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SHIFT REDUCING SUBSPACES AND IRREDUCIBLE-INVARIANT SUBSPACES GENERATED BY WANDERING VECTORS AND APPLICATIONS

CARLOS S. KUBRUSLY AND NHAN LEVAN

Abstract. We introduce the notions of elementary reducing subspaces and elementary irreducible-invariant subspaces—generated from wandering vectors—of a shift operator of countably infinite multiplicity, defined on a separable Hilbert space $\mathcal{H}$. Necessary and sufficient conditions for a set of shift wandering vectors to span a wandering subspace are obtained. These lead to characterizations of shift reducing subspaces and shift irreducible-invariant subspaces, as well as a new decomposition of $\mathcal{H}$ into orthogonal sum of elementary reducing subspaces. Applications of elementary reducing subspaces to Wavelet Expansion, and of elementary irreducible-invariant subspaces to Wavelet Multiresolution Analysis will be discussed.

1. Introduction

Let $U : \mathcal{H} \to \mathcal{H}$ be a linear bounded operator on a separable Hilbert space $\mathcal{H}$—with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. A closed subspace $\mathcal{W}$ of $\mathcal{H}$ is called *wandering subspace* for U, or simply, U*-wandering*, if [4, 12],

$$\mathcal{W} \perp U^m \mathcal{W}, \quad m > 0. \tag{1.1}$$

If the operator U is unitary, then (1.1) is equivalent to,

$$U^m \mathcal{W} \perp U^{m'} \mathcal{W}, \quad \forall\, m, m' \in \mathbb{Z} \quad \text{whenever} \quad m \neq m'. \tag{1.2}$$

Similarly, $w \in \mathcal{H}$ is a U*-wandering vector* if it spans a U-wandering subspace [11].

We now recall the "Wandering Subspace" definition of Hilbert space shift operators [4, 12]. Note: by a "shift" we mean a "bilateral shift".

Definition 1. A shift $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator for which there is a wandering subspace $\mathcal{W}$ such that $\mathcal{H}$ admits the "wandering subspace decomposition"

$$\mathcal{H} = \bigoplus_{m \in \mathbb{Z}} U^m \mathcal{W}. \tag{1.3}$$

The wandering subspace $\mathcal{W}$ is then called *generating* and its dimension is the *multiplicity* of U.

In the following we will be dealing with shifts of countably infinite multiplicity. What is interesting is the fact that the mutually orthogonal subspaces $\{U^m \mathcal{W}\}_{m \in \mathbb{Z}}$ are neither U-invariant nor U^*-invariant. Moreover, they do not even include any U-invariant or U^*-invariant subspace. Hence, they can serve as building blocks for

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shift reducing subspaces, as well as for shift irreducible-invariant subspaces. Also, it is worth noting that a shift wandering subspace cannot be reducing, while a shift reducing subspace cannot be wandering.

An invariant subspace can be uniquely decomposed into direct sum of an irreducible invariant subspace and a reducing subspace [4]. In this paper we present an "elementary" characterization of shift reducing subspaces, as well as that of shift irreducible-invariant subspaces. Our characterizations are elementary in the sense that they neither rely on the functional calculus of normal operators [5], nor on the functional representation of shifts [4], but are based on shift wandering vectors.

Let $U : \mathcal{H} \to \mathcal{H}$ be a shift, and let $\mathcal{W}$ be a U-wandering subspace. It is easy to see that the subspace

$$\mathcal{M}_{ir} := \bigvee_{m=0}^{\infty} U^m \mathcal{W}, \tag{1.4}$$

is U-irreducible-invariant [4]. More is true. Halmos [4] has shown that, if $\mathcal{M}$ is a U-irreducible-invariant subspace, then there is a U-wandering subspace $\mathcal{W}$ which is such that $\mathcal{M} = \bigvee_{m=0}^{\infty} U^m \mathcal{W}$. Similar result can be stated for U^*-irreducible-invariant subspaces. Our characterization of shift irreducible-invariant subspaces begins with Halmos' results. We then show that a U-irreducible-invariant subspace can be represented by elementary irreducible-invariant subspaces—generated from U-wandering vectors.

Let $\{\psi_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$ be an orthonormal set and define

$$\mathcal{W} := \overline{\text{span}}\{\psi_n\}_{n\in\mathbb{Z}} = \bigvee_{n\in\mathbb{Z}} \psi_n. \tag{1.5}$$

It is easy to see that if $\mathcal{W}$ is U-wandering, then so are the vectors $\{\psi_n\}_{n\in\mathbb{Z}}$. The converse is not true! It turns out that a sufficient condition for the orthonormal U-wandering vectors $\{\psi_n\}_{n\in\mathbb{Z}}$ to span a U-wandering subspace $\mathcal{W}$ is that the elementary U-reducing subspaces be orthogonal. This is shown in Theorem 1. Theorem 2 gives a decomposition of shift reducing subspace into an orthogonal sum of elementary reducing subspaces. In addition to the familiar wandering subspace decomposition (1.3) of $\mathcal{H}$, we show that $\mathcal{H}$ can also be decomposed into an orthogonal sum of elementary reducing subspaces. These are developed in Section 2, while application of shift elementary reducing subspaces to Wavelet Expansion is taken up in Section 3. Section 4 discusses representation of shift irreducible-invariant subspaces in terms of elementary irreducible-invariant subspaces. Finally, Section 5 connects shift elementary irreducible-invariant subspaces to shift outgoing and incoming subspaces, as well as to Wavelet Multiresolution Analysis (MRA). We then show a decomposition of the Wavelet MRA Time Operator [1] into "elementary" Time Operators.

We close the paper with a discussion on advantages of "representation-free" Hilbert space shift operators which is the "icon" of our paper.

2. Shift Elementary Reducing Subspaces

Let $U : \mathcal{H} \to \mathcal{H}$ be a shift of countably infinite multiplicity. We begin with the following lemma.

Lemma 1. *Let $\mathcal{W}$ be as defined by* (1.5).

(i) *If*

$$\mathcal{W}_m := U^m\mathcal{W} = U^m \bigvee_{n\in\mathbb{Z}} \psi_n, \quad m \in \mathbb{Z}, \tag{2.1}$$

then

$$\mathcal{W}_m = \bigvee_{n\in\mathbb{Z}} U^m\psi_n, \quad m \in \mathbb{Z}. \tag{2.2}$$

(ii) *Moreover, if $\mathcal{W}$ is a U-wandering subspace, then*

$$U^m\psi_n \perp U^{m'}\psi_{n'}, \quad \textit{whenever} \quad m \neq m', \quad \forall\, n, n' \in \mathbb{Z}. \tag{2.3}$$

In particular,

$$U^m\psi_n \perp U^{m'}\psi_n, \quad \textit{whenever} \quad m \neq m', \quad \forall\, n \in \mathbb{Z}. \tag{2.4}$$

(i.e., ψ_n, *for $n \in \mathbb{Z}$, are U-wandering vectors*).

Proof. (i) Recall that

$$U^m\mathrm{span}\{\psi_n\}_{n\in\mathbb{Z}} \;=\; \mathrm{span}\{U^m\psi_n\}_{n\in\mathbb{Z}}, \quad m \in \mathbb{Z}, \tag{2.5}$$

and

$$U^m\,\overline{\mathrm{span}}\{\psi_n\}_{n\in\mathbb{Z}} \;\subseteq\; \overline{U^m\mathrm{span}\{\psi_n\}}_{n\in\mathbb{Z}} \tag{2.6}$$

since U^m is continuous ([5], Problem 3.46). Moreover,

$$\overline{U^m\overline{\mathrm{span}}\{\psi_n\}}_{n\in\mathbb{Z}} \;=\; U^m\overline{\mathrm{span}}\{\psi_n\}_{n\in\mathbb{Z}} \tag{2.7}$$

because U^{-m} is continuous—inverse image of closed sets are closed. Therefore, by (2.6) and (2.7),

$$U^m\overline{\mathrm{span}}\{\psi_n\}_{n\in\mathbb{Z}} \;\subseteq\; \overline{U^m\mathrm{span}\{\psi_n\}_{n\in\mathbb{Z}}} \;\subseteq\; \overline{U^m\overline{\mathrm{span}}\{\psi_n\}_{n\in\mathbb{Z}}} \;=\; U^m\overline{\mathrm{span}}\{\psi_n\}_{n\in\mathbb{Z}}.$$

Hence

$$U^m\overline{\mathrm{span}}\{\psi_n\}_{n\in\mathbb{Z}} \;=\; \overline{U^m\mathrm{span}\{\psi_n\}_{n\in\mathbb{Z}}}. \tag{2.8}$$

It then follows from this and from (2.5) that

$$U^m\overline{\mathrm{span}}\{\psi_n\}_{n\in\mathbb{Z}} \;=\; \overline{U^m\mathrm{span}\{\psi_n\}_{n\in\mathbb{Z}}} \;=\; \overline{\mathrm{span}}\{U^m\psi_n\}_{n\in\mathbb{Z}}.$$

Thus (2.2) is proven.

(ii) We have

$$U^m\psi_n \in U^m\mathcal{W}, \quad \forall\, m, n \in \mathbb{Z}$$

by (2.1). Therefore, for arbitrary m, n and m',

$$U^m\psi_n \perp U^{m'}\mathcal{W}, \quad \text{whenever} \quad m \neq m',$$

by (1.2)—since $\mathcal{W}$ is U-wandering. Hence

$$U^m\psi_n \perp U^{m'}\psi_{n'}, \quad \forall\, n' \in \mathbb{Z},$$

by (2.2). In particular,

$$U^m\psi_n \perp U^{m'}\psi_n, \quad \text{whenever} \quad m \neq m', \quad \forall\, n \in \mathbb{Z},$$

(i.e., ψ_n are U-wandering vectors). This finishes the proof. □

Lemma 2. *Define the elementary reducing subspaces*

$$\mathcal{H}_n := \bigvee_{m\in\mathbb{Z}} U^m\psi_n, \quad n \in \mathbb{Z}. \tag{2.9}$$

If

$$\mathcal{H}_n \perp \mathcal{H}_{n'} \quad \textit{whenever} \quad n \neq n',$$

then

$$U^m\psi_n \perp U^{m'}\psi_{n'}, \quad \textit{whenever} \quad n \neq n', \quad \forall\, m, m' \in \mathbb{Z}.$$

Proof. We have by assumption and by definition of $\mathcal{H}_n$,

$$\bigvee_{k\in\mathbb{Z}} U^k\psi_n \;\perp\; \bigvee_{k\in\mathbb{Z}} U^k\psi_{n'}, \quad \text{whenever} \quad n \neq n'.$$

But, for each m and n,

$$U^m\psi_n \in \bigvee_{k\in\mathbb{Z}} U^k\psi_n.$$

Similarly, for each m' and n',

$$U^{m'}\psi_{n'} \in \bigvee_{k\in\mathbb{Z}} U^k\psi_{n'}.$$

It then follows that

$$U^m\psi_n \perp U^{m'}\psi_{n'}, \quad \text{whenever} \quad n \neq n', \quad \forall\, m, m' \in \mathbb{Z},$$

and the Lemma is proven. □

We are now ready to prove the following theorem.

Theorem 1. *Let $U : \mathcal{H} \to \mathcal{H}$ be a shift of countably infinite multiplicity, and $\{\psi_n\}_{n\in\mathbb{Z}}$ be an orthonormal set in $\mathcal{H}$. Let $\mathcal{W}$ be spanned by $\{\psi_n\}_{n\in\mathbb{Z}}$, and $\mathcal{H}_n$, $n \in \mathbb{Z}$, be spanned by $\{U^m\psi_n\}_{m\in\mathbb{Z}}$.*

(i) *If $\mathcal{W}$ is a U-wandering subspace then ψ_n, $n \in \mathbb{Z}$, are U-wandering vectors. Moreover, the elementary reducing subspaces $\mathcal{H}_n$, $n \in \mathbb{Z}$, are mutually orthogonal.*

(ii) *If ψ_n, $n \in \mathbb{Z}$, are U-wandering vectors, and $\mathcal{H}_n$, $n \in \mathbb{Z}$, are mutually orthogonal, then $\mathcal{W}$ is a U-wandering subspace.*

Proof. The first part of part (i) is already covered by Lemma 1(ii), while orthogonality of the subspaces $\mathcal{H}_n$, $n \in \mathbb{Z}$, follows readily from (2.3):

$$U^m\psi_n \perp U^{m'}\psi_{n'}, \quad \text{whenever} \quad m \neq m', \quad \forall\, n, n' \in \mathbb{Z},$$

and since for $m = m'$ we already have $\psi_n \perp \psi_{n'}$, whenever $n \neq n'$.

For part (ii) we first note that, it follows from Lemma 2 and from the assumption that ψ_n are U-wandering vectors,

$$U^m\psi_n \perp U^{m'}\psi_{n'}, \quad \forall\, n, n' \in \mathbb{Z}, \quad \text{whenever} \quad m \neq m'.$$

Therefore,

$$U^m\psi_n \perp \bigvee_{k\in\mathbb{Z}} U^{m'}\psi_k, \quad \forall\, n \in \mathbb{Z}, \quad \text{whenever} \quad m \neq m'.$$

Hence,

$$\bigvee_{k\in\mathbb{Z}} U^m\psi_k \perp \bigvee_{k\in\mathbb{Z}} U^{m'}\psi_k, \quad \text{whenever} \quad m \neq m',$$

or

$$U^m\mathcal{W} \perp U^{m'}\mathcal{W}, \quad \text{whenever} \quad m \neq m';$$

i.e., $\mathcal{W}$ is U-wandering. This completes the proof of Theorem 1. □

A consequence of Theorem 1 is:

Theorem 2. *Let $\mathcal{M}_{re}$ be the closed U-reducing subspace*

$$\mathcal{M}_{re} := \bigvee_{m\in\mathbb{Z}} U^m\mathcal{W}, \tag{2.10}$$

where the U-wandering subspace $\mathcal{W}$ is spanned by an orthonormal set $\{\psi_n\}_{n\in\mathbb{Z}}$. Then

$$\mathcal{M}_{re} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n, \tag{2.11}$$

where $\mathcal{H}_n$, $n\in\mathbb{Z}$, are the elementary reducing subspaces spanned by $\{U^m\psi_n\}_{m\in\mathbb{Z}}$, $n\in\mathbb{Z}$.

Proof. First, recall that $\mathcal{M}_{re}$ in fact reduces U since it is clearly invariant for every power of U, and since $U^* = U^{-1}$; thus it is invariant for U and U^*, and hence reduces U. Now we have

$$\mathcal{M}_{re} = \bigvee_{m\in\mathbb{Z}} U^m\mathcal{W} = \bigvee_{m\in\mathbb{Z}} \mathcal{W}_m = \bigvee_{m\in\mathbb{Z}}\bigvee_{n\in\mathbb{Z}} U^m\psi_n \tag{2.12}$$

by Lemma 1(i); see (2.1). Next recall that, since $\mathcal{H}_n \perp \mathcal{H}_m$ for $n\neq m$ (Theorem 1),

$$\bigvee_{n\in\mathbb{Z}}\bigvee_{m\in\mathbb{Z}} U^m\psi_n = \overline{\Big(\sum_{n=-\infty}^{\infty}\bigvee_{m\in\mathbb{Z}} U^m\psi_n\Big)} \cong \bigoplus_{n=-\infty}^{\infty}\bigvee_{m\in\mathbb{Z}} U^m\psi_n = \bigoplus_{n=-\infty}^{\infty}\mathcal{H}_n, \tag{2.13}$$

where $\cong$ means *unitarily equivalent.* Similarly, since $\mathcal{W}_n \perp \mathcal{W}_m$ for $n\neq m$, by (1.2), (2.1) and (2.2), it also follows that

$$\bigvee_{m\in\mathbb{Z}}\bigvee_{n\in\mathbb{Z}} U^m\psi_n = \overline{\Big(\sum_{m=-\infty}^{\infty}\bigvee_{n\in\mathbb{Z}} U^m\psi_n\Big)} \cong \bigoplus_{m=-\infty}^{\infty}\bigvee_{n\in\mathbb{Z}} U^m\psi_n = \bigoplus_{m=-\infty}^{\infty}\mathcal{W}_m. \tag{2.14}$$

But U is a unitary operator so that $\{U^m\psi_n\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for the Hilbert space $\bigvee_{m,n\in\mathbb{Z}} U^m\psi_n$ according to Lemma 1(ii) and Lemma 2. Thus, by unconditional convergence of the Fourier Series,

$$\bigvee_{m\in\mathbb{Z}}\bigvee_{n\in\mathbb{Z}} U^m\psi_n = \bigvee_{m,n\in\mathbb{Z}} U^m\psi_n = \bigvee_{n\in\mathbb{Z}}\bigvee_{m\in\mathbb{Z}} U^m\psi_n. \tag{2.15}$$

Therefore,

$$\bigoplus_{m\in\mathbb{Z}} \mathcal{W}_m \;\cong\; \mathcal{M}_{re} \;\cong\; \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n.$$

This finishes the proof of the theorem by writing $=$ for $\cong$, as usual. □

The next proposition follows at once from Theorems 1 and 2.

Proposition 1. (i) *If the U-wandering subspace $\mathcal{W}$ of Theorem* 1(i) *is also generating then, in addition to the U-wandering subspace decomposition,*

$$\mathcal{H} = \bigoplus_{m\in\mathbb{Z}} U^m\mathcal{W}, \tag{2.16}$$

and $\mathcal{H}$ also admits the elementary reducing subspaces decomposition,

$$\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n. \tag{2.17}$$

Hence, the operator U admits the decomposition

$$U = \bigoplus_{n=-\infty}^{\infty} U_n, \tag{2.18}$$

where each $U_n := U|\mathcal{H}_n$ is a shift of multiplicity 1 and whose generating-wandering subspace is $\operatorname{span}\{\psi_n\} = \bigvee \psi_n$.

(ii) *If, in addition to the conditions of Theorem* 1(ii)*, the subspaces $\mathcal{H}_n$, $n \in \mathbb{Z}$, span $\mathcal{H}$, then the U-wandering subspace $\mathcal{W}$ is generating.*

Proof. If $\mathcal{W}$ is U-wandering and generating then the subspace $\mathcal{M}_{re}$ is all of $\mathcal{H}$. Therefore (2.17) follows readily from Theorem 2, while (2.18) is self-evident. For Proposition 1(ii) we first note that, by Theorem 1(ii), $\mathcal{W}$ is already U-wandering. It remains to show that it is generating. Suppose $\mathcal{H}_n$, $n \in \mathbb{Z}$ span $\mathcal{H}$, and recall that $\mathcal{H}_n \perp \mathcal{H}_m$ for $n \neq m$. Then

$$\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n = \bigoplus_{n\in\mathbb{Z}} \bigvee_{m\in\mathbb{Z}} U^m\psi_n \cong \bigvee_{n,m\in\mathbb{Z}} U^m\psi_n. \tag{2.19}$$

Therefore, as in the proof of Theorem 2,

$$\mathcal{H} = \bigoplus_{m\in\mathbb{Z}} U^m\mathcal{W}; \tag{2.20}$$

i.e., $\mathcal{W}$ is a U-generating-wandering subspace, which completes the proof. □

Recall that a unitary operator is a direct sum of infinitely many unitary operators [10], which happens in particular for a (bilateral) shift of infinite multiplicity. Note that this well-known result (see e.g., [10], p.46) is also evident from (2.18), as summarized below.

Corollary 1. *A shift of countably infinite multiplicity is a direct sum of infinitely many shifts of multiplicity* 1.

It is plain from (2.16) that, each $h \in \mathcal{H}$ can be written as,

$$h = \sum_{m=-\infty}^{\infty} U^m w_m, \tag{2.21}$$

where

$$w_m \in \mathcal{W} \quad \text{and} \quad \sum_{m=-\infty}^{\infty} \|w_m\|^2 = \|h\|^2. \tag{2.22}$$

Therefore,

$$Uh \;=\; \sum_{m=-\infty}^{\infty} U^{m+1} w_m \;=\; \sum_{m'=-\infty}^{\infty} U^{m'} w_{m'-1}. \tag{2.23}$$

Let $\Phi\colon \mathcal{H} \to \ell^2(-\infty,\infty;\mathcal{W})$ be the map defined by

$$\Phi h = \{w_m\}_{m\in\mathbb{Z}}. \tag{2.24}$$

Then it is plain that Φ is unitary, and the shift U goes into the right shift S_r on $\ell^2(-\infty,\infty;\mathcal{W})$,

$$S_r\{w_m\}_{m\in\mathbb{Z}} = \{w_{m-1}\}_{m\in\mathbb{Z}}. \tag{2.25}$$

The shift action of U on $\mathcal{H}$, under the decomposition (2.17), is transparent since

$$h = \sum_{n=-\infty}^{\infty} h_n, \quad \forall\, h \in \mathcal{H}, \tag{2.26}$$

where

$$h_n \in \mathcal{H}_n \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \|h_n\|^2 = \|h\|^2. \tag{2.27}$$

Therefore,

$$Uh = \sum_{n=-\infty}^{\infty} Uh_n. \tag{2.28}$$

Then, since each $\mathcal{H}_n$ is reducing, the action of U takes place on $\mathcal{H}_n$. Now let us expand h_n in terms of the orthonormal basis $\{U^m\psi_n\}_{m\in\mathbb{Z}}$ of $\mathcal{H}_n$. We have

$$h_n = \sum_{m=-\infty}^{\infty} \langle h_n, U^m\psi_n\rangle U^m\psi_n \ \in \ \mathcal{H}_n. \tag{2.29}$$

Hence,

$$Uh_n = \sum_{m=-\infty}^{\infty} \langle h_n, U^m\psi_n\rangle U^{m+1}\psi_n = \sum_{m'=-\infty}^{\infty} \langle h_n, U^{m'-1}\psi_n\rangle U^{m'}\psi_n. \tag{2.30}$$

Thus $U|\mathcal{H}_n := U_n$ goes into the right shift S_n, of multiplicity 1, defined by

$$S_n\{\langle h_n, U^m\psi_n\rangle\}_{m\in\mathbb{Z}} = \{\langle h_n, U^{m-1}\psi_n\rangle\}_{m\in\mathbb{Z}}, \quad n \in \mathbb{Z}. \tag{2.31}$$

Consequently, U goes into the shift S which is direct sum of infinitely many shifts S_n of multiplicity 1—on the Hilbert space $\ell^2\big(-\infty,\infty;\bigvee\{\langle h_n, U^m\psi_n\rangle\}_{m\in\mathbb{Z}}\big)$, $n \in \mathbb{Z}$,

$$S := \bigoplus_{n=-\infty}^{\infty} S_n. \tag{2.32}$$

3. Shift Elementary Reducing Subspaces in Wavelet Expansion

We now turn to application of elementary shift reducing subspaces $\mathcal{H}_n$, $n \in \mathbb{Z}$, to Wavelet Expansion.

Let D denote the dilation-by-2 operator defined on the function space $\mathcal{L}^2(\mathbb{R})$ by

$$Df = g, \quad g(\cdot) = \sqrt{2}\, f(2(\cdot)), \quad f(\cdot) \in \mathcal{L}^2(\mathbb{R}). \tag{3.1}$$

It is plain that D is unitary. Moreover, it is a shift of countably infinite multiplicity. Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ and define the functions

$$\psi_n(\cdot) := \psi((\cdot) - n) = T^n\psi(\cdot), \quad n \in \mathbb{Z}, \tag{3.2}$$

where T is the translation-by-1 operator on $\mathcal{L}^2(\mathbb{R})$ defined by

$$Tf = g, \quad g(\cdot) = f((\cdot) - 1), \tag{3.3}$$

and it is also a shift of countably infinite multiplicity. Now let $\psi_{m,n}(\cdot)$ be "generated" from $\psi_n(\cdot)$ by

$$\psi_{m,n}(\cdot) := D^m\psi_n(\cdot) = \sqrt{2}^{\,m}\psi_n(2^m(\cdot)) = \sqrt{2}^{\,m}\psi(2^m(\cdot) - n). \tag{3.4}$$

for $m, n \in \mathbb{Z}$. We recall the following definition from [9].

Definition 2. If the functions $\psi_{m,n}(\cdot)$, $m, n \in \mathbb{Z}$, are orthonormal and span the function space $\mathcal{L}^2(\mathbb{R})$, then $\psi(\cdot)$ is called an orthonormal wavelet or, simply a wavelet, and $\psi_{m,n}(\cdot)$ are called wavelet functions—generated from $\psi(\cdot)$.

It follows easily from the above that.

Lemma 3. *Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ be a T-wandering vector,*

$$T^n\psi(\cdot) \perp T^{n'}\psi(\cdot), \quad \textit{whenever} \quad n \neq n', \quad n, n' \in \mathbb{Z}.$$

Then $\psi(\cdot)$ is a wavelet if and only if the orthonormal functions $\psi_n(\cdot)$, $n \in \mathbb{Z}$, defined by (3.2)*, span a generating D-wandering subspace*

$$\mathcal{W}(\psi) := \bigvee_{n\in\mathbb{Z}} \psi_n(\cdot) = \bigvee_{n\in\mathbb{Z}} T^n\psi(\cdot). \tag{3.5}$$

It follows at once from this Lemma that, for a given wavelet $\psi(\cdot)$ there corresponds a D-wandering subspace decomposition of the function space $\mathcal{L}^2(\mathbb{R})$,

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} D^m\mathcal{W}(\psi) = \bigoplus_{m\in\mathbb{Z}} \mathcal{W}_m(\psi), \tag{3.6}$$

where

$$\mathcal{W}_m(\psi) := D^m\mathcal{W}(\psi), \quad m \in \mathbb{Z}. \tag{3.7}$$

Therefore, any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ admits the orthogonal decomposition

$$f(\cdot) = \sum_{m\in\mathbb{Z}} D^m w_m(\cdot), \tag{3.8}$$

where

$$w_m \in \mathcal{W}(\psi) \quad \text{and} \quad \sum_{m\in\mathbb{Z}} \|w_m\|^2 = \|f\|^2. \tag{3.9}$$

Let $P_{\mathcal{W}_m(\psi)}$ be the orthogonal projections onto the subspaces $\mathcal{W}_m(\psi)$, then it follows from (3.8) that

$$P_{\mathcal{W}_m(\psi)}\big(f(\cdot)\big) = D^m w_m(\cdot), \quad m \in \mathbb{Z}, \tag{3.10}$$

since, by definition, $D^m w_m \in \mathcal{W}_m(\psi)$. Therefore, since D is unitary,

$$w_m(\cdot) = D^{*\,m} P_{\mathcal{W}_m(\psi)}\big(f(\cdot)\big), \quad m \in \mathbb{Z}. \tag{3.11}$$

From which it follows that

$$\langle w_m(\cdot), \psi_n(\cdot)\rangle = \big\langle D^{*\,m} P_{\mathcal{W}_m(\psi)}\big(f(\cdot)\big), \psi_n(\cdot)\big\rangle = \big\langle f(\cdot), P_{\mathcal{W}_m(\psi)}\big(D^m\psi_n(\cdot)\big)\big\rangle.$$

Therefore,

$$\langle w_m(\cdot), \psi_n\rangle = \langle f(\cdot), \psi_{m,n}(\cdot)\rangle, \quad m, n \in \mathbb{Z}, \tag{3.12}$$

since $D^m\psi_n$ $(= \psi_{m,n}(\cdot))$ already lives in $\mathcal{W}_m(\psi)$. Now, since the orthonormal set $\{\psi_n(\cdot)\}_{n\in\mathbb{Z}}$ spans $\mathcal{W}(\psi)$, we also have

$$w_m(\cdot) = \sum_{n\in\mathbb{Z}} \langle w_m(\cdot), \psi_n(\cdot)\rangle\, \psi_n(\cdot), \quad m \in \mathbb{Z}. \tag{3.13}$$

Therefore, by (3.12),

$$w_m(\cdot) = \sum_{n\in\mathbb{Z}} \langle f(\cdot), \psi_{m,n}(\cdot)\rangle\, \psi_n(\cdot), \quad m \in \mathbb{Z}. \tag{3.14}$$

This can be rewritten as

$$w_m(\cdot) = \sum_{n\in\mathbb{Z}} \langle D^{*\,m} f(\cdot), \psi_n(\cdot)\rangle\, \psi_n(\cdot), \quad m \in \mathbb{Z}, \tag{3.15}$$

which implies that

$$w_m(\cdot) = P_{\mathcal{W}(\psi)}\big(D^{*\,m} f(\cdot)\big), \quad m \in \mathbb{Z}. \tag{3.16}$$

Therefore, by (3.10),

$$P_{\mathcal{W}_m(\psi)}\big(f(\cdot)\big) = D^m w_m(\cdot) = D^m P_{\mathcal{W}(\psi)}\big(D^{*\,m} f(\cdot)\big), \quad m \in \mathbb{Z}, \tag{3.17}$$

or

$$P_{\mathcal{W}_m(\psi)} f(\cdot) = \sum_{n\in\mathbb{Z}} \langle f(\cdot), \psi_{m,n}(\cdot)\rangle\, \psi_{m,n}(\cdot) = D^m P_{\mathcal{W}(\psi)}\big(D^{*\,m} f(\cdot)\big), \quad m \in \mathbb{Z}. \tag{3.18}$$

In Wavelet Theory [9], the subspace $\mathcal{W}_m(\psi)$ is called "*scale-2^m-time-shift detail subspace*", while $P_{\mathcal{W}_m(\psi)}\big(f(\cdot)\big)$ is "*scale-2^m-time-shift detail variations of $f(\cdot)$*". The function $D^m f(\cdot)\ \big(= \sqrt{2}^{\,m} f(2^m(\cdot)\big)$ is referred to as ***$f(\cdot)$ at scale 2^m***, while $D^{*\,m} f(\cdot)\ \big(= \frac{1}{\sqrt{2}^{\,m}} f(\frac{1}{2^m}(\cdot))\big)$ is ***$f(\cdot)$ at resolution 2^{-m}***. We conclude from (3.20):

Proposition 2. *Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ be a wavelet. Then the projections $P_{\mathcal{W}_m(\psi)}$ and $P_{\mathcal{W}(\psi)}$ are unitarily equivalent, with D^m acting as the equivalence operator,*

$$P_{\mathcal{W}_m(\psi)} = D^m P_{\mathcal{W}(\psi)}\, D^{*\,m}, \quad m \in \mathbb{Z}.$$

From which it follows that

$$P_{\mathcal{W}_{m+1}(\psi)} = D P_{\mathcal{W}_m(\psi)}\, D^*, \quad m \in \mathbb{Z}.$$

In other words, the scale-2^{m+1}-time-shift detail variations of $f(\cdot)$ is equal to the scale-2^m-time-shift detail variations at scale 2 of $f(\cdot)$—at resolution 2^{-1}.

We have from (3.10) and (3.16),

$$f(\cdot) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, \psi_{m,n}\rangle\, \psi_{m,n}(\cdot). \tag{3.19}$$

This is the "usual" wavelet expansion over all scales of time-shift detail variations of $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$.

We now turn to another type of expansion which is a consequence of the "elementary D-reducing subspaces decomposition" of the function space $\mathcal{L}^2(\mathbb{R})$. Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ be a wavelet, then we have, by Proposition 1,

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n = \bigoplus_{n\in\mathbb{Z}} \bigvee_{m\in\mathbb{Z}} D^m \psi_n(\cdot). \tag{3.20}$$

Hence, $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ now admits the orthogonal decomposition,

$$f(\cdot) = \sum_{n\in\mathbb{Z}} h_n(\cdot), \tag{3.21}$$

where

$$h_n \in \mathcal{H}_n \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \|h_n\|^2 = \|f\|^2, \tag{3.22}$$

and

$$h_n(\cdot) = P_{\mathcal{H}_n}\big(f(\cdot)\big), \quad n \in \mathbb{Z}. \tag{3.23}$$

Here $P_{\mathcal{H}_n}$ is the orthogonal projection onto $\mathcal{H}_n$. Therefore, since the orthonormal set $\{D^m\psi_n(\cdot)\}_{m\in\mathbb{Z}}$ spans $\mathcal{H}_n$,

$$h_n(\cdot) = \sum_{m\in\mathbb{Z}} \langle h_n, D^m\psi_n\rangle D^m\psi_n(\cdot) \tag{3.24}$$

or, from (3.23),

$$h_n(\cdot) = \sum_{m\in\mathbb{Z}} \langle P_{\mathcal{H}_n}(f), D^m\psi_n\rangle D^m\psi_n(\cdot). \tag{3.25}$$

Therefore, as before,

$$h_n(\cdot) = \sum_{m\in\mathbb{Z}} \langle f, P_{\mathcal{H}_n}(D^m\psi_n)\rangle D^m\psi_n(\cdot). \tag{3.26}$$

Consequently,

$$h_n(\cdot) = P_{\mathcal{H}_n}\big(f(\cdot)\big) = \sum_{m\in\mathbb{Z}} \langle f, \psi_{m,n}\rangle\, \psi_{m,n}(\cdot), \quad n \in \mathbb{Z}. \tag{3.27}$$

We call the subspace $\mathcal{H}_n$ the *n-time-shift-scale detail subspace*, while $P_{\mathcal{H}_n}\big(f(\cdot)\big)$ is the *n-time-shift-scale detail variations of* $f(\cdot)$. Now, let us rewrite (3.27) as

$$h_n(\cdot) \;=\; P_{\mathcal{H}_n}\big(f(\cdot)\big) \;=\; \sum_{m\in\mathbb{Z}} P_{\psi_{m,n}}\big(f(\cdot)\big) \tag{3.28}$$

$$= \sum_{m\in\mathbb{Z}} \langle f, D^m\psi_n\rangle D^m\psi_n(\cdot) \tag{3.29}$$

$$= \sum_{m\in\mathbb{Z}} D^m\langle D^{*\,m} f, \psi_n\rangle\, \psi_n(\cdot), \quad n \in \mathbb{Z}. \tag{3.30}$$

But

$$\langle D^{*\,m} f, \psi_n\rangle\, \psi_n(\cdot) = P_{\psi_n}\big(D^{*\,m} f(\cdot)\big). \tag{3.31}$$

Therefore (3.30) can be rewritten as

$$P_{\mathcal{H}_n}\big(f(\cdot)\big) = \sum_{m\in\mathbb{Z}} D^m P_{\psi_n} D^{*\,m}\big(f(\cdot)\big). \tag{3.32}$$

It then follows from (3.28) and (3.32) that

$$P_{\psi_{m,n}}\big(f(\cdot)\big) = D^m P_{\psi_n} D^{*\,m}\big(f(\cdot)\big), \quad m,n \in \mathbb{Z}, \quad \forall\, f \in \mathcal{L}^2(\mathbb{R}). \tag{3.33}$$

We have therefore proved the following proposition.

Proposition 3. *Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ be a wavelet. Then the projections $P_{\psi_{m,n}}$ and $P_{\psi_{0,n}}$ are unitarily equivalent, with D^m acting as the equivalence operator,*

$$P_{\psi_{m,n}} \;=\; D^m P_{\psi_{0,n}} D^{*\,m}, \quad m \in \mathbb{Z},$$

where $\psi_{0,n}(\cdot) := \psi((\cdot) - n) = \psi_n(\cdot)$. Therefore,

$$P_{\psi_{m+1,n}} \;=\; D P_{\psi_{m,n}} D^*, \quad m \in \mathbb{Z}.$$

In other words, the scale-2^{m+1}-time-shift-n detail variation of $f(\cdot)$ is equal to the scale-2^m-time-shift-n detail variation at scale 2 of $f(\cdot)$—at resolution 2^{-1}.

This proposition is an analog of Proposition 2. We will have more to say about the projections $P_{\mathcal{W}_m(\psi)}$ and $P_{\psi_{m,n}}$ in Section 5.

It follows from (3.21) and (3.27) that

$$f(\cdot) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \langle f, \psi_{m,n}\rangle\, \psi_{m,n}(\cdot). \tag{3.34}$$

This shows that, with respect to a wavelet $\psi(\cdot)$, a function $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ is summation of all its time-shift-n-scale detail variations, as well as what we have seen above, summation of all its scale-2^m-time-shift detail variations.

Now, for each $m \in \mathbb{Z}$ and each $n \in \mathbb{Z}$,

$$\mathcal{W}_m(\psi) \cap \mathcal{H}_n(\psi) = \{\psi_{m,n}\}. \tag{3.35}$$

Then, since the orthogonal complements of $\{\psi_{m,n}\}$ in $\mathcal{W}_m(\psi)$ and in $\mathcal{H}_n(\psi)$, respectively, are orthogonal, we have

$$P_{\psi_{m,n}} f(\cdot) = P_{\mathcal{W}_m(\psi)} P_{\mathcal{H}_n(\psi)} f(\cdot) = P_{\mathcal{H}_n(\psi)} P_{\mathcal{W}_m(\psi)} f(\cdot). \tag{3.36}$$

This implies that, for $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$, its "detail-variations at scale-2^m and time-shift-n" can be obtained in two ways. Either by projecting its "time-shift-n detail-variations" onto the "scale-2^m-time-shift detail subspace," or by projecting its "scale-2^m-time-shift detail variations" onto the "time-shift-n-scale detail subspace." These explain the existence of the two wavelet expansions (3.21) and (3.36).

Preliminary results of this section were reported in [8].

4. Shift Elementary Irreducible-Invariant Subspaces

We now turn to shift irreducible-invariant subspaces. We begin by recalling Halmos' results [4], together with their "adjoint" version.

Proposition 4. *Let $\mathcal{W}$ be a U-wandering subspace, then the subspace $\mathcal{M}_{ir}$ (respectively, $\mathcal{M}_{*\,ir}$) defined by*

$$\mathcal{M}_{ir} := \bigvee_{m=0}^{\infty} U^m \mathcal{W} \quad \Big(\textit{respectively, } \mathcal{M}_{*\,ir} := \bigvee_{m=-\infty}^{-1} U^m \mathcal{W}\Big) \tag{4.1}$$

is U(respectively, U^*)-irreducible-invariant. Conversely, if $\mathcal{M}$ (respectively, $\mathcal{M}_*$) is U(respectively, U^*)-irreducible-invariant, then there exists a U-wandering subspace $\mathcal{W}$ so that $\mathcal{M} = \bigvee_{m=0}^{\infty} U^m \mathcal{W}$ (respectively, $\mathcal{M}_* := \bigvee_{m=-\infty}^{-1} U^m \mathcal{W}$).

Let $\mathcal{M}_{ir}$ be U-irreducible-invariant and let $\mathcal{W}$ be the corresponding U-wandering subspace. Suppose $\mathcal{W}$ is spanned by an orthonormal set $\{\psi_n\}_{n\in\mathbb{Z}}$. Then,

$$\mathcal{M}_{ir} = \bigvee_{m=0}^{\infty} U^m \mathcal{W} = \bigvee_{m=0}^{\infty} \bigvee_{n\in\mathbb{Z}} U^m \psi_n. \tag{4.2}$$

Therefore, as in the proof of Theorem 2, we have

$$\mathcal{M}_{ir} = \bigvee_{m=0}^{\infty} \bigvee_{n\in\mathbb{Z}} U^m \psi_n = \bigvee_{n\in\mathbb{Z}} \bigvee_{m=0}^{\infty} U^m \psi_n. \tag{4.3}$$

Define the subspaces

$$\mathcal{H}_{(0),n} := \bigvee_{m=0}^{\infty} U^m \psi_n, \quad n \in \mathbb{Z}. \tag{4.4}$$

Here the subscript (0) means that the integer m on the right hand side ranges from 0 onward. It is evident that the subspaces $\{\mathcal{H}_{(0),n}\}_{n\in\mathbb{Z}}$ are U- irreducible-invariant. Moreover, they are also mutually orthogonal, $\mathcal{H}_{(0),n} \perp \mathcal{H}_{(0),n'}$, whenever $n \neq n'$. Therefore, as in the proof of Theorem 2,

$$\mathcal{M}_{ir} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{(0),n}. \tag{4.5}$$

In exactly the same way we obtain for the U^*-irreducible-invariant subspace $\mathcal{M}_{*\,ir}$,

$$\mathcal{M}_{*\,ir} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(-1)}, \tag{4.6}$$

where

$$\mathcal{H}_n^{(-1)} := \bigvee_{m=-\infty}^{-1} U^m \psi_n, \quad n \in \mathbb{Z}, \tag{4.7}$$

and the superscript (-1) indicates that the upper bound of m is -1. Moreover, the U^*-irreducible-invariant subspaces $\{\mathcal{H}_n^{(-1)}\}_{n\in\mathbb{Z}}$ are also mutually orthogonal.

Halmos' results can now be restated as.

Theorem 3. *A U(respectively, U^*)-irreducible-invariant subspace $\mathcal{M}_{ir}$ (respectively, $\mathcal{M}_{*\,ir}$) admits the orthogonal decomposition*

$$\mathcal{M}_{ir} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{(0),n}, \quad \Big(\textit{respectively, } \mathcal{M}_{*\,ir} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(-1)}\Big),$$

where

$$\mathcal{H}_{(0),n} := \bigvee_{m=0}^{\infty} U^m \psi_n \quad \Big(\textit{respectively, } \mathcal{H}_n^{(-1)} := \bigvee_{m=-\infty}^{-1} U^m \psi_n\Big), \quad n \in \mathbb{Z},$$

are orthogonal elementary U(respectively, U^*)-irreducible-invariant subspaces, and $\{\psi_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the U-wandering subspace $\mathcal{W}$.

Sequences of U and U^*-irreducible-invariant subspaces can be generated from $\mathcal{M}_{ir}$ and $\mathcal{M}_{*\,ir}$, respectively. Such sequences play a key role in Wavelet Multiresolution Analysis as will be seen in the next section.

5. Shift Elementary Irreducible-Invariant Subspaces in Wavelet Multiresolution Analysis(MRA)

We begin by recalling the "Incoming-Outgoing Subspaces" definition of shifts, see for instance [6] and the references therein.

Definition 3. A shift $U\colon \mathcal{H} \to \mathcal{H}$ is a unitary operator for which there is an outgoing (respectively, incoming) subspace V^o (respectively, V^i) satisfying the following conditions:

(i)^o $U\,V^o \subset V^o$ (respectively, (i)^i $U^*V^i \subset V^i$),
(ii) $\bigcap_{m=-\infty}^{\infty} U^m V^{o,i} = \{0\}$,
(iii) $\overline{\bigcup}_{m=-\infty}^{\infty} U^m V^{o,i} = \mathcal{H}$,

where, in (ii) and (iii), $V^{o,i}$ can be either V^o or V^i.

We must note that Definition 3 was, originally, the Lax-Phillips definition of an *outgoing subspace* (respectively, *incoming subspace*) for a unitary operator U [7]. However, since U is actually a shift, Definition 3 can simultaneously serve as that of a Hilbert space shift operator [3]. Thus, it is appropriate to refer to it as "Incoming-Outgoing Definition" of shifts.

We now recall some basic facts relating incoming, outgoing, and wandering subspaces of shifts.

Proposition 5. *Let $U\colon \mathcal{H} \to \mathcal{H}$ be a shift, and let V^o (respectively, V^i) be U-outgoing (respectively, U-incoming). Then* [7],

$$V^o := \bigoplus_{m=0}^{\infty} U^m \mathcal{W} \quad \Big(\text{respectively, } V^i := \bigoplus_{m=-\infty}^{-1} U^m \mathcal{W}\Big),$$

where the subspace

$$\mathcal{W} := V^o \ominus U\, V^o, \quad \Big(\text{respectively, } \mathcal{W} := V^i \ominus U^* V^i\Big)$$

is U-generating-wandering. Moreover, V^o (respectively, V^i) is also U-irreducible-invariant (respectively, U^*-irreducible-invariant) [4].

An easy consequence of the above is:

Lemma 4. *Let $U\colon \mathcal{H} \to \mathcal{H}$ be a shift and V_0 be a closed subspace of $\mathcal{H}$. Let $\{V_p\}_{p\in\mathbb{Z}}$ be the subspaces generated from V_0 by*

$$V_{p+1} = UV_p, \quad \big(\text{respectively, } V_{p+1} = U^* V_p\big), \quad p \in \mathbb{Z}.$$

Then $\{V_p\}_{p\in\mathbb{Z}}$ satisfies the following properties,

(i) $V_{p+1} \subset V_p, \quad p \in \mathbb{Z},$
(ii) $\bigcap_{p=-\infty}^{\infty} V_p = \{0\},$
(iii) $\overline{\bigcup}_{p=-\infty}^{\infty} V_p = \mathcal{H},$

if and only if V_0 is U-outgoing (respectively, U-incoming). Similarly, if condition (i) *is replaced by*

(i') $V_p \subset V_{p+1}, \quad p \in \mathbb{Z},$

then $\{V_p\}_{p\in\mathbb{Z}}$ satisfies (i'), (ii), (iii) *if and only if V_0 is U-incoming (respectively, U-outgoing).*

It follows from this Lemma that Definition 3 can be restated in terms of the sequences of subspaces $\{U^m V^o\}_{m\in\mathbb{Z}}$ or $\{U^m V^i\}_{m\in\mathbb{Z}}$ as follows.

Definition 4. A shift $U\colon \mathcal{H} \to \mathcal{H}$ is a unitary operator for which there is an outgoing (respectively, incoming) subspace V^o (respectively, V^i) satisfying the following conditions:

(i)^o $U^{m+1} V^o \subset U^m V^o$ (respectively, (i)^i $U^{*\,m} V^i \subset U^{*\,m+1} V^i$),
(ii) $\bigcap_{m=-\infty}^{\infty} U^m V^{o,i} = \{0\},$
(iii) $\overline{\bigcup}_{m=-\infty}^{\infty} U^m V^{o,i} = \mathcal{H},$

where, in (ii) and (iii), $V^{o,i}$ can be either V^o or V^i.

The Wandering Subspace Definition of shifts (i.e., Definition 1) can be restated, in the spirit of Definition 4 as follows.

Definition 5. A shift $U\colon \mathcal{H} \to \mathcal{H}$ is a unitary operator for which there is a generating wandering subspace $\mathcal{W}$ satisfying the following conditions:

(i)^w $U^m \mathcal{W} \perp U^n \mathcal{W}, \quad m, n \in \mathbb{Z},$
(ii) $\bigcap_{m=-\infty}^{\infty} U^m \mathcal{W} = \{0\},$
(iii) $\overline{\bigcup}_{m=-\infty}^{\infty} U^m \mathcal{W} = \mathcal{H}.$

Remark 1. Comparing Definitions 4 and 5 we see that the subspaces $\{U^m V^o\}_{m\in\mathbb{Z}}$ or $\{U^m V^i\}_{m\in\mathbb{Z}}$, and $\{U^m \mathcal{W}\}_{m\in\mathbb{Z}}$ differ only in properties (i)^{o,i} and (i)^w. These two properties show the difference between outgoing subspace V^o or incoming subspace V^i, and the generating wandering subspace $\mathcal{W}$. Equivalence between the two definitions are evident from Proposition 5.

To proceed, we now recall the definition of Wavelet Multiresolution Analysis (MRA) [9], which plays an important role in Wavelet Theory.

Definition 6. A sequence of "*approximation subspaces*" $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ of the function space $\mathcal{L}^2(\mathbb{R})$ is a Wavelet MRA, with *scaling function* $\phi(\cdot)$, if the following conditions hold:

(o) $\{\phi((\cdot)-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the subspace $\mathcal{V}_0(\phi)$,
(i) $\mathcal{V}_m(\phi) \subset \mathcal{V}_{m+1}(\phi), \quad m \in \mathbb{Z} \qquad (\text{or}, \quad \mathcal{V}_{m+1}(\phi) \subset \mathcal{V}_m(\phi), \quad m \in \mathbb{Z})$,
(ii) $\bigcap_{m=-\infty}^{\infty} \mathcal{V}_m(\phi) = \{0\}$,
(iii) $\overline{\bigcup}_{m=-\infty}^{\infty} \mathcal{V}_m(\phi) = \mathcal{L}^2(\mathbb{R})$,
(iv) $v(\cdot) \in \mathcal{V}_m(\phi) \iff v(2(\cdot)) \in \mathcal{V}_{m+1}(\phi), \quad m \in \mathbb{Z}$
$(\text{or}, \quad v(\cdot) \in \mathcal{V}_m(\phi) \iff v(\frac{1}{2}(\cdot)) \in \mathcal{V}_{m+1}(\phi), \quad m \in \mathbb{Z})$.

Remark 2. Definition 6(i)-(iii) can also be expressed in terms of the projections P_m from $\mathcal{L}^2(\mathbb{R})$ onto the subspaces $\mathcal{V}_m(\phi)$. This was pointed out by Antoniou and Gustafson [1]. We will have more to say about this later—in connection with the Time Operator of Wavelet MRA.

We must note that Definition 6(o) is "particular" to wavelets and has nothing to do with the shift operator D. It was introduced as a mechanism for constructing a wavelet $\psi(\cdot)$ from a given scaling function $\phi(\cdot)$. Also, Definition 6(iv) can be expressed in terms of D and D^* as follows.

$$\mathcal{V}_{m+1}(\phi) = D\,\mathcal{V}_m(\phi) \qquad (\text{or}, \quad \mathcal{V}_{m+1}(\phi) = D^*\mathcal{V}_m(\phi)), \quad m \in \mathbb{Z}. \tag{5.1}$$

Therefore, if $\mathcal{V}_{m+1}(\phi) = D\,\mathcal{V}_m(\phi)$, then Definition 6(i) becomes

$$D^*\mathcal{V}_m(\phi) \subset \mathcal{V}_m(\phi) \qquad (\text{or}, \quad D\,\mathcal{V}_m(\phi) \subset \mathcal{V}_m(\phi)), \quad m \in \mathbb{Z}; \tag{5.2}$$

i.e., $\mathcal{V}_m(\phi)$ is D^*-invariant (or, D-invariant). Similarly, if $\mathcal{V}_{m+1}(\phi) = D^*\mathcal{V}_m(\phi)$, then Definition 6(i) becomes

$$D\,\mathcal{V}_m(\phi) \subset \mathcal{V}_m(\phi) \qquad (\text{or}, \quad D^*\mathcal{V}_{m+1}(\phi) \subset \mathcal{V}_{m+1}(\phi)), \quad m \in \mathbb{Z}; \tag{5.3}$$

i.e., $\mathcal{V}_m(\phi)$ is D-invariant (or, D^*-invariant).

We therefore conclude from Definition 6 and Lemma 4:

Proposition 6. *A Wavelet MRA—with scaling function $\phi(\cdot)$—is a sequence of decreasingly-nested (respectively, increasingly-nested) subspaces $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ of $\mathcal{L}^2(\mathbb{R})$; i.e.,*

$$\mathcal{V}_m(\phi) \subset \mathcal{V}_{m+1}(\phi) \qquad (\textit{respectively}, \quad \mathcal{V}_{m+1}(\phi) \subset \mathcal{V}_m(\phi)), \quad m \in \mathbb{Z},$$

generated, either from,

(i) *a D-incoming subspace $\mathcal{V}^i(\phi)$ by*

$$\mathcal{V}_0(\phi) := \mathcal{V}^i(\phi), \;\; \mathcal{V}_p(\phi) = D^m\mathcal{V}^i(\phi) \iff \mathcal{V}_{m+1}(\phi) = D\,\mathcal{V}_m(\phi), \quad m \in \mathbb{Z}, \tag{5.4}$$

$$(\textit{respectively, by} \quad \mathcal{V}_m(\phi) = {D^*}^m\mathcal{V}^i(\phi) \iff \mathcal{V}_{m+1}(\phi) = D^*\mathcal{V}_m(\phi), \quad m \in \mathbb{Z}),$$

or from,

(ii) *a D-outgoing subspace* $\mathcal{V}^o(\phi)$ *by*

$$\mathcal{V}_0(\phi) := \mathcal{V}^o(\phi),\ \mathcal{V}_m(\phi) = {D^*}^m\mathcal{V}^o(\phi) \iff \mathcal{V}_{m+1}(\phi) = D^*\mathcal{V}_m(\phi), \quad m \in \mathbb{Z}, \tag{5.5}$$

(*respectively, by* $\mathcal{V}_m(\phi) = D^m\mathcal{V}_0^o(\phi) \iff \mathcal{V}_{m+1}(\phi) = D\,\mathcal{V}_m(\phi), \quad m \in \mathbb{Z}$),

where $\mathcal{V}^i(\phi)$ *or* $\mathcal{V}^o(\phi)$ *are spanned by the orthonormal basis* $\{\phi((\cdot) - n)\}_{n\in\mathbb{Z}}$.

We shall refer to a Wavelet MRA generated from an incoming subspace (respectively, an outgoing subspace) as an Incoming Wavelet MRA (respectively, an Outgoing Wavelet MRA). From now on, without lack of generality, we only consider Incoming Wavelet MRA.

Assumption 1. *Let* $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ *be an Incoming Wavelet MRA—with incoming subspace* $\mathcal{V}^i(\phi)$ *generated from a scaling function* $\phi(\cdot)$*—satisfying,*

$$\mathcal{V}_0(\phi) := \mathcal{V}^i(\phi) = \bigvee_{n\in\mathbb{Z}} \phi((\cdot) - n), \tag{5.6}$$

$$\mathcal{V}_{m+1}(\phi) = D\,\mathcal{V}_m(\phi) = D^{m+1}\mathcal{V}_0(\phi), \quad m \in \mathbb{Z}, \tag{5.7}$$

and

$$\mathcal{V}_m(\phi) \subset \mathcal{V}_{m+1}(\phi), \quad m \in \mathbb{Z}. \tag{5.8}$$

We have, by Propositions 5,

$$\mathcal{V}_0(\phi) := \mathcal{V}^i(\phi).$$

Then, by Proposition 6,

$$\mathcal{V}_0(\phi) := \mathcal{V}^i(\phi) = \bigoplus_{p=-\infty}^{-1} D^p\mathcal{W}(\psi), \tag{5.9}$$

where, as before, the D-generating-wandering subspace $\mathcal{W}(\psi)$ is spanned by the orthonormal wavelet functions $\{\psi_n(\cdot) := \psi((\cdot) - n)\}_{n\in\mathbb{Z}}$—generated from a wavelet $\psi(\cdot)$,

$$\mathcal{W}(\psi) := \bigvee_{n\in\mathbb{Z}} \psi((\cdot) - n). \tag{5.10}$$

It then follows that

$$\mathcal{V}_m(\phi) = D^m\mathcal{V}^i(\phi) = \bigoplus_{p=-\infty}^{m-1} D^p\mathcal{W}(\psi). \tag{5.11}$$

Remark 3. We must note that equation (5.9)—without the functions $\phi(\cdot)$ and $\psi(\cdot)$—is the "usual" representation of an incoming subspace for a shift operator which, in our case, is the dilation-by-2 operator D. However, with the shift operator D and only when $\psi(\cdot)$ is an orthonormal wavelet, then $\mathcal{W}$ is characterized by (5.10). As a consequence, in (5.9) the incoming subspace $\mathcal{V}_0$ is now "depending" on $\psi(\cdot)$ and is represented by the orthogonal subspaces $\{D^p\mathcal{W}(\psi)\}_{-\infty<p\le-1}$. However, with equation (5.6), or Definition 5(o), the subspace $\mathcal{V}_0$ is also required to be spanned by the orthonormal set $\{\phi((\cdot) - n)\}_{n\in\mathbb{Z}}$. Thus, in Wavelet Theory, the incoming (or, outgoing) subspace $\mathcal{V}_0$ depends on both a wavelet $\psi(\cdot)$ and a scaling function $\phi(\cdot)$. This is the key idea which resulted in a procedure for constructing a wavelet $\psi(\cdot)$ from a given scaling function $\phi(\cdot)$ [9].

We now obtain an alternate representation for the approximation subspaces $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$. First, by Lemma 1(i), (5.9) can be rewritten as

$$\mathcal{V}_0(\phi) = \mathcal{V}^i(\phi) = \bigoplus_{p=-\infty}^{-1} \bigvee_{n\in\mathbb{Z}} D^p\psi((\cdot)-n). \tag{5.12}$$

Then, since $\mathcal{V}^i(\phi)$ is D^*-irreducible-invariant, it follows from Theorem 3 that

$$\mathcal{V}_0(\phi) = \mathcal{V}^i(\phi) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(-1)}(\psi), \tag{5.13}$$

where, as before,

$$\mathcal{H}_n^{(-1)}(\psi) := \bigvee_{p=-\infty}^{-1} D^p\psi((\cdot)-n), \quad n\in\mathbb{Z}. \tag{5.14}$$

Next, we have from (5.7),

$$\mathcal{V}_m(\phi) = D^m\mathcal{V}_0(\phi) = \bigoplus_{n\in\mathbb{Z}} D^m\,\mathcal{H}_n^{(-1)}(\psi), \quad m\in\mathbb{Z}. \tag{5.15}$$

But, by (5.13),

$$D^m\,\mathcal{H}_n^{(-1)}(\psi) = \bigvee_{p=-\infty}^{-1} D^{m+p}\psi((\cdot)-n) = \bigvee_{\nu=-\infty}^{m-1} D^\nu\psi((\cdot)-n), \quad m,n\in\mathbb{Z}. \tag{5.16}$$

Let us define the D^*-irreducible-invariant subspaces

$$\mathcal{H}_n^{(m-1)}(\psi) := D^m\,\mathcal{H}_n^{(-1)}(\psi) = \bigvee_{p=-\infty}^{m-1} D^p\psi((\cdot)-n), \quad m,n\in\mathbb{Z}. \tag{5.17}$$

Then it follows from this, (5.16), and (5.15) that

$$\mathcal{V}_m(\phi) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(m-1)}(\psi), \quad m\in\mathbb{Z}. \tag{5.18}$$

Now it is plain that, for each fixed n, the subspaces $\{\mathcal{H}_n^{(m)}(\psi)\}_{m\in\mathbb{Z}}$ are also nested,

$$\mathcal{H}_n^{(m)}(\psi) \subset \mathcal{H}_n^{(m+1)}(\psi), \quad m\in\mathbb{Z}. \tag{5.19}$$

Moreover, by (2.9), they are subspaces of the elementary reducing subspace $\mathcal{H}_n$,

$$\mathcal{H}_n := \bigvee_{p\in\mathbb{Z}} D^p\psi((\cdot)-n), \quad n\in\mathbb{Z}, \tag{5.20}$$

which we have referred to as a *n-time-shift-scale detail subspace.* Therefore the subspace $\mathcal{H}_n^{(m)}(\psi)$ can be called a *n-time-shift-scale-*2^m *detail subspace.* It is easy to see that, for each fixed n, the subspaces $\{\mathcal{H}_n^{(m)}\}_{m\in\mathbb{Z}}$—of $\mathcal{H}_n$— also inherit the Wavelet MRA properties of the original Wavelet MRA $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ on $\mathcal{L}^2(\mathbb{R})$.

We summarize the above in the next proposition.

Proposition 7. *Let $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ be an Incoming Wavelet MRA satisfying Assumption* 1. *Then the approximation subspaces $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ admit the orthogonal decomposition*

$$\mathcal{V}_m(\phi) := \bigvee_{n\in\mathbb{Z}} D^m\phi((\cdot)-n)) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(m-1)}(\psi), \quad m\in\mathbb{Z}, \tag{5.21}$$

where the elementary D^*-irreducible-invariant subspaces $\{\mathcal{H}_n^{(m)}(\psi)\}_{m\in\mathbb{Z}}$, are defined by (5.17)

$$\mathcal{H}_n^{(m)}(\psi) = \bigvee_{p=-\infty}^{m} D^p\psi((\cdot)-n), \quad m,n\in\mathbb{Z}. \tag{5.22}$$

Moreover, for each fixed n, the subspaces $\{\mathcal{H}_n^{(m)}\}_{m\in\mathbb{Z}}$ form an "elementary" Wavelet MRA on the elementary reducing subspace $\mathcal{H}_n$.

We close the paper with a decomposition of the Time Operator of Wavelet MRA. First, recall that Definition 6 is equivalent to [1].

Definition 7. Let $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ be closed subspaces of the function space $\mathcal{L}^2(\mathbb{R})$, and let P_m be the projections onto $\mathcal{V}_m(\phi)$. Then $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ is a Wavelet MRA, with *scaling function* $\phi(\cdot)$, if the following conditions hold:

(o) $\{\phi((\cdot)-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the subspace $\mathcal{V}_0(\phi)$,
(i') $P_m < P_{m+1}$,
(ii') $P_{-\infty} = \lim_{m\to-\infty} P_m = 0$,
(iii') $P_{+\infty} = \lim_{m\to+\infty} P_m = I$,
(iv') $P_{m+1} = DP_mD^*$.

(In (i') $<$ means inclusion of ranges; in (ii') and (iii') we have strong convergence).

Antoniou and Gustafson [2] have shown that this Definition not only defined a Wavelet MRA, but it also allowed them to define the Time Operator of Wavelet MRA, as the self-adjoint operator $\mathcal{T}$ with dense domain $\mathcal{D}(\mathcal{T})$,

$$\mathcal{T} := \sum_{m\in\mathbb{Z}} m(P_{m+1}-P_n). \tag{5.23}$$

We have seen in Propositions 2 and 3 that the projections $P_{\mathcal{W}_m(\psi)}$ and $P_{\psi_{m,n}}$ also have property (iv') of Definition 7,

$$P_{\mathcal{W}_{m+1}(\psi)} = DP_{\mathcal{W}_m(\psi)}D^*, \quad m\in\mathbb{Z}. \tag{5.24}$$

and

$$P_{\psi_{m+1,n}} = DP_{\psi_{m,n}}D^*, \quad m\in\mathbb{Z}. \tag{5.25}$$

These suggest that $P_{\mathcal{W}_m(\psi)}$ and $P_{\psi_{m,n}}$ should, somehow, be "related" to the Time Operator $\mathcal{T}$. Indeed, since

$$P_{m+1} - P_m = P_{\mathcal{W}_m(\psi)}, \tag{5.26}$$

and since each subspace $\mathcal{W}_m(\psi)$ is spanned by the orthonormal set $\{\psi_{m,n}(\cdot)\}_{n\in\mathbb{Z}}$, it follows from (5.23) that [2],

$$\mathcal{T}f(\cdot) = \sum_{m\in\mathbb{Z}} m\,P_{\mathcal{W}_m(\psi)}\big(f(\cdot)\big), \qquad f(\cdot)\in\mathcal{L}^2(\mathbb{R}), \tag{5.27}$$

$$= \sum_{m\in\mathbb{Z}} m \sum_{n\in\mathbb{Z}} \langle f(\cdot),\psi_{m,n}(\cdot)\rangle\,\psi_{m,n}(\cdot), \tag{5.28}$$

$$= \sum_{m\in\mathbb{Z}} m \sum_{n\in\mathbb{Z}} P_{\psi_{m,n}}\big(f(\cdot)\big). \tag{5.29}$$

Equations (5.27) and (5.29) provide connections to the Time Operator $\mathcal{T}$ of the projections $P_{\mathcal{W}_m}(\psi)$ and $P_{\psi_{m,n}}$. More is true. As in the proof of Theorem 2, the

right hand side of (5.28) can be rewritten as

$$\mathcal{T}f(\cdot) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} m \,\langle f(\cdot), \psi_{m,n}(\cdot)\rangle \,\psi_{m,n}(\cdot), \quad f(\cdot) \in \mathcal{L}^2(\mathbb{R}), \tag{5.30}$$

or,

$$\mathcal{T}f(\cdot) = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}_n f(\cdot), \tag{5.31}$$

where $\mathcal{T}_n$ are defined by

$$\mathcal{T}_n f(\cdot) := \sum_{m\in\mathbb{Z}} m \,\langle f(\cdot), \psi_{m,n}(\cdot)\rangle \,\psi_{m,n}(\cdot) = \sum_{m\in\mathbb{Z}} m \, P_{\psi_{m,n}} f(\cdot). \tag{5.32}$$

We therefore conclude that:

Proposition 8. *The Time Operator $\mathcal{T}$ of Wavelet MRA admits the decomposition*

$$\mathcal{T} = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}_n, \tag{5.33}$$

where $\mathcal{T}_n$—called "elementary" Time Operators—are defined on $\mathcal{H}_n$ by

$$\mathcal{T}_n := \sum_{m\in\mathbb{Z}} m \, P_{\psi_{m,n}}, \quad n \in \mathbb{Z}. \tag{5.34}$$

Consequently, each wavelet function $\psi_{m,n}(\cdot)$ is age-eigenvector [2] *of $\mathcal{T}_n$.*

We note that the Time Operator $\mathcal{T}_n$ agrees with the fact that, from Proposition 7, the subspaces $\{\mathcal{H}_n^{(m)}\}_{m\in\mathbb{Z}}$ form a Wavelet MRA on $\mathcal{H}_n$.

We refer to [1] and [2] for further results on Time Operator of Wavelets, and connections between Wavelet Theory and Wandering Subspace Theory, and other parts of Mathematics.

Finally, we must note that our approach to shifts is "non-conventional", in the sense that, instead of dealing with specific shift representation on $\ell^2(-\infty, \infty; \mathcal{W})$ such as that of (2.25)

$$S_r\{w_m\}_{m\in\mathbb{Z}} = \{w_{m-1}\}_{m\in\mathbb{Z}},$$

we deal with the shift U on $\mathcal{H}$ first, via its wandering subspaces $\mathcal{W}_n$, then via its elementary reducing subspaces $\mathcal{H}_n$. An advantage of this "representation-free" approach, of course, is the fact that we can get back to S_r, via the vectors $U^m\psi_n$, which are actually "age eigenvectors" of the associated Time Operator $\mathcal{T}$. Our key results (Theorem 1 and Proposition 1) are clearly consequence of the representation-free approach. These results cannot be derived from the representation (2.25), even though S_r is related to U via the unitary operator Φ defined by (2.24). This is due to the fact that the conventional approach does not allow one to connect shifts with Time Operator or with Wavelets, since it has nothing to do with the age eigenvectors $U^m\psi_n$ which, in the case of wavelets, are precisely the wavelet functions $\psi_{m,n}(\cdot)$.

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References

1. I. Antoniou and K. Gustafson, *Wavelets and stochastic processes*, Math. Comput. Simulation **49** (1999), 81–104.
2. I.E. Antoniou and K.E. Gustafson, *The time operator of wavelets*, Integrability and Chaos in Discrete Systems, Brussels, 1997 (Chaos Solitons Fractals **11**, 2000, 443–452).
3. R. Beals, *Topics in Operator Theory* (The University of Chicago Press, Chicago, 1971).
4. P.R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. **208** (1961), 102–112.
5. C.S. Kubrusly, *Elements of Operator Theory* (Birkhäuser, Boston, 2001).
6. C.S. Kubrusly and N. Levan, *Dual-shift decomposition of Hilbert space*, Semigroup of Operators: Theory and Applications, Rio de Janeiro, 2001 (Optimization Software, Los Angeles, 2002, 135–144).
7. P.D. Lax and R.S. Phillips, *Scattering Theory* (Academic Press, New York, 1967).
8. N. Levan and C.S. Kubrusly, *A wavelet "time-shift-detail" decomposition*, Math. Comput. Simulation **63** (2003), 73-78.
9. S. Mallat, *A Wavelet Tour of Signal Processing* (Academic Press, San Diego, 1998).
10. H. Radjavi and P. Rosenthal, *Invariant Subspaces* (Springer, New York, 1973).
11. D. Sarason, *Invariant subspaces* (Topics in Operator theory, Math. Surveys, No. 13, Amer. Math. Soc., Providence, 1974, 1–47).
12. B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space* (North-Holland, Amsterdam, 1970).

Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil
E-mail address: carlos@ele.puc-rio.br

Department of Electrical Engineering, University of California in Los Angeles, Los Angeles, CA 90024-1594, USA
E-mail address: levan@ee.ucla.edu