

A WAVELET “TIME-SHIFT-DETAIL” DECOMPOSITION

N. LEVAN AND C.S. KUBRUSLY

ABSTRACT. We show that, with respect to an orthonormal wavelet $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ is, on the one hand, the sum of its “layers of details” over all time-shifts, and on the other hand, the sum of its layers of details over all scales. The latter is well known and is a consequence of a wandering subspace decomposition of $\mathcal{L}^2(\mathbb{R})$ which, in turn, resulted from a wavelet Multiresolution Analysis (MRA). The former has not been discussed before. We show that it is a consequence of a decomposition of $\mathcal{L}^2(\mathbb{R})$ in terms of reducing subspaces of the dilation-by-2 shift operator.

1. INTRODUCTION

An element $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ —with the usual inner product $[\cdot, \cdot]$ and norm $\|\cdot\|$ —is called an orthonormal wavelet if the functions

$$(1.1) \quad \psi_{m,n}(\cdot) := 2^{\frac{m}{2}} \psi(2^m(\cdot) - n), \quad m, n \in \mathbb{Z}$$

—called wavelet orthonormal functions, form a basis for $\mathcal{L}^2(\mathbb{R})$ [5]. Therefore, corresponding to a given wavelet $\psi(\cdot)$, any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ admits the orthogonal decomposition

$$(1.2) \quad f(\cdot) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [f(\cdot), \psi_{m,n}(\cdot)] \psi_{m,n}(\cdot).$$

Now, for each $m, n \in \mathbb{Z}$, the projection of $f(\cdot)$ onto $\psi_{m,n}(\cdot)$ is

$$(1.3) \quad [f(\cdot), \psi_{m,n}(\cdot)] \psi_{m,n}(\cdot) = [f(\cdot), 2^{\frac{m}{2}} \psi(2^m(\cdot) - n)] 2^{\frac{m}{2}} \psi(2^m(\cdot) - n).$$

This can be considered as a *detail variation* of $f(\cdot)$ —at scale 2^m and at time-shift n . For each $m \in \mathbb{Z}$, the projection of $f(\cdot)$ onto the *scale-detail subspace* $W_m(\psi)$ defined by

$$(1.4) \quad W_m(\psi) := \bigvee_{n \in \mathbb{Z}} \{\psi_{m,n}\} = \overline{\text{span}}\{\psi_{m,n}\}_{n \in \mathbb{Z}}$$

is the partial sum on the RHS of (1.2)

$$(1.5) \quad \sum_{n=-\infty}^{\infty} [f(\cdot), \psi_{m,n}] \psi_{m,n}(\cdot).$$

This, in turn, can be regarded as a “*layer of details*” (LOD) of $f(\cdot)$ —at scale 2^m [5]. Consequently, any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ is the sum of all its LOD over all scales.

Date: July 8, 2002.

2000 Mathematics Subject Classification. 42C05, 47A15.

Key words and phrases. Wavelet; scale and time-shift details; shift-wandering subspace decomposition; shift reducing subspaces decomposition.

The second author’s work was supported in part by Brazilian National Research Council (CNPq).

Now, instead of LOD at scale 2^m , we define the LOD—at time-shift n —of $f(\cdot)$ as the sum

$$(1.6) \quad \sum_{m=-\infty}^{\infty} [f(\cdot), \psi_{m,n}(\cdot)] \psi_{m,n}(\cdot).$$

Then, is it true that any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ is the sum of its LOD—over all time-shifts? In other words, is it true that

$$(1.7) \quad f(\cdot) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [f(\cdot), \psi_{m,n}] \psi_{m,n}(\cdot)?$$

The answer is affirmative as we shall show. This implies that a wavelet approximation can be carried out either by scale-details summation or by time-shift-details summation.

We begin by recalling basic facts of Hilbert space bilateral shifts. The main result is given in Theorem 1.

2. MAIN RESULTS

Let H be a separable Hilbert space with inner product $[\cdot, \cdot]_H$ and norm $\|\cdot\|_H$. Let $U: H \rightarrow H$ be a bounded linear operator. A closed subspace W is a wandering subspace for U if it is orthogonal to all its images under positive powers of U [3],

$$(2.1) \quad W \perp U^n W, \quad \forall n > 0.$$

In addition, if the subspaces $U^n W$, $n \geq 0$, span H , then W is a generating wandering subspace for U .

A bounded linear operator $U: H \rightarrow H$ is a bilateral shift, or simply a shift, if it is unitary and it admits a generating wandering subspace W . In other words, a unitary operator U on H is a shift if and only if H admits the “wandering subspace” decomposition

$$(2.2) \quad H = \bigoplus_{m=-\infty}^{\infty} U^m W.$$

The dimension of W is called multiplicity of the shift.

We must note that if S is a completely nonunitary isometry on H then $\ker S^*$ is the unique generating wandering subspace and

$$(2.3) \quad H = \bigoplus_{m=0}^{\infty} S^m W.$$

Therefore, S is now a unilateral shift.

To proceed, let $D: \mathcal{L}^2(\mathbb{R}) \rightarrow \mathcal{L}^2(\mathbb{R})$ be the dilation-by-2 (or dyadic scaling) operator defined by

$$(2.4) \quad Df = g, \quad g(t) = \sqrt{2}f(2t),$$

and $T: \mathcal{L}^2(\mathbb{R}) \rightarrow \mathcal{L}^2(\mathbb{R})$ be the translation-by-1 operator defined by

$$(2.5) \quad Tf = g, \quad g(t) = f(t-1).$$

It is easy to see that both D and T are shifts of infinite multiplicity. Moreover, the wavelet orthonormal functions $\psi_{m,n}(\cdot)$ generated from an orthonormal wavelet $\psi(\cdot)$ can now be written as

$$(2.6) \quad \psi_{m,n}(\cdot) = 2^{\frac{m}{2}} \psi(2^m(\cdot) - n) = D^m T^n \psi(\cdot), \quad m, n \in \mathbb{Z}.$$

Define the closed subspace

$$(2.7) \quad W(\psi) := \bigvee_{n \in \mathbb{Z}} \{T^n \psi\} = \overline{\text{span}}\{T^n \psi\}_{n \in \mathbb{Z}}.$$

Then, since $\{\psi_{m,n}\}_{m,n \in \mathbb{Z}}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$, [1],

$$(2.8) \quad \mathcal{L}^2(\mathbb{R}) = \bigoplus_{m=-\infty}^{\infty} D^m W(\psi) = \bigoplus_{m=-\infty}^{\infty} D^m \bigvee_{n \in \mathbb{Z}} \{T^n \psi\} = \bigoplus_{m=-\infty}^{\infty} W_m(\psi),$$

where

$$(2.9) \quad W_m(\psi) = D^m W(\psi), \quad m \in \mathbb{Z}.$$

We note that the subspaces $W_m(\psi)$ are neither D -invariant nor D^* -invariant.

Remark 1. The subspaces $W_m(\psi)$ were defined in (1.4). This and (2.6) yield

$$W_m(\psi) = \bigvee_{n \in \mathbb{Z}} \{D^m T^n \psi\}.$$

They were redefined in (2.9), which together with (2.7) yields

$$W_m(\psi) = D^m \bigvee_{n \in \mathbb{Z}} \{T^n \psi\}.$$

The next proposition shows that there is no ambiguity here; both expressions for $W_m(\psi)$ coincide.

Proposition 1. *The following identity holds for every integer $m \in \mathbb{Z}$.*

$$(2.10) \quad D^m \bigvee_{n \in \mathbb{Z}} \{T^n \psi\} = \bigvee_{n \in \mathbb{Z}} \{D^m T^n \psi\}.$$

Proof. Since $D: \mathcal{L}^2(\mathbb{R}) \rightarrow \mathcal{L}^2(\mathbb{R})$ is linear, continuous and invertible, it follows (by the Banach Continuous Inverse Theorem) that D^m is linear and continuous for every $m \in \mathbb{Z}$. Recall that

$$(2.11) \quad D^m \text{span}\{T^n \psi\}_{n \in \mathbb{Z}} = \text{span}\{D^m T^n \psi\}_{n \in \mathbb{Z}}$$

and

$$(2.12) \quad D^m \overline{\text{span}}\{T^n \psi\}_{n \in \mathbb{Z}} \subseteq \overline{D^m \text{span}\{T^n \psi\}_{n \in \mathbb{Z}}},$$

since D^m is continuous [4, Problem 3.46]. Moreover,

$$(2.13) \quad \overline{D^m \text{span}\{T^n \psi\}_{n \in \mathbb{Z}}} = D^m \overline{\text{span}}\{T^n \psi\}_{n \in \mathbb{Z}}$$

by the fact that D^{-m} is also continuous. Therefore, by (2.12) and (2.13),

$$\begin{aligned} D^m \overline{\text{span}}\{T^n \psi\}_{n \in \mathbb{Z}} &\subseteq \overline{D^m \text{span}\{T^n \psi\}_{n \in \mathbb{Z}}} \\ &\subseteq \overline{D^m \overline{\text{span}}\{T^n \psi\}_{n \in \mathbb{Z}}} = D^m \overline{\text{span}}\{T^n \psi\}_{n \in \mathbb{Z}}. \end{aligned}$$

Hence

$$D^m \overline{\text{span}}\{T^n \psi\}_{n \in \mathbb{Z}} = \overline{D^m \text{span}\{T^n \psi\}_{n \in \mathbb{Z}}}.$$

It follows from this and from (2.11) that

$$D^m \overline{\text{span}}\{T^n \psi\}_{n \in \mathbb{Z}} = \overline{D^m \text{span}\{T^n \psi\}_{n \in \mathbb{Z}}} = \overline{\text{span}}\{D^m T^n \psi\}_{n \in \mathbb{Z}}.$$

This proves (2.10). $\square$

We now derive a second wavelet decomposition of $\mathcal{L}^2(\mathbb{R})$ into orthogonal sum of reducing subspaces for the shift D . For this we begin by defining, for each $n \in \mathbb{Z}$, the subspace

$$(2.14) \quad H_n(\psi) := \bigvee_{m \in \mathbb{Z}} \{\psi_{m,n}\} = \bigvee_{m \in \mathbb{Z}} \{D^m T^n \psi\} = \overline{\text{span}}\{D^m T^n \psi\}_{m \in \mathbb{Z}}$$

—called *time-shift detail* subspace, which is invariant for every power of D . Since $D^* = D^{-1}$, and since m runs over $\mathbb{Z}$, it follows that $H_n(\psi)$ is also D^* -invariant. Hence $H_n(\psi)$ reduces D . Moreover, since $\psi_{m,n}(\cdot) \perp \psi_{m,p}(\cdot)$ whenever $n \neq p$,

$$H_n(\psi) \perp H_p(\psi) \quad \text{for } n \neq p.$$

We now show.

Theorem 1. *Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ be an orthonormal wavelet. Then the space $\mathcal{L}^2(\mathbb{R})$ admits the orthogonal decomposition*

$$(2.15) \quad \mathcal{L}^2(\mathbb{R}) = \bigoplus_{n=-\infty}^{\infty} H_n(\psi),$$

where the subspaces

$$H_n(\psi) := \bigvee_{m \in \mathbb{Z}} \{\psi_{m,n}\}, \quad n \in \mathbb{Z}$$

are reducing for D .

Proof. Recall that, since $H_n(\psi) \perp H_p(\psi)$ for $n \neq p$,

$$(2.16) \quad \bigoplus_{n=-\infty}^{\infty} \bigvee_{m \in \mathbb{Z}} \{D^m T^n \psi\} \cong \overline{\left(\sum_{n=-\infty}^{\infty} \bigvee_{m \in \mathbb{Z}} \{D^m T^n \psi\} \right)} = \bigvee_{n \in \mathbb{Z}} \bigvee_{m \in \mathbb{Z}} \{D^m T^n \psi\},$$

where $\cong$ means *unitarily equivalent*. Similarly, as $W_m(\psi) \perp W_p(\psi)$ for $m \neq p$ according to the orthogonal direct sum in (2.8), we get from (2.10) that

$$(2.17) \quad \bigoplus_{m=-\infty}^{\infty} \bigvee_{n \in \mathbb{Z}} \{D^m T^n \psi\} \cong \bigvee_{m \in \mathbb{Z}} \bigvee_{n \in \mathbb{Z}} \{D^m T^n \psi\}.$$

Since $\{\psi_{m,n}\}_{m,n \in \mathbb{Z}}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$, it follows from (2.6) that

$$\mathcal{L}^2(\mathbb{R}) = \bigvee_{m,n \in \mathbb{Z}} \{\psi_{m,n}\} = \bigvee_{m,n \in \mathbb{Z}} \{D^m T^n \psi\}.$$

Thus, by unconditional convergence of the Fourier Series,

$$(2.18) \quad \bigvee_{m \in \mathbb{Z}} \bigvee_{n \in \mathbb{Z}} \{D^m T^n \psi\} = \bigvee_{m,n \in \mathbb{Z}} \{D^m T^n \psi\} = \bigvee_{n \in \mathbb{Z}} \bigvee_{m \in \mathbb{Z}} \{D^m T^n \psi\},$$

and therefore, according to (2.14),

$$(2.19) \quad \mathcal{L}^2(\mathbb{R}) \cong \bigoplus_{n=-\infty}^{\infty} \bigvee_{m \in \mathbb{Z}} \{D^m T^n \psi\} = \bigoplus_{n=-\infty}^{\infty} H_n(\psi).$$

This completes the proof of the Theorem by writing $=$ for $\cong$, as usual. $\square$

We conclude from the above that.

Proposition 2. *With respect to an orthonormal wavelet $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$, any $f(\cdot)$ in $\mathcal{L}^2(\mathbb{R})$ admits the “scale-detail” decomposition*

$$(2.20) \quad f(\cdot) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [f(\cdot), \psi_{m,n}(\cdot)] \psi_{m,n}(\cdot),$$

as well as the “time-shift-detail” decomposition

$$(2.21) \quad f(\cdot) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [f(\cdot), \psi_{m,n}(\cdot)] \psi_{m,n}(\cdot).$$

We must note that the decomposition (2.8) is a wavelet wandering subspace decomposition of $\mathcal{L}^2(\mathbb{R})$ which gives rise to the decomposition (1.2) of $f(\cdot)$ in $\mathcal{L}^2(\mathbb{R})$. Moreover, it is also consequence of a wavelet MRA which is defined as follows [5].

A sequence of subspaces $\{V_m(\phi), m \in \mathbb{Z}\}$ of $\mathcal{L}^2(\mathbb{R})$ is a wavelet MRA, with scaling function $\phi(\cdot)$, if the following conditions hold:

- (i) $V_m(\phi) \subset V_{m+1}(\phi), m \in \mathbb{Z}$,
- (ii) $\bigcap_{m=-\infty}^{\infty} V_m(\phi) = \{0\}$,
- (iii) $\bigcup_{m=-\infty}^{\infty} V_m(\phi) = \mathcal{L}^2(\mathbb{R})$,
- (iv) $v(\cdot) \in V_m(\phi) \iff v(2(\cdot)) \in V_{m+1}(\phi), m \in \mathbb{Z}$,
- (v) $\{\phi(\cdot - n), n \in \mathbb{Z}\}$ is an orthonormal basis of the subspace $V_0(\phi)$.

Condition (v) is “native” only to wavelet, while conditions (i)-(iv), on the one hand, define the shift operator D [2], and on the other hand define, in general, an incoming subspace V_0 for the shift operator D —“à la” Lax-Phillips Scattering Theory. Then with condition (v), V_0 depends on $\phi(\cdot)$, hence it is written as $V_0(\phi)$. We refer to the work of Antoniou and Gustafson [1] for these and other interesting connections between wavelet MRA and various parts of Mathematics.

What is interesting, from invariant subspace view point, is that by conditions (i) and (iv), each $V_m(\phi)$ is a D^* -invariant subspace. Moreover, it is also irreducible, i.e., it does not contain any nontrivial reducing subspace of D . This is due to the fact that $V_m(\phi)$ can be expressed in terms of the subspaces $W_k(\psi)$, $-\infty < k \leq m-1$, as [5]

$$(2.22) \quad V_m(\phi) = \bigoplus_{k=-\infty}^{m-1} W_k(\psi),$$

and we have noted above that $W_k(\psi)$ are neither D -invariant nor D^* -invariant.

The decomposition (2.15), on the contrary, cannot be derived from a wavelet MRA since the subspaces $H_n(\psi)$ are reducing subspaces for D . Reducing subspaces of shifts are well understood. Thus the decomposition (2.15) of $\mathcal{L}^2(\mathbb{R})$ —can be called a wavelet reducing subspaces decomposition—provides further understanding of wavelets as well as their relationships to shift operators.

We close by noting that, for each $m \in \mathbb{Z}$ and each $n \in \mathbb{Z}$,

$$(2.23) \quad W_m(\psi) \cap H_n(\psi) = D^m T^n \psi = \psi_{m,n},$$

which is simply the detail at scale- 2^m and at time-shift- n . Therefore, the projection of $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ onto $\psi_{m,n}$

$$(2.24) \quad P_{\psi_{m,n}} f(\cdot) = [f(\cdot), \psi_{m,n}(\cdot)] \psi_{m,n}(\cdot)$$

is the detail variation at scale- 2^m and at time-shift- n . Then, since the orthogonal complements of $\{\psi_{m,n}\}$ in $W_m(\psi)$ and in $H_n(\psi)$, respectively, are orthogonal, we also have

$$(2.25) \quad P_{\psi_{m,n}} f(\cdot) = P_{W_m(\psi)} P_{H_n(\psi)} f(\cdot) = P_{H_n(\psi)} P_{W_m(\psi)} f(\cdot).$$

This again explains why the scale-detail decomposition (2.20) and the time-shift-detail decomposition (2.21) are decompositions of the same $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$. Consequently, (2.2) and (2.15) are two orthogonal decompositions of the same space $\mathcal{L}^2(\mathbb{R})$.

ACKNOWLEDGMENT

We wish to thank Professor K. Gustafson for his interesting comments.

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UNIVERSITY OF CALIFORNIA IN LOS ANGELES, LOS ANGELES, CA 90024-1594, USA
E-mail address: `levan@ee.ucla.edu`

CATHOLIC UNIVERSITY OF RIO DE JANEIRO, 22453-900, RIO DE JANEIRO, RJ, BRAZIL
E-mail address: `carlos@ele.puc-rio.br`