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DUAL-SHIFT DECOMPOSITION OF HILBERT SPACE

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ABSTRACT. We introduce the notion of "dual-shift" decomposition of a separable Hilbert space on which two unilateral shifts are defined. Such a decomposition is then obtained for the function space $L^2[0,1]$ on which the two unilateral shifts are "derived" from the dilation-by-2 and the translation-by-1 bilateral shifts on $L^2(\mathbf{R})$. We then use Multiresolution Analysis of Wavelet Theory to show existence of Haar system type orthonormal base for $L^2[0,1]$. Finally, we combine these with the dual-shift decomposition to obtain a "refined" decomposition for $L^2[0,1]$.

1. Introduction

We introduce the concept of orthogonal decomposition of a Hilbert space, with respect to two discrete unilateral shift semigroups defined on the space.

Let S and V be unilateral shifts defined on a separable Hilbert space H. We know that H admits the *wandering subspace* decompositions: $H = \bigoplus_{k=0}^{\infty} S^k \ker(S^*)$, and $H = \bigoplus_{k=0}^{\infty} V^k \ker(V^*)$.

Is it possible to decompose H into a "similar" orthogonal decomposition—involving both S and V simultaneously? For instance,

$$H = \bigoplus_{k=1}^{\infty} S^k \ \ker(S^*) \ \oplus \ \bigoplus_{k=1}^{\infty} V^k \ \ker(V^*).$$

If such a decomposition exists then we refer to it as a "Dual-Shift Decomposition" of the Hilbert space H. It is worth noticing that we can always get a decomposition as above if we allow the shifts S and V to act on different Hilbert spaces. In fact, the Orthogonal Projection Theorem ensures that $H = M \oplus M^{\perp}$ for any subspace M of H. Since M and $M^{\perp}$ are Hilbert spaces, it is enough to consider their wandering subspace decompositions in terms of a unilateral shift S on M and a unilateral shift V on $M^{\perp}$, respectively. However, our "Dual-Shift Decomposition" requires that S and V are unilateral shifts acting on the same Hilbert space H.

We begin by showing necessary and sufficient conditions for a dual-shift decomposition to exist. Then we derive such a decomposition for the function space $L^2[0,1]$—with respect to two unilateral shifts, denoted by D_u and D_{1u}, which are related to the dilation-by-2 and the translation-by-1 bilateral shifts on $L^2(\mathbf{R})$.

Let $\psi(.)$ be an orthonormal wavelet—living in $L^2[0,1]$—which "comes" from a scaling function $\phi(.) \in L^2[0,1]$. We show that the orthonormal wavelet functions $\psi_{mn}(.) := (D^m T^n \psi)(.)$—living in $L^2[0,1]$—together with the scaling function $\phi(.)$ form an orthonormal basis for $L^2[0,1]$! This is shown by means of the Multiresolution Analysis (MRA) associated with $\phi(.)$ and $\psi(.)$

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An example of the above is the celebrated Haar system in $L^2[0,1]$. This system is derived from the Haar wavelet $\psi_H(.)$

$$\begin{aligned}\psi_H(t) &= 1, \quad 0 \le t \le \frac{1}{2}, \\ &= -1, \quad \frac{1}{2} < t \le 1,\end{aligned}$$

and the associated Haar scaling function $\phi_H(.) = \chi_{[0,1]}(.)$, where $\chi_{[0,1]}(.)$ is the characteristic function of $[0,1]$.

Finally, we derive from the Haar system on $L^2[0,1]$ similar systems for the subspaces $L^2[0,\frac{1}{2}]$ and $L^2(\frac{1}{2},1]$. These Haar systems will then be combined with the dual-shift decomposition to yield a "refined" decomposition for the function space $L^2[0,1]$.

2. Main Results

In the following we will be dealing with separable Hilbert spaces. Inner product and norm are denoted by $[.,.]$ and by $||\,.\,||$, respectively. We begin by deriving an orthogonal decomposition for a separable Hilbert space H on which two isometries, with special properties, are defined.

Lemma 1. *Let S and V be isometries on a Hilbert space H. The following assertions are pairwise equivalent.*

(a) $\operatorname{ran}(S) = \ker(V^*)$.
(b) $\operatorname{ran}(V) = \ker(S^*)$.
(c) $SS^* + VV^* = I$.

Proof. The equivalence between (a) and (b) follows at once by recalling that $\ker(T) = \operatorname{ran}(T^*)^\perp$ for every operator T on H, and for every linear manifold M of $H : \overline{M}^\perp = M^\perp$, and $M^{\perp\perp} = \overline{M}$, and the fact that isometries have a closed range. Suppose any of the equivalent assertions (a) and (b) holds true and take an arbitrary $x = u + v$ in $H = \operatorname{ran}(S) + \operatorname{ran}(S)^\perp = \ker(V^*) + \operatorname{ran}(V)$ so that $u \in \operatorname{ran}(S) = \ker(V^*)$ and $v = \operatorname{ran}(V) = \ker(S^*)$. Thus

$$(SS^* + VV^*)x = SS^*u + SS^*v + VV^*u + VV^*v = SS^*Sy + VV^*Vz = Sy + Vz,$$

for some y and z in H such that $u = Sy$ and $v = Vz$. Therefore $(SS^* + VV^*)x = u + v = x$; that is assertion (c) holds true. Conversely, suppose (c) holds true. If $u \in \operatorname{ran}(S)$ so that $u = Sy$ for some $y \in H$, then $SS^*u = SS^*Sy = Sy = u$, and hence $u = SS^*u + VV^*u = u + VV^*u$, so that $u \in \ker(VV^*) = \ker(V^*)$; that is, $\operatorname{ran}(S) \subseteq \ker(V^*)$. On the other hand, if $v \in \ker(V^*)$, then $v = (SS^* + VV^*)V = SS^*v \in \operatorname{ran}(S)$ and so $\ker(V^*) \subseteq \operatorname{ran}(S)$. Hence (c) implies (a). This finishes the proof. □

Recall that a unilateral shift S_u on a Hilbert space H is an isometry which is such that H admits the orthogonal decomposition [1,6]

$$H = \bigoplus_{k=0}^{\infty} S_u^k \ker(S_u^*). \tag{2.1}$$

The subspace $\ker(S_u^*)$ is called the generating *wandering subspace* of S_u, while its dimension is the *multiplicity* of the unilateral shift.

Now, let us rewrite (2.1) as

$$H = \ker(S_u^*) \oplus \bigoplus_{k=1}^{\infty} S_u^k \ker(S_u^*). \tag{2.2}$$

Therefore,

$$\operatorname{ran}(S_u) = \bigoplus_{k=1}^{\infty} S_u^k \ker(S_u^*) = \bigoplus_{k=0}^{\infty} S_u^k S_u \ker(S_u^*). \tag{2.3}$$

This shows that the restriction of S_u to its range space $\operatorname{ran}(S_u)$ is also a unilateral shift whose wandering subspace is $S_u \ker(S_u^*)$.

It follows from Lemma 1 and from (2.2) that if there exists a second unilateral shift V_u (say) which, together with S_u, satisfy the conditions of Lemma 1, then the space H admits the orthogonal decomposition

$$H = \operatorname{ran}(V_u) \oplus \operatorname{ran}(S_u), \tag{2.4}$$

$$= \bigoplus_{k=1}^{\infty} V_u^k \ker(V_u^*) \oplus \bigoplus_{k=1}^{\infty} S_u^k \ker(S_u^*). \tag{2.5}$$

We summarize the above in the next theorem.

Theorem 1. *Let S_u and V_u be unilateral shifts on a Hilbert space H such that* $\operatorname{ran}(V_u) = \ker(S_u^*)$. *Then H admits the "dual-shift" decomposition*

$$H = \bigoplus_{k=1}^{\infty} V_u^k \ker(V_u^*) \oplus \bigoplus_{k=1}^{\infty} S_u^k \ker(S_u^*).$$

To proceed, we recall that a bilateral shift U on a Hilbert space H is a unitary operator for which there exists a generating wandering subspace W_g such that

$$U^m W_g \perp U^{m'} W_g, \quad m \neq m', \tag{2.6}$$

and, since it is generating, H admits the orthogonal decomposition [1,6]

$$H = \bigoplus_{k=-\infty}^{\infty} U^k W_g. \tag{2.7}$$

We must note that generating wandering subspace of a bilateral shift need not be unique! Also, an alternate definition of bilateral shifts is [2].

Definition 1. A bilateral shift $U : H \to H$ is a unitary operator for which there is a subspace V_0 satisfying the following conditions:

(i)$_o$ $UV_0 \subset V_0$,
((i)$_i$ $U^* V_0 \subset V_0$,)
(ii) $\bigcap_{m=-\infty}^{\infty} U^m V_0 = \{0\}$,
(iii) $\overline{\bigcup}_{m=-\infty}^{\infty} U^m V_0 = H$.

This Definition is actually the Lax-Phillips definition of outgoing subspace (respectively, incoming subspace) V_0 for a unitary operator U [3].

Proposition 1. *Let V_0 be an outgoing subspace of a unitary operator U then: $V_0 = \bigoplus_{m=0}^{\infty} U^m W_g$, and $H = \bigoplus_{m=-\infty}^{\infty} U^m W_g$, where $W_g = V_0 \ominus UV_0$ is a generating wandering subspace for U. Hence, U is a bilateral shift operator on H, [3,4].*

Moreover, V_0 is an irreducible invariant subspace of U [4]. Conversely, if an invariant subspace M of a bilateral shift U is irreducible, then there is a wandering subspace W for U so that $M = \bigoplus_{m=0}^{\infty} U^m W$, [4].

An easy consequence of the above is.

Lemma 2. *Let V_0 be a closed subspace of H and define $V_m := U^m V_0$, ($V_m := U^{*\,m} V_0$), $m \in \mathbf{Z}$, where $U : H \to H$ is a unitary operator. The set $\{V_m,\ m \in \mathbf{Z}\}$ satisfies the following properties:*
(i) $V_{m+1} \subset V_m,\ m \in \mathbf{Z}$,
(ii) $\bigcap_{m=-\infty}^{\infty} V_m = \{0\}$,
(iii) $\overline{\bigcup_{m=-\infty}^{\infty} V_m} = H$,
if and only if V_0 is an outgoing (respectively, incoming) subspace for U. Similarly, if condition (i) is replaced by
(i') $V_m \subset V_{m+1},\ m \in \mathbf{Z}$,
then $\{V_m,\ m \in \mathbf{Z}\}$ satisfies (i'), (ii), (iii) if and only if V_0 is an incoming (respectively, outgoing) subspace for U.

The above lead us to the concept of *Multiresolution Analysis* (MRA) of Wavelet Theory [5]. For this we begin by defining, on $L^2(\mathbf{R})$, the dilation-by-2 operator D

$$Df = g, \quad g(.) = \sqrt{2} f(2(.)), \tag{2.8}$$

and its adjoint operator D^*

$$D^* f = g, \quad g(.) = \frac{1}{\sqrt{2}} f(\frac{(.)}{2}), \tag{2.9}$$

and the translation-by-1 operator T,

$$Tf = g, \quad g(.) = f((.) - 1), \tag{2.10}$$

and its adjoint

$$Tf = g, \quad g(.) = f((.) + 1). \tag{2.11}$$

It is easy to see that both D and T are unitary operators—more precisely, bilateral shifts—on $L^2(\mathbf{R})$.

We have [5].

Definition 2. A sequence of subspaces $\{V_m(\phi),\ m \in \mathbf{Z}\}$ of the function space $L^2(\mathbf{R})$ is a MRA, with *scaling function* $\phi(.)$, if the following conditions hold:
(i) $V_{m+1}(\phi) \subset V_m(\phi), \quad m \in \mathbf{Z}$,
(ii) $\bigcap_{m=-\infty}^{\infty} V_m(\phi) = \{0\}$,
(iii) $\overline{\bigcup_{m=-\infty}^{\infty} V_m(\phi)} = L^2(\mathbf{R})$,
(iv) $v(.) \in V_m(\phi) \Leftrightarrow v(\frac{1}{2}(.)) \in V_{m+1}(\phi), \quad m \in \mathbf{Z}$,
(v) $\{\phi((.) - n),\ n \in \mathbf{Z}\}$ is an orthonormal basis of the subspace $V_0(\phi)$.

It is clear that Definition 2(iv) can be expressed in terms of D^* as

$$V_{m+1}(\phi) = D^* V_m(\phi), \quad m \in \mathbf{Z}, \tag{2.12}$$

while, Definition 2(v) is "native" only to MRA and has nothing to do with the fact that D is a bilateral shift.

We conclude from Lemma 2 and Definition 2 that.

Proposition 2. *A MRA is a sequence of decreasingly-nested subspaces* $\{V_m(\phi),\ m \in \mathbf{Z}\}$ *of the function space* $L^2(\mathbf{R})$, *i.e.,* $V_m(\phi) \subset V_{m+1}(\phi)$, $m \in \mathbf{Z}$, *generated from an incoming subspace* $V_0(\phi)$ *for the bilateral shift* D, *i.e.,* $V_m(\phi) = D^{*\,m}V_0(\phi)$, $m \in \mathbf{Z}$, *where* $V_0(\phi)$ *is, in turn, generated by a scaling function* $\phi(.)$, *i.e.,* $V_0(\phi) = \overline{\text{span}}\ \{\phi((.) - n),\ n \in \mathbf{Z}\}$.

We now derive a dual-shift decomposition for the function space $L^2[0,1]$, considered as a subspace of the function space $L^2(\mathbf{R})$. This, we shall see, involves two unilateral shifts "deriving" from the bilateral shift operator D defined on $L^2(\mathbf{R})$.

To see how D behaves on $L^2[0,1]$, we consider

$$\int_0^1 |f(t)|^2 dt = \int_0^{\frac{1}{2}} |\sqrt{2}f(2\tau)|^2 d\tau, \quad f(.) \in L^2[0,1].$$

This shows that D is an isometry sending $L^2[0,1]$ to $L^2[0,\frac{1}{2}]$. Hence the subspace $L^2[0,1]$ is D-invariant. To proceed, let us identify the subspace $L^2[0,\frac{1}{2}]$ with the subspace $\{f(.) \in L^2[0,1] : f(.) = 0\ a.e.\ on\ (\frac{1}{2},1]\}$ of $L^2[0,1]$. Then the part of D on $L^2[0,1]$, i.e., $D|L^2[0,1] := D_u : L^2[0,1] \to L^2[0,1]$ is an isometry whose range space is the subspace $L^2[0,\frac{1}{2}]$ of $L^2[0,1]$.

We therefore have.

Theorem 2. *The operator* D_u *is a unilateral shift on* $L^2[0,1]$, *with wandering subspace* $L^2(\frac{1}{2},1]$. *Therefore,*

$$L^2[0,1] = \bigoplus_{m=0}^{\infty} D_u^m L^2(\frac{1}{2},1]. \tag{2.13}$$

Proof. The proof follows readily from the fact that the part of a bilateral shift is a unilateral shift [6], and since $\ker(D_u^*) = L^2(\frac{1}{2},1]$. □

Corollary 1. D_u *is a unilateral left shift on* $L^2[0,\frac{1}{2}]$ *with wandering subspace* $L^2(\frac{1}{4},\frac{1}{2}]$, *and*

$$L^2[0,\frac{1}{2}] = \bigoplus_{m=1}^{\infty} D_u^m L^2(\frac{1}{2},1] = \bigoplus_{m=0}^{\infty} D_u^m L^2(\frac{1}{4},\frac{1}{2}]. \tag{2.14}$$

Proof. We have from (2.13): $L^2[0,\frac{1}{2}] = \bigoplus_{m=0}^{\infty} D_u^{m+1}\, L^2(\frac{1}{2},1] = \bigoplus_{m=0}^{\infty} D_u^m L^2(\frac{1}{4},\frac{1}{2}]$. This proves the Corollary. □

Next, we construct a second unilateral shift which together with D_u will yield a dual-shift decomposition for $L^2[0,1]$. For this we define the operator

$$D_1 = DT. \tag{2.15}$$

It is easy to see that

$$DT^2 = TD. \tag{2.16}$$

Therefore,

$$D_1 := DT = TDT^*, \tag{2.17}$$

i.e., D_1 is T-unitarily equivalent to D. Therefore it is also a bilateral shift on $L^2(\mathbf{R})$ and has infinite multiplicity.

Now, the space $L^2[0,1]$ is invariant under D_1 since

$$\int_0^1 |f(t)|^2 dt = \int_{\frac{1}{2}}^1 |\sqrt{2}f(2\tau-1)|^2 d\tau.$$

Therefore, as in the case of the unilateral shift D_u, we identify the subspace $L^2(\frac{1}{2},1]$ with the subspace $\{f(.) \in L^2[0,1] : f(.) = 0,\ a.e.\ on\ [0,\frac{1}{2}]\}$ of $L^2[0,1]$. Then, the part D_{1u} of D_1 on $L^2[0,1]$,

$$D_{1u} := D_1|L^2[0,1] \tag{2.18}$$

is an isometry on $L^2[0,1]$, and its range space is the subspace $L^2(\frac{1}{2},1]$. We therefore conclude that.

Theorem 3. *The operator D_{1u} is a unilateral shift on $L^2[0,1]$, with wandering subspace $L^2[0,\frac{1}{2}]$. Therefore $L^2[0,1]$ admits the orthogonal decomposition*

$$L^2[0,1] = \bigoplus_{m=0}^{\infty} D_{1u}^m\, L^2[0,\frac{1}{2}]. \tag{2.19}$$

As before, we also have.

Corollary 2. *D_{1u} is a unilateral right shift on $L^2(\frac{1}{2},1]$ with wandering subspace $L^2(\frac{1}{2},\frac{3}{4}]$, and*

$$L^2(\frac{1}{2},1] = \bigoplus_{m=1}^{\infty} D_{1u}^m\, L^2[0,\frac{1}{2}] = \bigoplus_{m=0}^{\infty} D_{1u}^m\, L^2(\frac{1}{2},\frac{3}{4}]. \tag{2.20}$$

It then follows easily from the above that.

Theorem 4. *With respect to the unilateral shifts D_u and D_{1u}, the function space $L^2[0,1]$ admits the dual-shift decomposition*

$$L^2[0,1] = \bigoplus_{m=1}^{\infty} D_u^m\, L^2(\frac{1}{2},1] \oplus \bigoplus_{m=1}^{\infty} D_{1u}^m\, L^2[0,\frac{1}{2}], \tag{2.21}$$

$$= \bigoplus_{m=0}^{\infty} D_u^m L^2(\frac{1}{4},\frac{1}{2}] \oplus \bigoplus_{m=0}^{\infty} D_{1u}^m\, L^2(\frac{1}{2},\frac{3}{4}]. \tag{2.22}$$

Corollary 3. *With respect to the unilateral shifts D_u and D_{1u}, the space $L^2[0,1]$ admits the orthogonal decomposition*

$$L^2[0,1] = \bigoplus_{m=0}^{\infty} L^2(\frac{1}{2^{m+2}},\frac{1}{2^{m+1}}] \oplus \bigoplus_{m=0}^{\infty} L^2(1-\frac{1}{2^{m+1}},1-\frac{1}{2^{m+2}}].$$

To proceed we now recall the definition of an *orthonormal wavelet.*

Definition 3. An element $\psi(.)$ of the function space $L^2(\mathbf{R})$ is an orthonormal wavelet if

$$||\psi(.)|| = 1, \quad and \quad \psi((.)-l) \perp \psi((.)-n), \quad l \neq n, \quad l,n \in \mathbf{Z}. \tag{2.23}$$

Moreover, the subspace

$$W_g(\psi) := \overline{\text{span}}\,\{\psi((.)-n),\, n \in \mathbf{Z}\}, \tag{2.24}$$

is a generating wandering subspace of the dilation-by-2 operator D.

It follows at once from this definition that, corresponding to an orthonormal wavelet $\psi(.)$, the function space $L^2(\mathbf{R})$ admits the orthogonal decomposition

$$L^2(\mathbf{R}) = \bigoplus_{m=-\infty}^{\infty} D^m W_g(\psi), \tag{2.25}$$

Therefore the set

$$\{\psi_{mn}(.) := (D^m T^n \psi)(.),\ m,\, n \in \mathbf{Z}\} \tag{2.26}$$

is an orthonormal basis—called *wavelet orthonormal basis*—of $L^2(\mathbf{R})$, and each $\psi_{mn}(.)$ is called a *wavelet orthonormal function*—generated from the wavelet $\psi(.)$.

Let $\{V_m(\phi),\ m \in \mathbf{Z}\}$ be a MRA with scaling function $\phi(.)$. Moreover, suppose that

$$V_m(\phi) \subset V_{m+1}(\phi). \tag{2.27}$$

Let W_m be the orthogonal complement in $V_{m+1}(\phi)$ of $V_m(\phi)$,

$$V_{m+1}(\phi) = V_m(\phi) \oplus W_m, \quad m \in \mathbf{Z}. \tag{2.28}$$

Then it can be shown that there exists a wavelet $\psi(.)$ such that [5]

$$W_0 := \overline{\text{span}}\, \{\psi((.) - n), \quad n \in \mathbf{Z}\} := W_0(\psi), \tag{2.29}$$

and

$$W_m = D^m W_0 := W_m(\psi), \quad m \in \mathbf{Z}. \tag{2.30}$$

Moreover, $W_0(\psi)$ is a generating wandering subspace of the bilateral shift D. It is easy to see that

$$L^2(\mathbf{R}) = V_0(\phi) \oplus \bigoplus_{m=0}^{\infty} W_m(\psi), \tag{2.31}$$

$$= \bigoplus_{m=-\infty}^{\infty} D^m W_0(\psi). \tag{2.32}$$

Suppose now that an orthonormal wavelet $\psi(.)$ in $L^2(\mathbf{R})$ also belongs to the function space $L^2[0,1]$—considered as a subspace of $L^2(\mathbf{R})$. Then it is plain that

$$\psi(.) \in L^2[0,1] \quad \Rightarrow \quad (T^{n-1}\psi)(.) \in L^2[n-1, n], \quad n \geq 1. \tag{2.33}$$

This, in turn, implies that

$$\psi_{m\,n-1}(.) = (D^m T^{n-1}\psi)(.) \in L^2[\frac{n-1}{2^m}, \frac{n}{2^m}], \quad m \geq 0,\ n \geq 1. \tag{2.34}$$

Therefore, for $\psi_{m\,n-1}(.)$ to live in $L^2[0,1]$ we must require $m \geq 0$, and $1 \leq n \leq 2^m$,

$$\psi_{m\,n-1}(.) := (D^m T^{n-1}\psi)(.) \in L^2[0,1], \quad \forall\, m \geq 0, \quad 1 \leq n \leq 2^m. \tag{2.35}$$

We therefore have.

Lemma 3. *If an orthonormal wavelet $\psi(.)$ lives in $L^2[0,1]$, then the wavelet orthonormal functions $\psi_{m\,n-1}(.) := (D^m T^{n-1}\psi)(.)$, $\forall\, m \geq 0$, and $1 \leq n \leq 2^m$, also live in $L^2[0,1]$.*

The orthonormal set (2.35) is not a basis of $L^2[0,1]$! This is shown in the next Theorem.

Theorem 5. *Let $\phi(.) \in L^2[0,1]$ be a scaling function which results in the orthonormal wavelet $\psi(.) \in L^2[0,1]$. Then*

$$L^2[0,1] = \{\phi(.)\} \oplus \bigoplus_{m=0}^{\infty} Z_m(\psi), \tag{2.36}$$

where $Z_m(\psi)$ are finite dimensional subspaces defined by

$$Z_m(\psi) = \text{span}\,\{(D^m T^{n-1}\psi)(.), 1 \le n \le 2^m\}, \quad m \ge 0.$$

Therefore the set $\{\phi(.), (D^m T^{n-1}\psi)(.), 1 \le n \le 2^m,\ m \ge 0\}$, is an orthonormal basis of $L^2[0,1]$.

Proof. Let $\{V_m(\phi),\, m \in \mathbf{Z}\}$ be the MRA with scaling function $\phi(.)$ which results in the wavelet $\psi(.)$. Then from (2.31)

$$L^2(\mathbf{R}) = V_0(\phi) \oplus \bigoplus_{m=0}^{\infty} W_m(\psi). \tag{2.37}$$

Let P be the orthogonal projection from $L^2(\mathbf{R})$ onto $L^2[0,1]$. Then it is plain that

$$PL^2(\mathbf{R}) = L^2[0,1] = PV_0(\phi) \oplus \bigoplus_{m=0}^{\infty} PW_m(\psi). \tag{2.38}$$

But $PV_0(\phi) = P(\overline{\text{span}}\,\{\phi((.)-n),\ n \in \mathbf{Z}\}) = \{\phi(.)\}$, and $PW_m(\psi) = P(\overline{\text{span}}\,\{(D^m T^{n-1}\psi)(.),\ m,\ n \in \mathbf{Z}\}) = Z_m(\psi)$. Therefore (2.36) is proven. The rest of the Theorem then follows trivially. □

We must note that Theorem 5 gives a simple proof of the interesting fact that the orthogonal complement in $L^2[0,1]$ of the set of orthonormal wavelet functions $\psi_{m\,n-1}(.)$, generated from an orthonormal wavelet $\psi(.)$ living in $L^2[0,1]$, is the subspace spanned by the associated scaling function $\phi(.)$—also living in $L^2[0,1]$. An example of this is the well known Haar system to be discussed below.

The most well known, and the very first, orthonormal wavelet is the Haar wavelet $\psi_H(.)$ [5], defined by

$$\begin{aligned} \psi_H(t) &= 1, \quad 0 \le t \le \frac{1}{2}, \\ &= -1, \quad \frac{1}{2} < t \le 1. \end{aligned} \tag{2.39}$$

We recall that the Haar system [7] is the set of functions

$$h_1(t) := \chi_{[0,1]}(t), \quad t \in [0,1], \tag{2.40}$$

$$h_{2^m+n}(t) = \sqrt{2}^m \left(h_1(2^{m+1}t - 2n + 2) - h_1(2^{m+1}t - 2n + 1)\right), \quad t \in [0,1], \tag{2.41}$$

for $m \ge 0$, and $1 \le n \le 2^m$; and in (2.40) $\chi_{[0,1]}(.)$ denotes the characteristic function of the closed interval $[0,1]$. Moreover, the Haar system is a basis of all the function spaces $L^p[0,1],\ 1 \le p < \infty$. Here we only concentrate on $L^2[0,1]$.

We must note that, recently Antoniou and Gustafson [8] showed that the Haar system is also eigenbasis of the Time Operator of Statistical Physics.

It is plain that (2.41) can be written in terms of the operators D, T, and D_1 as,

$$h_{2^m+n}(t) = \frac{1}{\sqrt{2}}\,\{(D^{m+1}T^{2n-2}\,h_1)(t) - (D^{m+1}T^{2n-1}\,h_1)(t)\},\ t \in [0,1], \tag{2.42}$$

for $m \geq 0$, and $1 \leq n \leq 2^m$. This can be further written as

$$h_{2^m+n}(t) = (D^m T^{n-1}(D - D_1)\frac{\chi_{[0,1]}}{\sqrt{2}})(t),\ t \in [0,1],\ m \geq 0,\ 1 \leq n \leq 2^m. \quad (2.43)$$

From which it follows that, for $m = 0$, and hence $n = 1$,

$$h_{2^0+1}(t) = ((D - D_1)\frac{\chi_{[0,1]}}{\sqrt{2}})(t), \quad t \in [0,1]. \quad (2.44)$$

Therefore,

$$\begin{aligned} h_{2^0+1}(t) = h_2(t) &= 1, \quad 0 \leq t \leq \frac{1}{2}, \\ &= -1, \quad \frac{1}{2} < t \leq 1. \end{aligned} \quad (2.45)$$

But, this is precisely the Haar wavelet $\psi_H(.)$ defined by (2.39). Therefore $h_{2^m+n}(.)$ can simply be written in terms of $\psi_H(.)$ as

$$h_{2^m+n}(.) = (D^m T^{n-1}\psi_H)(.), \quad m \geq 0, \quad 1 \leq n \leq 2^m. \quad (2.46)$$

Therefore, by Lemma 3.

Lemma 4. *The Haar system* $\{h_1(.),\ h_{2^m+n}(.),\ m \geq 0,\ 1 \leq n \leq 2^m\}$ *on* $L^2[0,1]$ *consists of the function* $h_1(.) = \chi_{[0,1]}(.)$, *and the Haar wavelet orthonormal functions* $\psi^H_{m\,n-1}(.) := (D^m T^{n-1}\psi_H)(.)$, $m \geq 0$, $1 \leq n \leq 2^m$, *where, for* $m = 0, n = 1$, $\psi^H_{00}(.)$ *is the Haar wavelet* $\psi_H(.)$.

Now, the Haar scaling function $\phi_H(.)$ associated with the Haar wavelet $\psi_H(.)$ is the characteristic function $\chi_{[0,1]}(.)$ [5]. Therefore, by Theorem 5, the Haar system

$$\{\phi_H(.), \quad \psi^H_{m\,n-1}(.) := (D^m T^{n-1}\psi_H)(.), \quad 1 \leq n \leq 2^m, \quad m \geq 0\}, \quad (2.47)$$

is an orthonormal basis of $L^2[0,1]$. An easy consequence of Lemma 4 is.

Corollary 4. *(i) The Haar system for* $L^2[0,\frac{1}{2}]$ *is the set of orthonormal functions*

$$\{\chi_{[0,\frac{1}{2}]}(.), \quad \psi^H_{m\,n-1}(.) = (D^m T^{n-1}\psi_H)(.), \quad m \geq 1, \quad 1 \leq n \leq 2^{m-1}\}, \quad (2.48)$$

(ii) The Haar system for $L^2(\frac{1}{2},1]$ *is the set of orthonormal functions*

$$\{\chi_{(\frac{1}{2},1]}(.), \quad \psi^H_{m\,n-1}(.) = (D^m T^{n-1}\psi_H)(.), \quad m \geq 1, \quad 2^{m-1} < n \leq 2^m\}, \quad (2.49)$$

Returning to the dual-shift decomposition of $L^2[0,1]$ in Theorem 4. It follows that each $f(.) \in L^2[0,1]$ admits the orthogonal expansion

$$f(.) = \sum_{m=1}^{\infty} D^m_{1u}\, g_m(.) + \sum_{m=1}^{\infty} D^m_u\, h_m(.), \quad (2.50)$$

where $g_m(.) \in L^2[0,\frac{1}{2}]$, $h_m(.) \in L^2(\frac{1}{2},1]$, $\sum_{m=1}^{\infty} ||g_m(.)||^2 < \infty$, $\sum_{m=1}^{\infty} ||h_m(.)||^2 < \infty$. Also, if orthonormal bases of $L^2[0,\frac{1}{2}]$ and $L^2(\frac{1}{2},1]$ are available then $g_m(.)$ and $h_m(.)$ can, in turn, be expressed in terms of these bases. For instance, expanding $g_m(.)$ in terms of the orthonormal set (2.48) we get

$$g_m(.) = a_{m0}\chi_{[0,\frac{1}{2}]}(.) + \sum_{n=1}^{2^{m-1}} a_{mn}\psi^H_{m,n-1}(.). \quad (2.51)$$

Similarly, using (2.49) we get

$$h_m(.) = b_{m0}\chi_{(\frac{1}{2},1]}(.) + \sum_{n>2^{m-1}}^{2^m} b_{mn}\psi^H_{m\,n-1}(.). \tag{2.52}$$

Therefore, (2.50) becomes

$$\begin{aligned} f(.) &= \sum_{m=1}^{\infty} a_{m0}D_u^m\chi_{[0,\frac{1}{2}]}(t) + \sum_{m=1}^{\infty}\sum_{n=1}^{2^{m-1}} a_{mn}D_u^m\,\psi^H_{m\,n-1}(.) \\ &+ \sum_{m=1}^{\infty} b_{m0}D_{1u}^m\chi_{(\frac{1}{2},1]}(.) + \sum_{m=1}^{\infty}\sum_{n>2^{m-1}}^{2^m} b_{mn}\,D_{1u}^m\,\psi^H_{m\,n-1}(.)\}. \end{aligned} \tag{2.53}$$

We must note that other orthonormal bases of $L^2[0,\frac{1}{2}]$ and $L^2(\frac{1}{2},1]$ can be used. For instance, one can transform the Legendre Polynomials, which form an orthonormal basis of $L^2[-1,1]$, into bases for $L^2[0,\frac{1}{2}]$ and for $L^2(\frac{1}{2},1]$, using the operators D and D_1, since $DD_1L^2[-1,1] = L^2[0,\frac{1}{2}]$, and $D_1^2\,L^2[-1,1] = L^2(\frac{1}{2},1]$.

3. Conclusion

We introduced the concept of a Dual-Shift Decomposition of a Hilbert space. In particular we derived such a decomposition for the function space $L^2[0,1]$, using the two bilateral shifts of Wavelet Theory: the dilation-by-2 and translation-by-1 operators. Moreover, we derived "Haar-Like" systems for $L^2[0,\frac{1}{2}]$ and $L^2(\frac{1}{2},1]$—from the celebrated Haar system on $L^2[0,1]$.

We must note that in the process of deriving the above we show that a Multiresolution Analysis (MRA) of Wavelet Theory is actually "generated" from an outgoing or an incoming subspace—a concept of the Lax-Phillips Scattering Theory—of the bilateral shift dilation-by-2 operator. Moreover, using the MRA, we show that any wavelet living in $L^2[0,1]$ together with its scaling function do indeed generate a "Haar-Like" system for the space $L^2[0,1]$.

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