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CONTRACTIONS SATISFYING THE ABSOLUTE VALUE PROPERTY $|A|^2 \leq |A^2|$

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Abstract. Let $B(H)$ denote the algebra of operators on a complex Hilbert space H, and let $\mathcal{U}$ denote the class of operators $A \in B(H)$ which satisfy the absolute value condition $|A|^2 \leq |A^2|$. It is proved that if $A \in \mathcal{U}$ is a contraction, then either A has a nontrivial invariant subspace or A is a proper contraction and the nonnegative operator $D = |A^2| - |A|^2$ is strongly stable. A Putnam-Fuglede type commutativity theorem is proved for contractions A in $\mathcal{U}$, and it is shown that if normal subspaces of $A \in \mathcal{U}$ are reducing, then every compact operator in the intersection of the weak closure of the range of the derivation $\delta_A(X) = AX - XA$ with the commutant of A^* is quasinilpotent.

1. Introduction

Let H be an infinite-dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all operators on H (i.e., of all bounded linear transformations of H into itself). For any operator A in $B(H)$ set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |A|^2 - |A^*|^2$ (the self-commutator of A), and consider the following standard definitions: A is hyponormal if $|A^*|^2 \leq |A|^2$ (i.e., if $[A^*, A]$ is nonnegative or, equivalently, if $\|A^*x\| \leq \|Ax\|$ for every x in H), p-hyponormal (for some $0 < p \leq 1$) if $|A^*|^{2p} \leq |A|^{2p}$, quasihyponormal if $0 \leq A^*[A^*, A]A$, and paranormal if $\|Ax\|^2 \leq \|A^2x\|\,\|x\|$ for every x in H. Let $\mathcal{U}$ denote the class of operators A satisfying the absolute value condition $|A|^2 \leq |A^2|$, and let $\mathcal{H}(1)$, $\mathcal{H}(p)$, $\mathcal{Q}(1)$ and $\mathcal{K}$ denote, respectively, the classes consisting of hyponormal, p-hyponormal, quasihyponormal and paranormal operators. Then

$$\mathcal{H}(1) \subset \mathcal{Q}(1) \subset \mathcal{U} \subset \mathcal{K}$$

and

$$\mathcal{H}(1) \subset \mathcal{H}(p) \subset \mathcal{U} \subset \mathcal{K},$$

where all the inclusions are proper [8]. The class $\mathcal{U}$ has recently been studied in a number of papers (see [10], [16], [17] for further references). This note continues this study, concentrating mainly on contractions in $\mathcal{U}$. It is proved that if A is a contraction (i.e., if $\|A\| \leq 1$, which means that $\|Ax\| \leq \|x\|$ for every x in H) in $\mathcal{U}$, then either A has a nontrivial invariant subspace or A is a proper contraction (i.e., $\|Ax\| < \|x\|$ for every nonzero x in H) and the nonnegative operator $D = |A^2| - |A|^2$ is strongly stable (i.e., $D^n \xrightarrow{s} 0$; the power sequence $\{D^n\}$ converges strongly to 0). A Putnam-Fuglede type commutativity theorem is proved for contractions A in $\mathcal{U}$. We also prove that if normal subspaces of $A \in \mathcal{U}$ are reducing, then

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every compact operator in the intersection of the weak closure of the range of the derivation $\delta_A(X) = AX - XA$ with the commutant of A^* is quasinilpotent.

In the following we shall denote the spectrum, the point spectrum, the approximate point spectrum and the spectral radius of $A \in B(H)$ by $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$ and $r(A)$, respectively. The joint point spectrum of A, $\sigma_{jp}(A)$, is the set $\{\lambda \in \sigma_p(A)\colon (A-\lambda)x = 0 \Longleftrightarrow (A-\lambda)^*x = 0\}$. We shall denote the set of isolated points of $\sigma(A)$ which are eigenvalues of $A \in B(H)$ of finite algebraic multiplicity (respectively, finite geometric multiplicity) by $\sigma_{00}(A)$ (respectively, $\sigma_0(A)$). A contraction A is said to be completely nonunitary if there exists no nonzero reducing subspace M for A such that $A|_M$ is unitary, and an operator A is said to be pure (i.e., completely nonnormal) if there exists no nonzero reducing subspace M for A such that $A|_M$ is normal. A contraction A is of class $C_{0\cdot}$ if $\lim_{n\to\infty} \|A^n x\| = 0$ for every x in H (i.e., if $A^n \xrightarrow{s} 0$, which means that A is a strongly stable contraction); and it is said to be of class $C_{1\cdot}$ if $\lim_{n\to\infty} \|A^n x\| > 0$ (equivalently, if $A^n x \not\to 0$) for every nonzero x in H. Classes $C_{\cdot 0}$ and $C_{\cdot 1}$ are defined by considering A^* instead of A, and we define the classes $C_{\alpha\beta}$ (for $\alpha, \beta = 0, 1$) by $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$. A C_{00}-contraction A is said to be of class C_0 if there exists an inner function u such that $u(A) = 0$. (See [2] and [14] for more about these classes.)

2. An Invariant Subspace Theorem

The operators $A \in \mathcal{U}$ being paranormal, a number of the properties of $A \in \mathcal{U}$ follow from those of paranormal operators. Thus given $A \in \mathcal{U}$:

1. A is normaloid (i.e., $r(A) = \|A\|$) and the nonzero eigenvalues of A are normal eigenvalues (i.e., if $0 \neq \lambda \in \sigma_p(A)$ and $x \in H$ is a vector such that $Ax = \lambda x$, then $A^*x = \overline{\lambda}x$) [5, 6].
2. If $\sigma(A)$ is countable (in particular, if A is compact), then A is normal [12].
3. A satisfies Weyl's theorem (so that $\sigma_{00}(A) = \sigma_0(A)$) [5].
4. If A is a completely nonunitary contraction, then A is of class $C_{\cdot 0}$. Furthermore, if A is an injective pure contraction and the defect operator $(1 - A^*A)^{\frac{1}{2}}$ is of Hilbert-Schmidt class $\mathcal{C}_2$, then A is of class C_{10} [6, 7].
5. A can not be supercyclic [4].

There are, however, properties that operators $A \in \mathcal{U}$ have, which they share with hyponormal operators and which are not shared by paranormal operators. (Thus, whereas the tensor product $A \otimes B$ of operators of $A, B \in \mathcal{U}$ is again in $\mathcal{U}$ [10], the tensor product of paranormal operators is not necessarily a paranormal operator [13].) Recall that a contraction A is said to be a proper contraction if $\|Ax\| < \|x\|$ for every nonzero x in H. A strict contraction (i.e., a contraction A such that $\|A\| < 1$) is a proper contraction, but a proper contraction is not necessarily a strict contraction (although the concepts of strict and proper contraction coincide for compact operators). It was recently proved in [11] that *if a hyponormal contraction A has no nontrivial invariant subspace, then* (a) *A is a proper contraction and* (b) *its self-commutator $[A^*, A]$ is a strict contraction.* We start by extending item (a), and giving a counterpart of item (b), to contractions A in $\mathcal{U}$; but first we need the following auxiliary result.

Proposition 2.1. *If A is a contraction in $\mathcal{U}$, then the nonnegative operator $D = |A^2| - |A|^2$ is a contraction whose power sequence $\{D^n\}$ converges strongly to a projection P, and $AP = 0$.*

Proof. Take any x in H and any nonnegative integer n. If $A \in \mathcal{U}$, then $0 \le D$. Let $R = D^{\frac{1}{2}}$ be the unique nonnegative square root of D. Recall that $\|\,|A|^{\frac{1}{2}}\|^2 = \|A\|$ and $\|\,|A|x\| = \|Ax\|$ for every $x \in H$, for all A in $B(H)$. If, in addition, A is a contraction (so that $\|\,|A^2|^{\frac{1}{2}}\|^2 = \|A^2\| \le 1$), then

$$\begin{aligned}\langle D^{n+1}x\,;x\rangle &= \|R^{n+1}x\|^2 = \langle DR^n x\,;R^n x\rangle = \|\,|A^2|^{\frac{1}{2}}R^n x\|^2 - \|\,|A|R^n x\|^2\\ &\le \|R^n x\|^2 - \|AR^n x\|^2 \le \|R^n x\|^2 = \langle D^n x\,;x\rangle.\end{aligned}$$

Thus R (and so D) is a contraction (set $n = 0$), and $\{D^n\}$ is a decreasing sequence of nonnegative contractions. Hence $\{D^n\}$ converges strongly to a projection (i.e., to a self-adjoint idempotent), say, P. Moreover,

$$\sum_{n=0}^{m} \|AR^n x\|^2 \le \sum_{n=0}^{m} \Big(\|R^n x\|^2 - \|R^{n+1}x\|^2\Big) = \|x\|^2 - \|R^{m+1}x\|^2 \le \|x\|^2$$

for all nonnegative integers m and every x in H. Therefore, $\|AR^n x\| \to 0$ as $n \to \infty$, and hence

$$APx = A\lim_n D^n x = \lim_n AR^{2n}x = 0,$$

for every $x \in H$, so that $AP = 0$. □

Theorem 2.2. *If a contraction A in $\mathcal{U}$ has no nontrivial invariant subspace, then* (a) *A is a proper contraction and* (b) *the nonnegative operator $D = |A^2| - |A|^2$ is a strongly stable contraction* (*so that $D \in C_{00}$*).

Proof. (a) If $A \in \mathcal{U}$, then $|A|^2 \le |A^2|$. By the Schwarz inequality,

$$\|Ax\|^2 = \langle |A|^2 x\,;x\rangle \le \langle |A^2|x\,;x\rangle \le \|\,|A^2|x\|\,\|x\| = \|A^2 x\|\,\|x\|$$

for every x in H. Put $M = \{x \in H\colon \|Ax\| = \|A\|\,\|x\|\}$, which is a subspace of H (reason: $M = \ker(|A|^2 - \|A\|^2)$, which is clearly a closed linear manifold of H). If x lies in M, then the above inequality ensures that

$$\|A\|\,\|Ax\|\,\|x\| = \|Ax\|^2 \le \|A^2x\|\,\|x\| \le \|A\|\,\|Ax\|\,\|x\|,$$

and hence $\|A(Ax)\| = \|A\|\,\|Ax\|$ so that Ax lies in M. That is, if $A \in \mathcal{U}$, then M is an invariant subspace for A. Now suppose A in $\mathcal{U}$ is a contraction. If A is a strict contraction, then it is trivially a proper contraction. If A is a nonstrict contraction (i.e., if $\|Ax\| \le \|x\|$ for every $x \in H$ and $\|A\| = 1$) and has no nontrivial invariant subspace, then $M = \{x \in H : \|Ax\| = \|x\|\} = \{0\}$. (Actually, since A has no nontrivial invariant subspace, and since M is an invariant subspace for A, it follows that M must be trivial: either $M = \{0\}$ or $M = H$; but if $M = H$, then A is an isometry, and isometries have nontrivial invariant subspaces.) Thus A is a proper contraction (i.e., $M = \{0\}$ implies $\|Ax\| < \|x\|$ for every nonzero $x \in H$).

(b) Let A be a contraction in $\mathcal{U}$. Proposition 2.1 says that D is a contraction, $D^n \xrightarrow{s} P$, and $AP = 0$ so that $PA^* = 0$ (recall: P is self-adjoint). If A has no nontrivial invariant subspace, then A^* has no nontrivial invariant subspace as well. Since $\ker P$ is a nonzero invariant subspace for A^* whenever $PA^* = 0$ and $A \ne 0$, it follows that $\ker P = H$. Hence $P = 0$ so that $D^n \xrightarrow{s} 0$. □

Remark 2.3. In general, proper contractions and strongly stable contractions are not related (there exist C_{00}-contractions that are not proper, and there exist proper contractions of class C_{11}), but every proper contraction is weakly stable [11]. Since weak stability coincides with strong stability for self-adjoint operators, it follows that every self-adjoint proper contraction is strongly stable, and hence (since it is self-adjoint) of class C_{00}. Clearly, $D = |A^2| - |A|^2$ is self-adjoint for every operator A in $B(H)$. If D is a proper contraction, then it is of class C_{00}. Is the converse true? Yes, it is. If $\{D^n\}$ converges strongly to 0, then D is a proper contraction. Indeed, if a self-adjoint operator D is strongly stable, then $\|D\| = r(D) \leq 1$ so that D^2 is a nonnegative contraction, and so is $(1 - D^2)$. If the contraction D is not proper, then there exists a nonzero x in H such that $\|Dx\|^2 = \|x\|^2$, and hence $\langle(1 - D^2)x\,; x\rangle = 0$. Thus $(1 - D^2)^{\frac{1}{2}}x = 0$ so that $D^2x = x$, which implies $\|D^{2n}x\| = \|x\| \neq 0$ for every nonnegative integer n, and therefore D is not strongly stable; a contradiction. Outcome: *A self-adjoint operator is a proper contraction if and only if it is a C_{00}-contraction.* This yields the following corollary of the above theorem.

Corollary 2.4. *If a contraction A in $\mathcal{U}$ has no nontrivial invariant subspace, then both A and $D = |A^2| - |A|^2$ are proper contractions.*

Corollary 2.5. *If a hyponormal contraction A has no nontrivial invariant subspace, then $D = |A^2| - |A|^2$ is a strict contraction.*

Proof. Let $\| \;\|_1$ denote the trace-norm. If A is a hyponormal operator without a nontrivial invariant subspace, then the Berger-Shaw Theorem ensures that the self-commutator $[A^*, A]$ is a trace-class operator, and so is

$$|A^2|^2 - |A|^4 = A^*(A^*A - AA^*)A = A^*[A^*, A]A$$

(the trace class is a two-sided ideal of $B(H)$). This implies that the nonnegative $D^2 = |D|^2$ also is trace-class. Indeed (cf. [3], p.294, inequality $(X.10)$),

$$\Big\| \Big| |A^2| - |A|^2 \Big|^2 \Big\|_1 \leq \Big\| |A^2|^2 - |A|^4 \Big\|_1$$

and therefore,

$$\|D^2\|_1 \leq \|A\|^2 \, \| \, [A^*, A] \, \|_1$$

so that D^2 is trace-class and, consequently, compact. Thus D is compact (the square root of a compact operator is again compact). If, in addition, A is a contraction, then D is strongly stable by Theorem 2.2 (reason: A lies in $\mathcal{U}$ because it is hyponormal). But for compact operators strong stability coincides with uniform stability (i.e., if K is compact, then $\|K^nx\| \to 0$ for every $x \in H$ if and only if $\|K^n\| \to 0$), and uniform stability means spectral radius less than one. Since D is self-adjoint, it follows that $\|D\| = r(D) < 1$; that is, D is a strict contraction. □

3. A Commutativity Theorem

Recall from [7] that a contraction A has $C_{\cdot 0}$ completely nonunitary part if and only if A satisfies the PF-property (i.e., if and only if $AX = XV^*$ for some isometry V and $X \in B(H)$ implies $A^*X = XV$). Let $\mathcal{P}$ denote the class of contractions B with $C_{\cdot 0}$ completely nonunitary part such that (a) $B \in \mathcal{P}$ implies that the restriction of B to an invariant subspace is again in $\mathcal{P}$; (b) $\sigma_p(B) = \sigma_{jp}(B)$; (c) $r(B) = \|B\|$; and (d) the defect operator $D_B = (1 - B^*B)^{\frac{1}{2}} \in \mathcal{C}_2$. (Trivially, isometries belong to the class $\mathcal{P}$.) Let $\mathcal{U}_0$ denote those $A \in \mathcal{U}$ for which normal subspaces are reducing. (An invariant subspace M of A is said to be a normal subspace of A if $A|_M$ is normal.) Let $\delta_{AB} : B(H) \to B(H)$, $\delta_{AA} = \delta_A$, denote the derivation $\delta_{AB}(X) = AX - XB$. The following theorem shows that the PF-property of contractions $A \in \mathcal{U}$ has a generalization to contractions $A \in \mathcal{U}_0$.

Theorem 3.1. *If A is a contraction in $\mathcal{U}_0$ such that $\delta_{AB}(X) = 0$ for some $B^* \in \mathcal{P}$ and $X \in B(H)$, then $\delta_{A^*B^*}(X) = 0$.*

The proof of the theorem proceeds through some steps, stated below as lemmas. The following lemma is well known for the case in which the contraction A is subnormal or hyponormal.

Lemma 3.2. *A C_0-contraction $A \in \mathcal{U}_0$ is normal (with pure point spectrum).*

Proof. As a C_0-contraction, A has a triangulation $A = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}$, where $\sigma(A_1) = \sigma_p(A_1)$ is a countable set contained in the unit disc $\mathbf{D}$ and $\sigma(A_2) = \sigma_a(A_2)$ is contained in the boundary $\partial\mathbf{D}$ of the unit disc [2, 14]. Since $A_1 \in \mathcal{U}_0$, the countability of $\sigma(A_1)$ implies that A_1 is normal and $A = A_1 \oplus A_2$. Clearly, $A_2 \in \mathcal{U}$. Hence, since $\sigma(A_2) \subseteq \partial\mathbf{D}$, $r(A_2) = 1 = r(A_2^{-1})$, which implies that A_2 is unitary. Since A is completely nonunitary, $A = A_1$. □

Lemma 3.3. *If A is a normal contraction and $B^* \in \mathcal{P}$ is a pure contraction, then the only solution $X \in B(H)$ to $\delta_{AB}(X) = 0$ is $X = 0$.*

Proof. Suppose there exists a nontrivial solution X to the equation $AX = XB$. Let $A_1 = A|_{\overline{\operatorname{ran} X}}$, $B_1^* = B^*|_{\ker^\perp X}$ and let $X_1 = \ker^\perp X \to \overline{\operatorname{ran} X}$ be the quasiaffinity defined by setting $X_1 x = Xx$ for each $x \in H$. Then A_1 is a subnormal contraction, $B_1^* \in \mathcal{P}$ is a $C_{\cdot 0}$ contraction and $A_1X_1 = X_1B_1$. Since subnormal contractions have $C_{\cdot 0}$ completely nonunitary part, it follows that both A_1 and B_1 are C_{00} completely nonunitary contractions. The hypothesis $D_{B^*} \in \mathcal{C}_2$ implies $D_{B_1^*} \in \mathcal{C}_2$. Hence B_1 is a C_0 contraction [2]. It now follows that A_1 is a C_0 contraction, which is quasisimilar to B_1 [14]. By Lemma 3.2, A_1 is normal and has pure point spectrum. Since quasisimilar C_0 contractions have the same spectrum, $\sigma(B_1^*) = \sigma_p(B_1^*)$. This, however, is impossible since $\sigma_p(B_1^*) = \sigma_{jp}(B_1^*)$, normal subspaces of B_1^* are reducing, and B_1^* is pure. Hence $X = 0$. □

Lemma 3.4. *If $A \in \mathcal{U}_0$ is a pure contraction and $B^* \in \mathcal{P}$ is a normal contraction, then the only solution $X \in B(H)$ to $\delta_{AB}(X) = 0$ is $X = 0$.*

Proof. The proof being similar to that of Lemma 3.3 is omitted. □

Lemma 3.5. *If $A \in \mathcal{U}_0$ and $B^* \in \mathcal{P}$ are pure contractions, then the only solution $X \in B(H)$ to $\delta_{AB}(X) = 0$ is $X = 0$.*

Proof. Once again the proof is similar to that of Lemma 3.3. Since A and B^* have $C_{\cdot 0}$ completely nonunitary parts, A_1 and B_1^* have $C_{\cdot 0}$ completely nonunitary parts. Thus A_1 and B_1 are quasisimilar C_0 contractions. Hence A_1 is normal and A has a normal part (which is a contradiction). □

Proof of Theorem 3.1. Decompose A and B^* into their normal and pure parts by $A = A_n \oplus A_p$ and $B^* = B_n^* \oplus B_p^*$, and let X have the corresponding matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then Lemmas 3.3, 3.4 and 3.5 imply that $X_{ij} = 0$ for all i, j except $i = j = 1$. Hence $(A_n \oplus A_p)(X_{11} \oplus 0) = (X_{11} \oplus 0)(B_n \oplus B_p)$. Since

$A_nX_{11} = X_{11}B_n$ if and only $A_n^*X_{11} = X_{11}B_n^*$ by the Putnam-Fuglede theorem (see [9]), the result follows. □

For each A in $B(H)$ let $\overline{R(\delta_A)}^w$ denote the weak closure of the range of the derivation δ_A (recall: $\overline{R(\delta_A)}^w = \overline{R(\delta_{AA})}^w$), and let $\{A\}'$ denote the commutant of A. Then every compact operator in $\overline{R(\delta_A)}^w \cap \{A\}'$ is quasinilpotent [15]. We close this paper with the following theorem which shows that compact operators in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ are quasinilpotent whenever either A or A^* lie in $\mathcal{U}_0$.

Theorem 3.6. *If A or A^* is an operator in $\mathcal{U}_0$, then every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent.*

Proof. We consider the case in which $A \in \mathcal{U}_0$; the proof of the other case follows from a similar argument. If B is an operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$, then B^* lies in $\overline{R(\delta_{A^*})}^w \cap \{A\}'$. We start by showing that zero is the unique possible eigenvalue of B^* whose eigenspace is finite-dimensional; that is,

$$\{\lambda \in \sigma_p(B^*)\colon\ \dim\ker(B^* - \lambda) < \infty\} \subseteq \{0\}.$$

Indeed, suppose there exists λ in $\sigma_p(B^*)$ such that $M = \ker(B^* - \lambda)$ is finite-dimensional. Then the subspace M is invariant under both A and B^*. The subspace M being finite-dimensional, the spectrum of the restriction A_1 of A to M consists of a finite number of points, and hence A_1 is normal. By hypothesis, normal subspaces of A reduce A. Therefore, $A = A_1 \oplus A_2$, where $A_2 = A|_{H\ominus M}$. Letting B^* have the representation $\begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}$, with respect to the decomposition $H = M \oplus (H \ominus M)$, it follows that $\lambda I_M \in \overline{R(\delta_{A_1^*})} \cap \{A_1^*\}'$. Recall from [1, pp.136–137] that if N is a normal operator, then $\overline{R(\delta_N)} \cap \{N\}'$ is nilpotent. Hence $\lambda = 0$.

If $B^* \in \overline{R(\delta_A^*)}^w \cap \{A\}'$ is compact, then it follows from the above inclusion that $\sigma(B^*) = \{0\}$. Hence B is quasinilpotent. □

References

1. J. Anderson, *On normal derivations*, Proc. Amer. Math. Soc. **38** (1973), 135–140.
2. H. Bercovici, *Operator Theory and Arithmetic in H^∞* (Mathematical Surveys and Monographs Vol. 26, Amer. Math. Soc., Providence, 1988).
3. Rajendra Bhatia, *Matrix Analysis* (Graduate Texts in Mathematics Vol. 169, Springer, New York, 1997).
4. P.S. Bourdon, *Orbits of hyponormal operators*, Michigan Math. J. **44** (1997), 345–353.
5. N.N. Chourasia and P.B. Ramanujan, *Paranormal operators on Banach spaces*, Bull. Austral. Math. Soc. **21** (1980), 161–168.
6. B.P. Duggal, *Roots of contractions with Hilbert-Schmidt defect operator and $C_{\cdot 0}$ completely non-unitary part*, Comment. Math., Ann. Soc. Math. Polonae, **36** (1996), 85–106.
7. B.P. Duggal, *On characterising contractions with C_{10} pure part*, Integral Equations Operator Theory **27** (1997), 314–323.
8. T. Furuta, *Invitation to Linear Operators* (Taylor & Francis, London, 2001).
9. P.R. Halmos, *A Hilbert Space Problem Book*, 2nd edn. (Graduate Texts in Mathematics Vol. 19, Springer, New York, 1982).
10. I.H. Jeon and B.P. Duggal, *On operators with an absolute value condition*, pre-print.
11. C.S. Kubrusly and N. Levan, *Proper contractions and invariant subspaces*, Internat. J. Math. Math. Sci. **28** (2001), 223–230.
12. Chaoxing Qiu, *Paranormal operators with countable spectrum are normal operators*, J. Math. Res. Exposition **7** (1987), 591–594.

13. T. Saitô, *Hyponormal operators and related topics*, Lectures on Operator Algebras (Lecture Notes in Mathematics Vol. 247, Springer, Berlin, 1972), 533–664.
14. B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space* (North-Holland, Amsterdam, 1970).
15. R.E. Weber, *Derivations and trace norm operators*, Proc. Amer. Math. Soc.**73** (1979), 79–82.
16. A. Uchiyama, *Weyl's theorem for class A operators*, Math. Inequal. Appl. **4** (2001), 143–150.
17. A. Uchiyama and K. Tanahashi, *On the Riesz idempotent of class A operators*, Math. Inequal. Appl. **5** (2002), 291–298.

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