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ON GENERATING WANDERING SUBSPACES FOR UNITARY OPERATORS

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Abstract. A bilateral shift U on a Hilbert space $\mathcal{H}$ induces an orthogonal decomposition of $\mathcal{H}$ consisting of reducing subspaces on which each direct summand of U is a bilateral shift of multiplicity one. This extends to a unitary operator that has a generating wandering subspace. It is shown that if U is unitary and $\mathcal{W}$ is U-wandering, then the span of all images of $\mathcal{W}$ under the integral powers of U is unitarily equivalent to a direct sum of reducing subspaces generated by one-dimensional spaces. This yields a double indexed orthonormal basis, and hence a basis with a "Fubini-like" property, where summation order can be interchanged. The case of irreducible-invariant subspaces is also considered.

1. Introduction

Throughout this paper $\mathcal{H}$ is a (complex, infinite-dimensional but not necessarily separable) Hilbert space. By a subspace we mean a *closed* linear manifold of $\mathcal{H}$. Let U be an operator on $\mathcal{H}$ (i.e., a bounded linear transformation of $\mathcal{H}$ into itself). Recall that an invertible operator U is one that has a bounded inverse U^{-1}. A unitary operator is an invertible isometry (or, equivalently, an isometry U such that $U^* = U^{-1}$, where U^* is the adjoint of U — isometries preserve inner product). Also recall that the (linear) span of a subset A of $\mathcal{H}$, denoted by $\operatorname{span} A$, is the linear manifold of $\mathcal{H}$ consisting of all (finite) linear combinations of vectors in A; its closure is a subspace of $\mathcal{H}$, usually denoted by $\bigvee A$. Let $\mathbb{Z}$ denote the set of all integers.

Proposition 1. *Take any set A of vectors in $\mathcal{H}$ and let m be an arbitrary integer in $\mathbb{Z}$. If U is an invertible operator on $\mathcal{H}$, then*

$$U^m \bigvee A = \bigvee U^m A.$$

Proof. Since U is invertible, U^m is well-defined for any (positive, negative or null) integer m and, since U^m is continuous,

$$U^m(\operatorname{span} A)^- \subseteq (U^m \operatorname{span} A)^-$$

(see e.g., [8], Problem 3.46). Moreover,

$$\big(U^m(\operatorname{span} A)^-\big)^- = U^m(\operatorname{span} A)^-$$

because U^{-m} is continuous (so that inverse image of closed sets are closed). Thus

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$$U^m(\operatorname{span} A)^- \subseteq (U^m \operatorname{span} A)^- \subseteq (U^m(\operatorname{span} A)^-)^- = U^m(\operatorname{span} A)^-,$$

and hence

$$U^m(\operatorname{span} A)^- = (U^m \operatorname{span} A)^-.$$

But U^m is linear and $\operatorname{span} A$ consists of finite linear combinations. Therefore,

$$U^m \operatorname{span} A = \operatorname{span}(U^m A).$$

The above two identities close the proof: $U^m(\operatorname{span} A)^- = (\operatorname{span}(U^m A))^-$. $\square$

Definition 1. A subspace $\mathcal{W}$ of $\mathcal{H}$ is a *wandering subspace* for an operator U on $\mathcal{H}$ (or a U-*wandering subspace*) if it is orthogonal to its images under all positive powers of U; that is,

$$\mathcal{W} \perp U^k\mathcal{W} \quad \text{for every integer } k \geq 1.$$

A vector x in $\mathcal{H}$ is a *wandering vector* for U (or a U-*wandering vector*) if the one-dimensional subspace $\operatorname{span}\{x\}$ is U-wandering.

Let Γ be an arbitrary (not necessarily countable) index set. Throughout the paper m and n are arbitrary indices in $\mathbb{Z}$, and α and β are arbitrary indices in Γ. Unless otherwise stated, sums and spans are supposed to rum over $\mathbb{Z}$ if indexed by m, or over Γ if indexed by α (uncountable sums are defined as usual; see, for instance, [3] or [8]).

Proposition 2. *If U is a unitary operator, then $\mathcal{W}$ is U-wandering if and only if*

$$U^m\mathcal{W} \perp U^n\mathcal{W} \quad \textit{whenever } m \neq n.$$

Proof. Let U be unitary and take arbitrary integers m and n in $\mathbb{Z}$. Since U is invertible and its inverse also is unitary, it follows that U^m is unitary. Recall that a unitary operator preserves inner product. Thus, for any subspaces $\mathcal{W}$ and $\mathcal{M}$ of $\mathcal{H}$,

$$\mathcal{W} \perp \mathcal{M} \iff U^m\mathcal{W} \perp U^m\mathcal{M}.$$

Then, for each integer $k \geq 1$, $\mathcal{W} \perp U^k\mathcal{W}$ if and only if $U^m\mathcal{W} \perp U^{m+k}\mathcal{W}$ so that

$$\mathcal{W} \perp U^k\mathcal{W} \iff U^m\mathcal{W} \perp U^n\mathcal{W}$$

whenever $m \neq n$. $\square$

Let $\{e_\alpha\}$ be an orthogonal set of unit vectors in $\mathcal{H}$ indexed by Γ. With $\bigvee$ denoting closure of span , as usual, put

$$\mathcal{W}_0 = \bigvee_\alpha e_\alpha = \big(\operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}\big)^-,$$

the subspace of $\mathcal{H}$ spanned by the orthonormal set $\{e_\alpha\}$. Equivalently, take any subspace $\mathcal{W}_0$ of $\mathcal{H}$ and let $\{e_\alpha\}$ be an orthonormal basis for $\mathcal{W}_0$ (cardinally of Γ is the orthogonal dimension of $\mathcal{W}_0$).

Proposition 3. *If U is unitary and $\mathcal{W}_0$ is U-wandering, then*

$$U^m e_\alpha \perp U^n e_\beta$$

for all distinct pairs of indices (m, α) and (n, β) (i.e., *whenever* $(m, \alpha) \neq (n, \beta)$).

Proof. If U is unitary, then $e_\alpha \perp e_\beta$ implies $U^m e_\alpha \perp U^m e_\beta$ whenever $\alpha \neq \beta$. It is clear that $U^m e_\alpha$ lies in $U^m \mathcal{W}_0$. Thus, if U is unitary and $\mathcal{W}_0$ is U-wandering, then Proposition 2 ensures that $U^m e_\alpha \perp U^n e_\beta$ whenever $m \neq n$. □

2. Main Result

Suppose U is an invertible operator on $\mathcal{H}$ and take the family of operators $\{U^m\}$ indexed by $\mathbb{Z}$. For each α in Γ consider the subspace of $\mathcal{H}$ spanned by all images of e_α under the integral powers of U,

$$\mathcal{H}_\alpha = \bigvee_m U^m e_\alpha,$$

which is separable (spanned by a countable set). In this case we say that $\mathcal{H}_\alpha$ is *generated* by the one-dimensional space $\operatorname{span}\{e_\alpha\}$. If U is unitary, then each $\mathcal{H}_\alpha$ reduces U. Indeed, each $\mathcal{H}_\alpha$ is invariant for every integral power of U and, since $U^* = U^{-1}$, each $\mathcal{H}_\alpha$ is invariant for U and U^*; that is, each $\mathcal{H}_\alpha$ reduces U.

Lemma 1. *The assertions below are equivalent.*

(a) $U^m e_\alpha \perp U^n e_\beta$ *whenever* $\alpha \neq \beta$.
(b) $\mathcal{H}_\alpha \perp \mathcal{H}_\beta$ *whenever* $\alpha \neq \beta$.

Now consider the following further assertions.

(c) U *is unitary and* $\mathcal{W}_0$ *is* U*-wandering.*
(d) U *is unitary and each* e_α *is* U*-wandering.*

Claim: (c) *implies* (a) *and* (a,d) *implies* (c).

Proof. Suppose $\alpha \neq \beta$. If $U^m e_\alpha \perp U^n e_\beta$ for every n, then $U^m e_\alpha \perp \bigvee_n U^n e_\beta$, which implies that $\bigvee_m U^m e_\alpha \perp \bigvee_n U^n e_\beta$ (reason: inner product is linear in the first argument and continuous). Therefore (a) implies (b). Conversely, if $\mathcal{H}_\alpha \perp \mathcal{H}_\beta$, then $U^m e_\alpha \perp U^n e_\alpha$ since $U^m e_\alpha \in \mathcal{H}_\alpha$ and $U^n e_\beta \in \mathcal{H}_\beta$. Hence (b) implies (a). And (c) implies (a) by Proposition 3. Now suppose (a) holds and U is unitary. Thus $e_\alpha \perp U^m e_\beta$ for every m. If each e_α is U-wandering, then $e_\alpha \perp U^k e_\beta$ for every $k \geq 1$ even if $\alpha = \beta$. Therefore, $\mathcal{W}_0 = \bigvee_\alpha e_\alpha \perp U^k e_\beta$ for all β whenever $k \geq 1$ so that $\mathcal{W}_0 \perp \bigvee_\beta U^k e_\beta$ for every $k \geq 1$. But Proposition 1 says that this is equivalent to $\mathcal{W}_0 \perp U^k \bigvee_\beta e_\beta = U^k \mathcal{W}_0$ for every $k \geq 1$; that is, $\mathcal{W}_0$ is U-wandering. Outcome: (a,d) implies (c). □

Let $\mathcal{W}_m$ be the image $\mathcal{W}_0$ under U^m,

$$\mathcal{W}_m = U^m \mathcal{W}_0 = U^m \bigvee_\alpha e_\alpha.$$

Since U is invertible, each $\mathcal{W}_m$ is a subspace of $\mathcal{H}$ (reason: if U is invertible, then U^{-m} is continuous and $\mathcal{W}_m$ is the inverse image of the subspace $\mathcal{W}_0$ under U^{-m}). Moreover (Proposition 1), if U is invertible, then $\mathcal{W}_m = \bigvee_\alpha U^m e_\alpha$. Furthermore (Proposition 2), if U is unitary, then $\mathcal{W}_0$ is U-wandering if and only if

$$\mathcal{W}_m \perp \mathcal{W}_n \quad \text{whenever } m \neq n.$$

Theorem 1. *If* U *is unitary and* $\mathcal{W}_0$ *is* U*-wandering, then*

$$\bigoplus_m \mathcal{W}_m = \bigoplus_\alpha \mathcal{H}_\alpha.$$

Proof. First recall that if $\{\mathcal{M}_\gamma\}$ is any indexed family of *pairwise orthogonal* subspaces of $\mathcal{H}$, then their direct sum (the Hilbert space consisting of all square-summable families of vectors in $\mathcal{H}$ with each vector in each $\mathcal{M}_\gamma$) is unitarily equivalent to their topological sum; that is,

$$\bigoplus_\gamma \mathcal{M}_\gamma \cong \Big(\sum_\gamma \mathcal{M}_\gamma\Big)^- = \bigvee_\gamma \mathcal{M}_\gamma,$$

where $\cong$ stands for unitary equivalence. From now on suppose U is unitary and $\mathcal{W}_0$ is U-wandering. Thus $\mathcal{W}_m \perp \mathcal{W}_n$ whenever $m \neq n$ according to Proposition 2. Therefore (cf. Proposition 1),

$$\bigoplus_m \mathcal{W}_m \cong \bigvee_m \mathcal{W}_m = \bigvee_m \bigvee_\alpha U^m e_\alpha.$$

On the other hand, Lemma 1 ensures that $\mathcal{H}_\alpha \perp \mathcal{H}_\beta$ whenever $\alpha \neq \beta$, and hence

$$\bigoplus_\alpha \mathcal{H}_\alpha \cong \bigvee_\alpha \mathcal{H}_\alpha = \bigvee_\alpha \bigvee_m U^m e_\alpha.$$

Now set $f_{m,\alpha} = U^m e_\alpha$ and observe from Proposition 3 that $\{f_{m,\alpha}\}$ is an orthonormal set indexed by $\mathbb{Z} \times \Gamma$. Consider the Hilbert space $\bigvee_{m,\alpha} f_{m,\alpha}$ spanned by this orthonormal set so that $\{f_{m,\alpha}\}$ is an orthonormal basis for it. By unconditional convergence of the Fourier series we get

$$\bigvee_m \bigvee_\alpha U^m e_\alpha = \bigvee_m \bigvee_\alpha f_{m,\alpha} = \bigvee_{m,\alpha} f_{m,\alpha} = \bigvee_\alpha \bigvee_m f_{m,\alpha} = \bigvee_\alpha \bigvee_m U^m e_\alpha.$$

Since unitarily equivalence is transitive,

$$\bigoplus_m \mathcal{W}_m \cong \bigoplus_\alpha \mathcal{H}_\alpha,$$

which completes the proof by writing $=$ for $\cong$ as usual. $\square$

Suppose U is a unitary operator on $\mathcal{H}$ and $\mathcal{W}$ is a wandering subspace for U. If the family of orthogonal (cf. Proposition 2) subspaces $\{U^m\mathcal{W}\}$ span the whole space $\mathcal{H}$; (i.e., if $\mathcal{H} = \bigoplus_m U^m\mathcal{W}$), then we say that $\mathcal{W}$ is a *generating* wandering subspace. Thus Theorem 1 says that if $\mathcal{W}_0$ is a generating wandering subspace for a unitary U, then $\mathcal{H}$ admits an orthogonal direct sum decomposition $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ consisting of reducing subspaces so that the unitary operator U is decomposed as $U = \bigoplus_\alpha U|_{\mathcal{H}_\alpha}$, where each $U|_{\mathcal{H}_\alpha}$ is unitary (direct summand of a unitary) acting on the subspace $\mathcal{H}_\alpha = \bigvee_m U^m e_\alpha$ generated by the one-dimensional subspace span$\{e_\alpha\}$. Hence, if the U-wandering subspace $\mathcal{W}_0$ is generating, then the orthonormal set $\{f_{m,\alpha}\}$ with $f_{m,\alpha} = U^m e_\alpha$ is a double indexed orthonormal basis for $\mathcal{H}$, and so (Fourier series)

$$x = \sum_m \sum_\alpha \langle x\,;f_{m,\alpha}\rangle f_{m,\alpha} = \sum_{m,\alpha} \langle x\,;f_{m,\alpha}\rangle f_{m,\alpha} = \sum_\alpha \sum_m \langle x\,;f_{m,\alpha}\rangle f_{m,\alpha}$$

for every x in $\mathcal{H}$ (where $\langle\ ;\ \rangle$ denotes the inner product on $\mathcal{H}$). Double indexed orthonormal bases yield a "Fubini-like" property allowing that summation order be interchanged, which has shown useful in wavelets theory (see Proposition 2 in [10]).

3. Applications

There are some alternative (but equivalent) ways of defining bilateral shifts on a Hilbert space (see e.g., [2], [4], [5], [6], [7], [11]). One of them goes as follows. An operator U acting on a Hilbert space $\mathcal{H}$ is a bilateral shift if there exists an infinite family $\{\mathcal{W}_m\}$ indexed by $\mathbb{Z}$ of nonzero pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_m \mathcal{W}_m$ (i.e., the orthogonal family $\{\mathcal{W}_m\}$ spans $\mathcal{H}$) and U maps each $\mathcal{W}_m$ isometrically onto $\mathcal{W}_{m+1}$. This ensures that $U|_{\mathcal{W}_m} : \mathcal{W}_m \to \mathcal{W}_{m+1}$ is a surjective isometry (i.e., a unitary transformation) so that the subspaces $\mathcal{W}_m$ are all unitarily equivalent and their common dimension is the multiplicity of U. It is readily verified that U is unitary and $\mathcal{W}_0$ is U-wandering. Therefore, $\mathcal{W}_0$ is a generating wandering subspace for U, and the multiplicity of U is precisely the orthogonal dimension of $\mathcal{W}_0$. Observe that $\mathcal{W}_0$ is separable if and only if $\mathcal{H}$ is, and recall that a shift of multiplicity μ (where μ is any cardinal number) is the direct sum of μ shifts of multiplicity one. Here is a straightforward corollary of Theorem 1.

Corollary 1. *Suppose U is a bilateral shift acting on a Hilbert space $\mathcal{H}$. Let $\{e_\alpha\}$ be an orthonormal basis for $\mathcal{W}_0$ indexed by Γ. Then* (a) *$\mathcal{H}$ admits the decomposition*

$$\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha,$$

where each $\mathcal{H}_\alpha = \bigvee_m U^m e_\alpha$ (a separable subspace of $\mathcal{H}$ generated by the one-dimensional space $\operatorname{span}\{e_\alpha\}$*) reduces U so that*

$$U = \bigoplus_\alpha U_\alpha,$$

with $U_\alpha = U|_{\mathcal{H}_\alpha}$. Moreover, (b) *each U_α is a bilateral shift of multiplicity one acting on $\mathcal{H}_\alpha$.*

Proof. Part (a) is an immediate consequence of Theorem 1, and part (b) is readily verified once $U_\alpha f_{m,\alpha} = U^{m+1} e_\alpha = f_{m+1,\alpha}$ for every m in $\mathbb{Z}$. Thus each U_α shifts the $\mathbb{Z}$-indexed orthonormal basis $\{f_{m,\alpha}\}$ for each $\mathcal{H}_\alpha$, and therefore U_α is a bilateral shift of multiplicity one on $\mathcal{H}_\alpha$. □

Consider the above setup where U is a bilateral shift on $\mathcal{H}$ and $\mathcal{W}_0 = \bigvee_\alpha e_\alpha$ is a generating wandering subspace for U. Recall that $\mathcal{W}_m = U^m \mathcal{W}_0$ and put

$$\mathcal{H}^+ = \bigvee_{k \in \mathbb{N}_0} \mathcal{W}_k.$$

$\mathbb{N}_0$ is the set of all *nonnegative* integers. This is a subspace of $\mathcal{H}$ spanned by the orthogonal ($\mathcal{W}_m \perp \mathcal{W}_n$ for $m \neq n$) sequence of subspaces $\{\mathcal{W}_k\}$. Thus we may write

$$\mathcal{H}^+ = \bigoplus_{k \in \mathbb{N}_0} \mathcal{W}_k.$$

It is known from [6] that *$\mathcal{H}^+ = \bigvee_{k \in \mathbb{N}_0} U^k \mathcal{W}_0$ is an irreducible-invariant subspace for U* (which means that $\mathcal{H}^+$ is U-invariant but not U^*-invariant) and, conversely, *if $\mathcal{M}$ is an irreducible invariant subspace for U, then there exists a wandering subspace $\mathcal{W}$ for U such that $\mathcal{M} = \bigvee_{k \in \mathbb{N}_0} U^k \mathcal{W}$.* Now, for each α in Γ, set

$$\mathcal{H}_\alpha^+ = \bigvee_{k \in \mathbb{N}_0} U^k e_\alpha.$$

Each $\mathcal{H}_\alpha^+$ is a separable (spanned by a countable set) subspace of $\mathcal{H}$ generated by the one-dimensional space span$\{e_\alpha\}$, which is clearly U-invariant. But none of them reduces U (i.e., none of them is U^*-invariant). Indeed, since U^* is invertible and $\{U^m e_\alpha\}$ is an orthonormal set (cf. Propositions 1 and 3),

$$U^*\mathcal{H}_\alpha^+ = \bigvee_{k\in\mathbb{N}_0} U^*U^k e_\alpha = \bigvee_{k\in\mathbb{N}_0} U^{k-1}e_\alpha \cong \bigoplus_{k\in\mathbb{N}_0} U^{k-1}e_\alpha = U^*e_\alpha \oplus \bigoplus_{k\in\mathbb{N}_0} U^k e_\alpha \cong U^*e_\alpha \oplus \mathcal{H}_\alpha^+$$

and so $U^*\mathcal{H}_\alpha^+ \not\subseteq \mathcal{H}_\alpha^+$ (since $U^*e_\alpha \neq 0$). Consider the family $\{\mathcal{H}_\alpha^+\}$ indexed by Γ.

Corollary 2. *$\{\mathcal{H}_\alpha^+\}$ is a family of pairwise orthogonal irreducible-invariant subspaces for U which decomposes the irreducible-invariant subspace $\mathcal{H}^+$:*

$$\mathcal{H}^+ = \bigoplus_\alpha \mathcal{H}_\alpha^+.$$

Proof. $\{e_\alpha\}$ is an orthonormal basis for $\mathcal{W}_0$. It follows by Proposition 3 that $\{U^k e_\alpha\}$ is an orthogonal family of vectors, and so $\{\mathcal{H}_\alpha^+\}$ is an orthogonal family of subspaces (see proof of Lemma 1). Moreover, it also follows that $\mathcal{H}^+ = \bigvee_{k\in\mathbb{N}_0} U^k \bigvee_\alpha e_\alpha$, and hence $\mathcal{H}^+ = \bigvee_{k\in\mathbb{N}_0}\bigvee_\alpha U^k e_\alpha$ by Proposition 1. But $\{U^k e_\alpha\}$ is an orthonormal basis for the Hilbert space $\bigvee_{k,\alpha} U^k e_\alpha$. Thus the same argument as in proof of Theorem 1 (unconditional convergence of the Fourier series) yields the desired result; that is, $\mathcal{H}^+ = \bigvee_\alpha \bigvee_{k\in\mathbb{N}_0} U^k e_\alpha = \bigvee_\alpha \mathcal{H}_\alpha^+ \cong \bigoplus_\alpha \mathcal{H}_\alpha^+$ since $\mathcal{H}_\alpha^+ \perp \mathcal{H}_\beta^+$ whenever $\alpha \neq \beta$. □

For applications of these decompositions in wavelets theory the reader is referred to [10] and [9] (also see [1]).

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