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ASYMPTOTICALLY PARTIALLY ISOMETRIC CONTRACTIONS

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ABSTRACT. If T is a Hilbert-space contraction, then the sequence $\{T^{*n}T^n\}_{n\geq 0}$ converges strongly to a nonnegative contraction, which is a projection if and only if T admits an orthogonal direct sum decomposition $T = G \oplus V$, where G is a strongly stable contraction and V is an isometry. Call this class of contractions *asymptotically partially isometric* (the discrete one-parameter semigroup $\{O \oplus V^n\}_{n\geq 0}$ of power partial isometries is such that $\{T^n - (O \oplus V^n)\}_{n\geq 0}$ converges strongly to zero). Two fundamental results ensure that this is quite a large class: (1) a contraction whose adjoint has property PF is asymptotically partially isometric, and (2) a contraction whose intersection of the continuous spectrum of its completely nonunitary direct summand with the unit circle has Lebesgue measure zero is asymptotically partially isometric. It is shown that if every biquasitriangular contraction is asymptotically partially isometric, then every contraction not in class $\mathcal{C}_{00}$ has a nontrivial invariant subspace.

1. Introduction

By an operator we mean a bounded linear transformation of a nonzero complex Hilbert space $\mathcal{H}$ into itself. A contraction is an operator T such that $\|T\| \leq 1$ (i.e., such that $\|Tx\| \leq \|x\|$ for every x in $\mathcal{H}$). Let T^* denote the adjoint of T, and let I be the identity operator. An isometry is a contraction V such that $V^*V = I$ (i.e., an operator V such that $\|Vx\| = \|x\|$ for every x in $\mathcal{H}$), and a coisometry is a contraction whose adjoint is an isometry. An operator U is unitary if it is both an isometry and a coisometry (equivalently, if it is a normal isometry, or a surjective isometry, or still an invertible isometry). If T is a contraction, then $\{T^{*n}T^n\}_{n\geq 0}$ is a bounded monotone sequence of self-adjoint operators (a nonincreasing sequence of nonnegative contractions, actually) so that it converges strongly:

$$T^{*n}T^n \xrightarrow{\;s\;} A$$

for some operator A. Basic properties of the strong limit A have been extensively investigated in current literature (see e.g. [24, p.40], [16], [4], [19], [13], [2], [14] and [11, Ch.3]). In particular, for every contraction T the strong limit A of $\{T^{*n}T^n\}_{n\geq 0}$ is a nonnegative contraction (i.e., $O \leq A \leq I$, where O stands for the null operator), which is nonstrict whenever it is not null (i.e., $\|A\| = 1$ whenever $A \neq O$). These are properties shared by (orthogonal) projections but A is not necessarily a projection (it is not necessarily idempotent).

Example 1. The unilateral weighted shift $T = \text{shift}\{(k+1)^{1/2}(k+2)^{-1}(k+3)^{1/2}\}_{k\geq 0}$ on ℓ^2_+ is a nonstrict proper contraction for which $A = \text{diag}\{(k+1)(k+2)^{-1}\}_{k\geq 0}$ is a completely nonprojective diagonal (cf. [14] or [11, pp.51,52]). In other words,

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$\|T\| = 1$ and $\|Tx\| < \|x\|$ for every nonzero x in ℓ^2_+ (i.e., T is a nonstrict proper contraction) because the weight sequence $\{w_k\}_{k\geq 0} = \{(k+1)^{1/2}(k+2)^{-1}(k+3)^{1/2}\}_{k\geq 0}$ is increasing in $[\sqrt{3/4}, 1)$ and converges to 1; and $Ax \neq A^2x$ for every nonzero x in ℓ^2_+ (i.e., A is completely nonprojective).

In fact, *A is a projection if and only if it commutes with T* (i.e., $A = A^2$ if and only if $AT = TA$ — cf. [2]; also see [14]). Since T^* is a contraction whenever T is, the sequence $\{T^nT^{*n}\}_{n\geq 0}$ converges strongly too. Let A_* be its strong limit,

$$T^nT^{*n} \xrightarrow{\;s\;} A_*,$$

which, of course, share the same properties of A (by replacing T with T^*). It can be verified that *A is a projection whenever A and A_* coincide* (i.e., $A = A_*$ implies $A = A^2$; cf. [14]).

A brief survey on the class of all contractions T for which A a projection is followed by an analysis on the role it plays towards a well-known invariant subspace problem. Such a class is fully characterized in Theorem 0 (Section 2) and, in light of such a characterization, we call those contractions *asymptotically partially isometric.* Two fundamental results which are enough to unfold many subclasses of it (such as cohyponormal, compact and algebraic contractions) are isolated in Propositions 1 and 2 of Section 3. We link this class to a classical open question on invariant subspaces in Section 4.

2. Contractions T for which A is a Projection

An operator T is strongly stable (notation: $T^n \xrightarrow{\;s\;} O$) if the power sequence $\{T^n\}_{n\geq 0}$ converges strongly to the null operator (i.e., $\|T^nx\| \to 0$ for every x in $\mathcal{H}$). Thus a strongly stable contraction is precisely a contraction of class $\mathcal{C}_{0\cdot}$ and, dually, a contraction whose adjoint is strongly stable is precisely a contraction of class $\mathcal{C}_{\cdot 0}$, so that a contraction T is of class $\mathcal{C}_{00}$ if and only if both T and T^* are strongly stable (see [24, p.72]). Since $\|T^nx\| \to \|A^{\frac{1}{2}}x\|$ for every x in $\mathcal{H}$, it follows that *a contraction T is strongly stable if and only if $A = O$.* On the opposite end lie the isometries: *a contraction T is an isometry if and only if $A = I$* (reason: $T^{*n}AT^n = A$ for every nonnegative integer n). These are the classes of contractions T for which A is a trivial projection. It is worth noticing that an operator T is a unilateral shift (of any multiplicity) if and only if it is a contraction for which $A = I$ and $A_* = O$ (i.e., an operator is a strongly stable coisometry if and only if it is a backward unilateral shift — see e.g. [11, p.88]; incidentally, this shows that the converse of the assertion "$A = A_*$ implies $A = A^2$ and $A_* = A_*^2$" fails).

Let $\mathcal{M}$ be a subspace (i.e., a closed linear manifold) of $\mathcal{H}$ and let V be an isometry on $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$, the orthogonal complement of $\mathcal{M}$. It is clear that the direct (orthogonal) sum $O \oplus V$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ is a partial isometry (a contraction that acts isometrically on the orthogonal complement of its kernel). In fact, this is the simplest nontrivial instance of a power partial isometry (a partial isometry for which all its powers are again partial isometries). It was proved in [8] that *every power partial isometry is a direct sum of a truncated unilateral shift, a unilateral shift, a backward unilateral shift, and a unitary operator* (where, of course, it is understood that not all four direct summands need to be present in every case). Note that the converse holds trivially because each possible direct summand is a power partial isometry. Since truncated shifts are nilpotent, it follows at once that every power partial isometry is a contraction for which $A = A^2$ and $A_* = A_*^2$ (indeed, $A = O \oplus I \oplus O \oplus I$ and $A_* = O \oplus O \oplus I \oplus I$ if all four direct summands are present). Thus the above italicized result from [8] can be seen as a special case of Theorem 0–b below, where the nilpotent direct summand is extended to a contraction of class $\mathcal{C}_{00}$.

Let T be a contraction on $\mathcal{H}$. If there exists an orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ on which $T = G \oplus V$ for some strongly stable contraction G on $\mathcal{M}$ and some isometry V on $\mathcal{M}^\perp$, then we say that T is an *asymptotically partially isometric contraction.* This means that the power sequence $\{T^n\}_{n\geq 0}$ approaches the sequence of power partial isometries $\{O \oplus V^n\}_{n\geq 0}$ in the strong operator topology; $\{T^n - (O \oplus V^n)\}_{n\geq 0}$ converges strongly to zero. We borrow the next result from [14] (part of it appeared in [2]). It ensures that *a contraction T is asymptotically partially isometric if and only if A is a projection.* (Recall the von Neumann-Wold decomposition: an isometry V is either a unilateral shift S_+, a unitary operator U, or a direct sum $V = S_+ \oplus U$.)

Theorem 0. *Let T be a contraction. If $A = A^2$, then*

(a) $$T = G \oplus S_+ \oplus U,$$

where G is a strongly stable contraction acting on $\ker A$*, S_+ is a unilateral shift on* $\ker(I - A) \cap \ker A_*$*, and U is a unitary operator on* $\ker(I - A) \cap \ker(I - A_*)$. *Moreover, if $A = A^2$ and $A_* = A_*^2$, then*

(b) $$T = B \oplus S_- \oplus S_+ \oplus U,$$

where B is a $\mathcal{C}_{00}$-contraction on $\ker A \cap \ker A_*$ *and S_- is a backward unilateral shift on* $\ker A \cap \ker(I - A_*)$. *Furthermore, if $A = A_*$, then*

(c) $$T = B \oplus U.$$

Proof. See [14] — also see [11, p.83]. □

Again, it is understood that any of the above direct summands may be missing and, if both summands S_- and S_+ are present, they may have distinct (finite or infinite) multiplicities. Note that the converse to each (a), (b) and (c) holds trivially. According to the Nagy-Foiaş-Langer decomposition for contractions [22], [15] (also see, for instance, [24, p.9] or [11, p.76]), every contraction T is uniquely decomposed as $T = C \oplus U$, where C is a completely nonunitary contraction and U is unitary. (Recall: a contraction is completely nonunitary if it has no nonzero unitary direct summand; equivalently, if the restriction of it to any nonzero reducing subspace is not unitary.) Thus, in particular, Theorem 0–a says that C is of class $\mathcal{C}_{0\cdot}$ (i.e., C is strongly stable) if and only if $A = A^2$ and the direct summand S_+ is missing in (a), and Theorem 0–c says that C is of class $\mathcal{C}_{00}$ if and only if $A = A_*$.

3. Two Wide Classes of Asymptotically Partially Isometric Contractions

Asymptotically partially isometric contractions are precisely those contractions T for which A is a projection (Theorem 0–a). Next we isolate two fundamental results (Propositions 1 and 2 below) which ensure that such a class is quite large.

Consider the following definition from [3] (see also [27] and [12]). A contraction T has property PF (a short for Putnam-Fuglede) if either T^* is not intertwined to any isometry or, if T^* is intertwined to some isometry V, then the same transformation that intertwines T^* to V also intertwines T to the coisometry V^*. In other words, let $\mathcal{K}$ be any nonzero complex Hilbert and let $X\colon \mathcal{H} \to \mathcal{K}$ be an arbitrary nonzero bounded linear transformation of $\mathcal{H}$ into $\mathcal{K}$. A contraction T on $\mathcal{H}$ has property PF if, whenever the equation $XT^* = VX$ holds for some isometry V on $\mathcal{K}$, then $XT = V^*X$. Here are two well-known fundamental facts about contractions with property PF: (1) *every isometry has property PF*, and (2) *if a coisometry has property PF, then it is unitary* — rather elementary proofs of these results appeared in [12]. It is worth remarking that, although property PF for contractions as posed above was introduced in [3], the problem of generalizing (in many directions) the classical Fuglede-Putnam Theorem (namely, if a bounded linear transformation intertwines a couple of normal operators, then it also intertwines their adjoints) has been considered by a large number of authors since [21] — for a review on pertinent literature the reader is referred to [1].

Proposition 1. *If a contraction T has property PF, then A_* is a projection.* (T *is asymptotically partially isometric whenever* T^* *is a contraction with property PF.*)

Proof. See [27] — also see [12]. □

Examples. Dominant contractions and paranormal contractions have property PF (see e.g. [3], [27], and the references therein), and so hyponormal contractions have property PF. Recall that an operator T is hyponormal, paranormal, or dominant if $O \leq T^*T - TT^*$, $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every x in $\mathcal{H}$, or $\operatorname{ran}(\lambda I - T) \subseteq \operatorname{ran}(\overline{\lambda} I - T^*)$ for every λ in the spectrum of T, respectively. These three classes are related as follows: every hyponormal operator is dominant and paranormal. An operator T is cohyponormal if T^* is hyponormal. Therefore (Proposition 1), if T^* is a dominant or a paranormal contraction (in particular, if T is a cohyponormal contraction), then T is asymptotically partially isometric (i.e., $A = A^2$). This clearly implies that $A = A^2$ and $A_* = A_*^2$ for every normal contraction, but for a normal contraction T we get $T^{*n}T^n = T^nT^{*n}$ for every nonnegative integer n so that $A = A_*$ trivially (which implies $A = A^2$).

Remarks. Perhaps a systematic investigation on asymptotically partially isometric contractions has been initiated after Putnam's paper [20]. It contains the first proof that a completely nonunitary cohyponormal contraction is strongly stable and, consequently, that if T^* is a hyponormal contraction, then $T = G \oplus U$, where G is a strongly stable contraction and U is unitary, so that A is a projection. Simplified and different proofs followed in [17] (see also [26, pp.113–116]) and in [13] (see also [11, pp.77–79]) by using a reverse approach. They first verified that A is a projection whenever T is a cohyponormal contraction and then concluded that a completely nonunitary cohyponormal contraction is strongly stable (thus stressing the role played by contractions for which A is a projection). In fact, this was extended to paranormal contractions in [17], and to dominant contractions in [5] and [25], which are classes of contractions that include the hyponormal one.

Characterization. Note that the converse of Proposition 1 fails. Indeed, if T is a backward unilateral shift (i.e., if $T = S_+^*$, where S_+ is a unilateral shift of any multiplicity), then A_* is a trivial projection ($A_* = I$) but T does not have property PF (it is a nonunitary coisometry). In fact, it was proved in [3] that *a contraction T has property PF if and only if its completely nonunitary direct summand is of* *class* $\mathcal{C}_{\cdot 0}$ (see also [12]). Corollaries: (1) *T and T^* have property PF if and only if their completely nonunitary direct summands are of class* $\mathcal{C}_{00}$ (i.e., *if and only if* $A = A_*$ — cf. Theorem 0–c), and (2) *if neither T nor T^* have property PF, then T has a nontrivial hyperinvariant subspace* [12].

Another approach to asymptotically partially isometric contractions, which evolves in a different direction and includes classes of contractions not related to the above examples, comes from an earlier result of Sz.-Nagy and Foiaş [23]. Let $\sigma(T)$ denote the spectrum of an operator T and consider its classical partition $\sigma(T) = \sigma_P(T) \cup \sigma_R(T) \cup \sigma_C(T)$, where $\sigma_P(T)$ is the point spectrum (i.e., the set of all eigenvalues of T), $\sigma_R(T) = \sigma_P(T^*)^* \backslash \sigma_P(T)$ is the residual spectrum, and $\sigma_C(T) = \sigma(T) \backslash \big(\sigma_P(T) \cup \sigma_R(T)\big)$ is the continuous spectrum. Let μ denote the Lebesgue measure on the unit circle Γ.

Proposition 2. *If T is a completely nonunitary contraction and $\mu\big(\sigma(T) \cap \Gamma\big) = 0$, then $A = A_* = O$* (i.e., *T is of class $\mathcal{C}_{00}$*).

Proof. See [23] — also see [24, p.85]. □

Corollary 1. *Let C be the completely nonunitary direct summand of an arbitrary contraction T. If $\mu\big(\sigma_C(C) \cap \Gamma\big) = 0$, then $A = A_*$ so that both T and T^* have property PF, and hence are asymptotically partially isometric.*

Proof. Let T be a contraction and consider its Nagy-Foiaş-Langer decomposition, viz. $T = C \oplus U$, where C is a completely nonunitary contraction and U is unitary (as always, any of the above direct summands may be missing). Every completely nonunitary contraction is weakly stable, and a weakly stable contraction C is such that $\sigma_P(C) \cup \sigma_R(C)$ is included in the open unit disc (see e.g. [11, pp.106,114]). Thus, according to Proposition 2, if $\mu\big(\sigma_C(C) \cap \Gamma\big) = 0$, then C and C^* are strongly stable and hence $A = A_* = O \oplus I$. But $A = A_*$ (which means that T and T^* have property PF) implies $A = A^2$ and $A_* = A_*^2$. □

Samples. Compact (countable spectrum) and algebraic (finite spectrum) contractions are asymptotically partially isometric. Quasinilpotent (one-point spectrum) contractions are also included but these are trivially asymptotically partially isometric; they lie in $\mathcal{C}_{00}$.

4. Biquasitriangular Contractions

Are they asymptotically partially isometric? From now on let $\mathcal{H}$ be a nonzero complex *separable* Hilbert space. An operator T on $\mathcal{H}$ is quasitriangular if there exists a sequence $\{P_n\}_{n\geq 1}$ of finite-rank projections on $\mathcal{H}$ that converges strongly to the identity operator and $\{(I - P_n)TP_n\}_{n\geq 1}$ converges uniformly to the null operator [6]. For a wide collection of results on quasitriangular operators the reader is referred to [18, pp.25–30] and [9, pp.163–192]. T is biquasitriangular if both T and T^* are quasitriangular. Since every operator on $\mathcal{H}$ with a countable spectrum is quasitriangular, it follows that the above samples (compact, algebraic and quasinilpotent) are all biquasitriangular.

Question 1. *Is every biquasitriangular contraction asymptotically partially isometric? (Is it true that if T and T^* are quasitriangular contractions, then $A = A^2$ and $A_* = A_*^2$?)*

If Question 1 has an affirmative answer, then a biquasitriangular contraction T admits a decomposition $T = B \oplus S_{-} \oplus S_{+} \oplus U$ (Theorem 0–b) but now just some direct summands might be missing; a unilateral shift S_{+} is not quasitriangular [6] (although the direct sum $S_{-} \oplus S_{+}$ may be [7]). Therefore, it is tempting to think that Question 1 might be reformulated as follows.

Question 1$'$. *Is it true that if T is a biquasitriangular contraction, then $A = A_{*}$?*

An affirmative answer to Question 1$'$ would imply that $T = B \oplus U$ (Theorem 0–c), which trivially implies an affirmative answer to Question 1. Recalling that U is biquasitriangular (it is normal), and that a (countable) direct sum of biquasitriangular operators is again biquasitriangular, the situation here is simpler; any direct summand might be missing. Note that Question 1$'$ can be equivalently stated as: *is the completely nonunitary direct summand of a biquasitriangular contraction of class $\mathcal{C}_{00}$?* Or, still equivalently, *is it true that if T is a biquasitriangular contraction, then T and T^{*} have property PF?*

Answer 1$'$. No. $T = S_{+} \oplus S_{+}^{*}$ is a biquasitriangular contraction for which $A \neq A_{*}$. Indeed, if S_{+} is a unilateral shift (of multiplicity one), then $S_{+} \oplus S_{+}^{*}$ is quasitriangular [7]. Since it is unitarily equivalent to its own adjoint, it follows that it is biquasitriangular. Hence $S_{+} \oplus S_{+}^{*}$ is a completely nonunitary biquasitriangular contraction which, of course, is not of class $\mathcal{C}_{00}$. In fact, if $T = S_{+} \oplus S_{+}^{*}$, then $A = I \oplus O$ and $A_{*} = O \oplus I$. Thus the contraction $S_{+} \oplus S_{+}^{*}$ supplies a negative answer to Question 1$'$, but not to Question 1; $S_{+} \oplus S_{+}^{*}$ is an asymptotically partially isometric biquasitriangular contraction.

Example 2. Let T be the unilateral weighted shift of Example 1, which is a hyponormal contraction (its positive weight sequence is increasing). Since T is not asymptotically partially isometric, we should verify whether it survives Question 1. Yes, it does; it is not quasitriangular (reason: $T^{*}T = \operatorname{diag}\{w_k^2\}_{k\geq 0} \geq (3/4)I$ and $\ker(T^{*}) \neq \{0\}$ — see e.g. [7]); and neither is $O \oplus T$ (see e.g. [6]).

Recall that if there exists an operator (on an infinite-dimensional complex separable Hilbert space) without a nontrivial invariant subspace, then it must be biquasitriangular (see e.g. [18, p.30]). Moreover, if a contraction T with $A = A^{2}$ and $A_{*} = A_{*}^{2}$ has no nontrivial invariant subspace, then it is of class $\mathcal{C}_{00}$ ($A = A_{*} = O$) by Theorem 0–b. That is, $T = B \in \mathcal{C}_{00}$ because the other possible direct summands S_{-}, S_{+} and U clearly have nontrivial invariant subspaces; isometries (and coisometries) have nontrivial invariant subspaces. The above two results show that Question 1 has at least one important consequence, namely, an affirmative answer to Question 1 leads to an affirmative answer to the following classical open question (see [10] for equivalent versions of it).

Question 2. *Does a contraction not in $\mathcal{C}_{00}$ have a nontrivial invariant subspace?*

Outcome. If every biquasitriangular contraction is asymptotically partially isometric, then every contraction not in $\mathcal{C}_{00}$ has a nontrivial invariant subspace.

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