* (b) *If a contraction is densely intertwined to a unilateral shift, then it has the bicommutant property.*

Note that item (b) says: *if a contraction T is such that $WT = SW$ for some operator W with dense range and some unilateral shift S, then T has the bicommutant property.* Combining Theorem 1 with Proposition 12 one has the following corollary.

Corollary 2. *Let $T \in \mathcal{THN}$ be a contraction such that D_T is Hilbert–Schmidt and normal subspaces of T are reducing.*

(a) *If T has no normal direct summand, then T is reflexive.*

(b) *If $\dim(\ker(\lambda I - T^*)) \neq 0$ for every $\lambda \in H \subset \mathbb{D}$ with $\sum_{\lambda \in H}(1 - |\lambda|) = \infty$, then T has the bicommutant property.*

Proof. (a) By Theorem 1, the hypotheses on T imply that $T \in C_{10}$. Hence T is reflexive by Proposition 12(a). To prove (b) we argue as follows. If we denote the cnu part of T by T_c, then T_c has Hilbert–Schmidt defect operator and (by Theorem 1) $T_c = T_n \oplus T_{10}$, where T_n is a normal $\mathcal{C}_0$-contraction and T_{10} is a $\mathcal{C}_{10}$-contraction. Observe that our hypothesis on H implies that the term T_{10} can not be missing. Since T_n is a $\mathcal{C}_0$-contraction, it follows that $\sum_{\lambda \in \sigma_P(T_n^*)}(1 - |\lambda|) < \infty$, and hence $\ker(\lambda I - T_{10}^*) \neq \{0\}$ holds for some $\lambda \in \mathbb{D}$. We infer by Proposition 7 that $\sigma_P(T_{10}^*)$ actually fills $\mathbb{D}$. Thus T_{10} can be densely intertwined to a unilateral shift S (see [15, p.92]), which implies that T can be also densely intertwined to S. Applying Proposition 12(b) we conclude that T has the bicommutant property. □

Remark 7. Consider the triangulation

$$T = \begin{pmatrix} T_{01} & * & * & * & * \\ O & T_{00} & * & * & * \\ O & O & T_{11} & * & * \\ O & O & O & \widetilde{T}_{00} & * \\ O & O & O & O & T_{10} \end{pmatrix}$$

of a contraction T, where $T_{ij} \in \mathcal{C}_{ij}$ and $\widetilde{T}_{00} \in \mathcal{C}_{00}$. If $T \in \mathcal{THN}$ is such that D_T is compact and normal subspaces of T reduce T, and if both T_{00} and $\widetilde{T}_{00}$ are injective with $\operatorname{ind} T_{00} = \operatorname{ind} \widetilde{T}_{00} = 0$, then it follows from the results of Section 3 that

$$T = T_{11} \oplus T_{10} \oplus D,$$

where T_{11} is unitary, T_{10} has no invertible parts and $D = T_{00} \oplus \widetilde{T}_{00}$ is the diagonal operator of Proposition 10. If the canonical isometry associated with T_{10} is not reductive, then T is reflexive [9, Corollary 2]. Is the canonical isometry associated with T_{10} nonreductive under the hypothesis that $D_{T_{10}}$ is compact?

As a final remark, recall that every contraction T has a unique (orthogonal) direct sum decomposition $T = T_u \oplus T_c$, where T_u is unitary and T_c is a cnu contraction (it is clear that either of the summands may be missing). This is the well-known Nagy–Foiaş–Langer decomposition for contractions [13, p.9]. Completely nonunitary direct summands of class $\mathcal{C}_{\cdot 0}$ have been characterized in [4] as follows. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. An operator T in $\mathcal{B}[\mathcal{H}]$ is said to satisfy the PF-property, short for Putnam–Fuglede commutativity property, if $TX = XV^*$ for some bounded linear transformation $X\colon \mathcal{K} \to \mathcal{H}$ and some isometry V in $\mathcal{B}[\mathcal{K}]$, implies $T^*X = XV$. The cnu direct summand of a contraction $T \in \mathcal{B}[\mathcal{H}]$ is of class $\mathcal{C}_{\cdot 0}$ if and only if T satisfies the PF-property ([4, Lemma 1] — see also [12]).

It is clear from Theorem 1 that $\mathcal{THN}$ contractions T with Hilbert–Schmidt defect operator, such that normal subspaces of T reduce T, have $\mathcal{C}_{\cdot 0}$ cnu direct summands. Combining this with the results from [4], we have the following (Putnam–Fuglede type) commutativity result.

Corollary 3. *Let T be a $\mathcal{THN}$ contraction with a Hilbert–Schmidt defect operator such that normal subspaces of T reduce T. If $TX = XV^*$ for some $X \in \mathcal{B}[\mathcal{H}]$ and some isometry V, then $T^*X = XV$, and $T|_{(\operatorname{ran} X)^-}$ and $V|_{(\ker X)^\perp}$ are unitarily equivalent unitaries.*

Acknowledgment

The final version of this paper owes a lot to the referees. In particular, the results in Remark 4, Proposition 5, Section 3 and the problem posed in Remark 7 are a consequence of their suggestions. It is our pleasure to thank them.

References

1. C. Apostol, *Operators quasisimilar to a normal operator*, Proc. Amer. Math. Soc., **53** (1975), 104–106.
2. H. Bercovici and L. Kérchy, *Quasisimilarity and properties of the commutant of C_{11}-contractions*, Acta Sci. Math. (Szeged), **45** (1983), 67–74.
3. H. Bercovici and L. Kérchy, *On the spectrum of C_{11}-contractions*, Proc. Amer. Math. Soc., **95** (1985), 412–418.
4. B.P. Duggal, *On characterising contractions with C_{10} pure part*, Integral Equations Operator Theory, **27** (1997), 314–323.
5. B.P. Duggal and S.V. Djordjević, *Generalized Weyl's theorem for a class of operators satisfying a norm condition*, Math. Proc. Royal Irish Acad., **104** (2004), 75–81 (corrigendum submitted).
6. H. G. Heuser, *Functional Analysis*, Wiley, Chichester, 1982.
7. L. Kérchy, *On invariant subspace lattices of C_{11}-contractions*, Acta Sci. Math. (Szeged), **43** (1981), 281–293.
8. L. Kérchy, *On the spectra of contractions belonging to special classes*, J. Funct. Anal., **67** (1986), 153–166.
9. L. Kérchy, *Injection of unilateral shifts into contractions with non-vanishing unitary asymptotes*, Acta Sci. Math. (Szeged), **61** (1995), 443–476.
10. C.S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory*, Birkhäuser, Boston, 1997.
11. C.S. Kubrusly, *Hilbert Space Operators*, Birkhäuser, Boston, 2003.

12. C.S. Kubrusly and B.P. Duggal, *Contractions with $C_{\cdot 0}$ direct summands*, Adv. Math. Sci. Appl., **11** (2001), 593–601.
13. B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.
14. K. Takahashi, *$C_{1\cdot}$-contractions with Hilbert–Schmidt defect operators*, J. Operator Theory, **12** (1984), 331–347.
15. K. Takahashi, *Contractions with the bicommutant property*, Proc. Amer. Math. Soc., **93** (1985), 91–95.
16. K. Takahashi and M. Uchiyama, *Every C_{00} contraction with Hilbert–Schmidt defect operator is of class C_0*, J. Operator Theory, **10** (1983), 331–335.
17. P.Y. Wu, *Reflexivity and double commutant property for hyponormal contractions*, Rev. Roumaine Math. Pures Appl., **28** (1983), 631–635.

8 Redwood Grove, Northfields Avenue, Ealing, London W5 4SZ, England, U.K.
E-mail address: bpduggal@yahoo.co.uk

CIMAT, Apdo. Postal 402, C.P. 36420 Guanajauto, GTO, Mexico
E-mail address: slavdj@cimat.mx

Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil
E-mail address: carlos@ele.puc-rio.br