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CONTRACTIONS WITH $\mathcal{C}_{\cdot 0}$ DIRECT SUMMANDS

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Abstract. Contractions with a $\mathcal{C}_{\cdot 0}$ completely nonunitary direct summand were characterized in [4] as those contractions with property PF. The main purpose of this paper is to isolate the essential feature behind such a characterization, namely, contractions T for which the strong limit of $\{T^n T^{*n}\}_{n\geq 1}$ is a projection, and to give a simple new proof of it by using only direct sum decomposition techniques. It is also proved a couple of corollaries for contractions with property PF that mirror the Fuglede-Putnam Theorem for normal operators.

Keywords: *$\mathcal{C}_{\cdot 0}$-contractions, direct sum decompositions, Fuglede-Putnam property.*

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1. Introduction

The class of Hilbert-space contractions with $\mathcal{C}_{\cdot 0}$ completely nonunitary direct summands was recently characterized in [4] as those contractions T for which either T^* is not intertwined to any isometry or, if T^* is intertwined to an isometry J, then the same intertwining transformation also intertwines T to the coisometry J^*. This necessary and sufficient condition was called "PF property" in [4] (PF for Putnam-Fuglede). Here we single out the essential feature behind such a characterization, viz. if a contraction T has property PF, then the strong limit A_* of $\{T^nT^{*n}\}_{n\geq 1}$ is a projection. A direct sum decomposition for contractions with $A_* = A_*^2$ was developed in [9]. Applying it to contractions with property PF yields a direct simple proof for the above-mentioned result. The paper is organized as follows. Notational preliminaries are introduced in § 2. Basic facts about PF property are considered in § 3. Contractions with $\mathcal{C}_{\cdot 0}$ completely nonunitary direct summands are characterized in § 4, and some applications come in § 5.

2. Preliminaries

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ stand for nonzero complex Hilbert spaces, and $\mathcal{B}[\mathcal{H},\mathcal{K}]$ stands for the Banach space of all bounded linear transformations of $\mathcal{H}$ into $\mathcal{K}$. If X lies in $\mathcal{B}[\mathcal{H},\mathcal{K}]$, then X^* in $\mathcal{B}[\mathcal{K},\mathcal{H}]$ denotes the adjoint of X. The range of $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ will be denoted by $\mathcal{R}(X)$ and its closure, which is a subspace (i.e. a closed linear manifold) of $\mathcal{K}$, by $\mathcal{R}(X)^-$. The null space (kernel) of $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, which is a subspace of $\mathcal{H}$, will be denoted by $\mathcal{N}(X)$. Set $\mathcal{B}[\mathcal{H}] = \mathcal{B}[\mathcal{H},\mathcal{H}]$ for short. If T lies in $\mathcal{B}[\mathcal{H}]$, then we say that T is an operator on $\mathcal{H}$. An operator T on $\mathcal{H}$ is strongly stable (notation: $T^n \xrightarrow{s} O$) if the power sequence $\{T^n\}_{n\geq 1}$ converges strongly to the null operator O (i.e. if $T^n x \to 0$ in $\mathcal{H}$ for every $x \in \mathcal{H}$). By a contraction we mean a operator T such that $\|T\| \leq 1$ (i.e. $\|Tx\| \leq \|x\|$ for every $x \in \mathcal{H}$). An isometry is a contraction T such that $\|Tx\| = \|x\|$ for every $x \in \mathcal{H}$ (equivalently, an operator T for which $T^*T = I$, the identity on $\mathcal{H}$), and T is a coisometry if T^* is an isometry. If T is an isometry and a coisometry, then it is a unitary operator. A contraction is of class $\mathcal{C}_{0\cdot}$ if it is strongly stable, and of class $\mathcal{C}_{\cdot 0}$ if its adjoint is strongly stable. On the other extreme, if a contraction T on $\mathcal{H}$ is such that $T^n x \not\to 0$ for every nonzero x in $\mathcal{H}$, then it is said to be of class $\mathcal{C}_{1\cdot}$ and, dually, if $T^{*n} x \not\to 0$ for every nonzero x in $\mathcal{H}$, of class $\mathcal{C}_{\cdot 1}$. These lead to the Nagy-Foiaş classes of contractions introduced in [12] (see also [13, p.72]), namely, $\mathcal{C}_{00}$, $\mathcal{C}_{01}$, $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$. The proposition below states a collection of well-known results that will be required in the sequel (see e.g. [5] or [7, Ch.3]).

Proposition 0. *If T is a contraction on $\mathcal{H}$, then $T^{*n}T^n \xrightarrow{s} A$* (i.e. the sequence $\{T^{*n}T^n\}_{n\geq 1}$ of operators on $\mathcal{H}$ converges strongly to an operator A on $\mathcal{H}$, which means that $\|(T^{*n}T^n - A)x\| \to 0$ for every $x \in \mathcal{H}$). *Moreover, A is a nonnegative contraction* (i.e. $O \leq A \leq I$), *$A = O$ if and only if T is strongly stable* (in fact, $\|T^n x\| \to \|A^{1/2}x\|$ for every $x \in \mathcal{H}$), *and $A = T^{*n}AT^n$ for every nonnegative integer n. Furthermore, associated with T and A there exists an isometry V on $\mathcal{R}(A)^-$ such that $A^{1/2}T = VA^{1/2}$.*

Remark: Since T^* is a contraction whenever T is, let A_* be the strong limit of $\{T^nT^{*n}\}_{n\geq 1}$ and let V_* be the associated isometry on $\mathcal{R}(A_*)^-$ so that all the above properties hold for T, A and V replaced with T^*, A_* and V_*, respectively.

Next we consider two consequences of Proposition 0 that will be needed in § 4 and § 5.

Proposition 1. *A contraction T in $\mathcal{B}[\mathcal{H}]$ is strongly stable* (i.e. $T \in \mathcal{C}_{0\cdot}$) *if and only if the unique solution X in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ to the equation $XT = JX$ for any isometry J in $\mathcal{B}[\mathcal{K}]$ is the trivial $X = O$.*

Proof. Take an arbitrary isometry J in $\mathcal{B}[\mathcal{K}]$ so that $\|J^n y\| = \|y\|$ for every positive integer n and every y in $\mathcal{K}$. If $T^n \xrightarrow{s} O$ and $XT = JX$ for some X in $\mathcal{B}[\mathcal{H},\mathcal{K}]$, then $\|Xx\| = \|J^n Xx\| = \|XT^n x\| \to 0$ for every x in $\mathcal{H}$, and hence $X = O$. Conversely, recall from Proposition 0 that $A^{1/2}T = VA^{1/2}$. If $T^n \overset{s}{\not\longrightarrow} O$ (i.e. if $A \neq O$), then set $\mathcal{K} = \mathcal{R}(A^{1/2})^- = \mathcal{R}(A)^- \neq \{0\}$ and consider the transformation X in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ defined by $Xx = A^{1/2}x$ for every x in $\mathcal{H}$. Thus $XT = VX$ and $X \neq O$. □

A nonzero transformation $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ intertwines an operator $T \in \mathcal{B}[\mathcal{H}]$ to an operator $S \in \mathcal{B}[\mathcal{K}]$ if $XT = SX$. In this case (i.e. if there exists a nonzero X intertwining T to S), then T is said to be intertwined to S. $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ is quasiinvertible if it is injective and has dense range (i.e. $\mathcal{N}(X) = \{0\}$ and $\mathcal{R}(X)^- = \mathcal{K}$). $T \in \mathcal{B}[\mathcal{H}]$ is a quasiaffine transform of $S \in \mathcal{B}[\mathcal{K}]$ if there exists a quasiinvertible $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ intertwining T to S. Proposition 1 can be rewritten in terms of intertwinement as follows. *A contraction is of class* $\mathcal{C}_{0\cdot}$ *if and only if it is not intertwined to any isometry.* Here is the $\mathcal{C}_{1\cdot}$ counterpart.

Proposition 2. *A contraction is of class* $\mathcal{C}_{1\cdot}$ *if and only if it is a quasiaffine transform of an isometry.*

Proof. Let T be a contraction on $\mathcal{H}$. If $T \in \mathcal{C}_{1\cdot}$, then $\mathcal{N}(A) = \mathcal{N}(A^{1/2}) = \{0\}$ so that $\mathcal{R}(A)^- = \mathcal{R}(A^{1/2})^- = \mathcal{H}$ (since A is self-adjoint) and $A^{1/2}T = VA^{1/2}$ (cf. Proposition 0). Hence T is a quasiaffine transform of the isometry V on $\mathcal{R}(A)^-$. Conversely, suppose there exists an injective $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ intertwining T to some isometry J on $\mathcal{K}$. Thus $0 < \|Xx\|$ whenever $x \neq 0$ (so that $\|X\| \neq 0$ because $\mathcal{H} \neq \{0\}$) and $XT^n = J^nX$ for every positive integer n. Hence $0 < \|Xx\| = \|J^nXx\| \leq \|X\|\|T^nx\|$ for each $n \geq 1$, and therefore $\lim \|T^nx\| > 0$, for every nonzero x in $\mathcal{H}$. That is, $T \in \mathcal{C}_{1\cdot}$. □

3. Property PF for Contractions

Definition [4]: A contraction $T \in \mathcal{B}[\mathcal{H}]$ has property PF if, whenever the equation

$$XT^* = JX$$

holds for some isometry $J \in \mathcal{B}[\mathcal{K}]$ and some $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, then

$$XT = J^*X.$$

That is, a contraction T on $\mathcal{H} \neq \{0\}$ has property PF if either T^* is not intertwined to any isometry on any $\mathcal{K} \neq \{0\}$ or, if $X \neq O$ intertwines T^* to an isometry J, then the same X also intertwines T to the coisometry J^*. Propositions 3 and 4 below state basic facts about contractions with property PF that will be needed in the sequel.

Proposition 3. *Every isometry has property* PF.

Proof. Suppose $T^*T = I$ (identity on $\mathcal{H}$) and $J^*J = I$ (identity on $\mathcal{K}$). If there exists $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ such that $XT^* = JX$, then $XT = J^*JXT = J^*XT^*T = J^*X$. □

Proposition 4. *If a nonunitary coisometry is a direct summand of a contraction T, then T does not have property* PF. *In particular, if a coisometry has property* PF*, then it is unitary.*

Proof. Suppose a contraction T on $\mathcal{H}$ has a coisometry as a direct summand. That is, there exists a proper subspace $\mathcal{M}$ of $\mathcal{H}$ that reduces T for which $T = S \oplus J^*$, where S is an operator on $\mathcal{M}$ and J is an isometry on $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$, the orthogonal complement of $\mathcal{M}$. Set $X = O \oplus J$ and $W = I \oplus J$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Since

$$XT^* = O \oplus J^2 = WX,$$

X intertwines T^* to the isometry W. If T has property PF, then

$$O \oplus JJ^* = XT = W^*X = O \oplus J^*J$$

so that J is a normal isometry (i.e. a unitary operator). □

The class of contractions T for which the strong limit A of $\{T^{*n}T^n\}_{n\geq 1}$ is a projection was investigated in [3] and [9]. It coincides with the class of contractions T that commute with A; that is, $A = A^2$ if and only if $AT = TA$ (cf. [3] and [9]). Next we give a simple new proof for a lemma that sets a link between this class and the class of contractions with property PF.

Lemma 1. *If a contraction T has property* PF, *then $A_* = A_*^2$.*

Proof. Let T be a contraction on $\mathcal{H}$. Consider the nonnegative contraction A_* on $\mathcal{H}$ and the isometry V_* on $\mathcal{R}(A_*)^-$ such that $A_*^{1/2}T^* = V_*A_*^{1/2}$ (cf. Proposition 0). Take an arbitrary nonnegative integer n. Thus

$$A_*^{1/2}T^{*n} = V_*^n A_*^{1/2}.$$

If T has property PF, then $A_*^{1/2}T = V_*^* A_*^{1/2}$ so that $A_*^{1/2}T^n = V_*^{*n}A_*^{1/2}$, and hence

$$A_*^{1/2}V_*^n = T^{*n}A_*^{1/2}$$

because $A_*^{1/2}$ is self-adjoint. But $A_* = T^nA_*T^{*n}$ so that

$$A_* = T^nA_*^{1/2}A_*^{1/2}T^{*n} = T^nA_*^{1/2}V_*^nA_*^{1/2} = T^nT^{*n}A_*^{1/2}A_*^{1/2},$$

and therefore $A_* = A_*^2$ (reason: $T^nT^{*n} \xrightarrow{s} A_*$). □

For another proof see [14]. Note that the converse fails: if T is a nonunitary coisometry (sample: a backward unilateral shift), then $A_* = I$ but T does not have property PF by Proposition 4. The above lemma plays a central role for proving the characterization of $\mathcal{C}_{\cdot 0}$ contractions in Theorem 1 below. The class of all contractions with property PF is quite large (see e.g. [4] and the references therein) and is included in the class of all contractions with $A_* = A_*^2$. This, among other evidences (cf. [3] and [9]), indicates that we need to know more about the class of contractions for which $A = A^2$ (or $A_* = A_*^2$). For instance, every hyponormal contraction is such that $A_* = A_*^2$ ([11], [10] and [8]), and hence every normal contraction is such that $A = A^2$ and $A_* = A_*^2$. This is also true for every compact contraction. In fact, normal or compact contractions are such that $A_* = A$, which implies $A = A^2$ [9].

4. Contractions with a $C_{\cdot 0}$ Direct Summand

The result in Theorem 1 below appeared in [4]. We shall give a new proof of it based entirely on direct sum decompositions for contractions. Observe from Proposition 1 (replacing T with T^*) that *a contraction $T \in \mathcal{B}[\mathcal{H}]$ is of class $\mathcal{C}_{\cdot 0}$ if and only if the unique solution $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ to the equation $XT^* = JX$ for any isometry $J \in \mathcal{B}[\mathcal{K}]$ is the trivial $X = O$.* Therefore, *if T is of class $\mathcal{C}_{\cdot 0}$, then it has property* PF *trivially.* Theorem 1 deals with the nontrivial converse for an arbitrary contraction. Recall that a contraction is completely nonunitary if it has no unitary direct summand.

Theorem 1. *The completely nonunitary direct summand of a contraction T is of class $\mathcal{C}_{\cdot 0}$ if and only if T has property* PF.

Proof. If a contraction T on $\mathcal{H}$ has property PF, then A_* is a projection by Lemma 1. Hence T^* can be decomposed as the direct sum of a strongly stable contraction G, a unilateral shift S_+, and a unitary operator U (cf. [9]), where any of the direct summands of the decomposition $T^* = G \oplus S_+ \oplus U$ may be missing (see also [7, p.83]). Thus

$$T \;=\; G^* \oplus S_+^* \oplus U^*,$$

where G^* is of class $\mathcal{C}_{\cdot 0}$, S_+^* is a completely nonunitary coisometry, and U^* is unitary. Since T has property PF, Proposition 4 ensures that S_+^* cannot be present in the above decomposition. Therefore,

$$T \;=\; G^* \oplus U^*$$

so that the completely nonunitary direct summand of T is of class $\mathcal{C}_{\cdot 0}$ (reason: G^* is completely nonunitary because G is strongly stable, and any direct summand of U^* is again unitary). To prove the converse consider the Nagy-Foiaş-Langer decomposition for a contraction T in $\mathcal{B}[\mathcal{H}]$, namely,

$$T \;=\; U \oplus C$$

on $\mathcal{H} = \mathcal{U} \oplus \mathcal{U}^\perp$, where $\mathcal{U} = \mathcal{N}(I - A) \cap \mathcal{N}(I - A_*)$, $U = T|_{\mathcal{U}}$ in $\mathcal{B}[\mathcal{U}]$ is unitary, and $C = T|_{\mathcal{U}^\perp}$ in $\mathcal{B}[\mathcal{U}^\perp]$ is a completely nonunitary contraction (the completely nonunitary direct summand of T). Suppose $XT^* = JX$ for some isometry J in $\mathcal{B}[\mathcal{K}]$ and some X in $\mathcal{B}[\mathcal{H}, \mathcal{K}]$. The von Neumann-Wold decomposition for isometries says that

$$J \;=\; W \oplus S_+$$

on $\mathcal{K} = \mathcal{W} \oplus \mathcal{W}^\perp$, where $\mathcal{W} = \mathcal{N}(I - A'_*)$ (with A'_* denoting the strong limit of $\{J^n J^{*n}\}_{n \geq 1}$), $W = J|_{\mathcal{W}}$ in $\mathcal{B}[\mathcal{W}]$ is unitary, and $S_+ = J|_{\mathcal{W}^\perp}$ in $\mathcal{B}[\mathcal{W}^\perp]$ is a unilateral shift. These are classical direct sum decompositions (see, for instance, [13, pp.3,9] or [7, pp.76,81]). The transformation X in $\mathcal{B}[\mathcal{H}, \mathcal{K}]$ can be written in terms of the orthogonal decompositions $\mathcal{H} = \mathcal{U} \oplus \mathcal{U}^\perp$ and $\mathcal{K} = \mathcal{W} \oplus \mathcal{W}^\perp$ as

$$X \;=\; \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

with X_{11} in $\mathcal{B}[\mathcal{U},\mathcal{W}]$, X_{12} in $\mathcal{B}[\mathcal{U}^\perp,\mathcal{W}]$, X_{21} in $\mathcal{B}[\mathcal{U},\mathcal{W}^\perp]$ and X_{22} in $\mathcal{B}[\mathcal{U}^\perp,\mathcal{W}^\perp]$. Since $XT^* = JX$ we get

$$X_{11}U^* = WX_{11}, \qquad X_{12}C^* = WX_{12},$$
$$X_{21}U^* = S_+X_{21}, \qquad X_{22}C^* = S_+X_{22}.$$

Proposition 3 ensures that

$$X_{11}U = W^*X_{11}$$

because U and W are isometries. Since $X_{21}^*S_+^* = UX_{21}^*$ and S_+^* is strongly stable, it follows by Proposition 1 that $X_{21}^* = O$ (and so $X_{21} = O$). Now suppose C is of class $\mathcal{C}_{\cdot 0}$ (i.e. C^* is strongly stable). Since W and S_+ are isometries, it also follows by Proposition 1 that $X_{12} = O$ and $X_{22} = O$. Therefore $XT = J^*X$, and hence T has property PF. □

5. Applications

Consider again the Nagy-Foiaş-Langer decomposition for a contraction T, viz.

$$T = U \oplus C,$$

where U and C are the unitary and completely nonunitary direct summands of T, respectively (each of them may be missing), and let A and A_* be the strong limits of $\{T^{*n}T^n\}_{n\geq 1}$ and $\{T^nT^{*n}\}_{n\geq 1}$ (cf. Proposition 0). We start with three straightforward corollaries of Theorem 1. Corollary 1 says that both T and T^* have property PF if and only if $A_* = A$. On the opposite end, Corollary 3 says that if none of T and T^* has property PF, then T has a nontrivial hyperinvariant subspace.

Corollary 1. *The following assertions are pairwise equivalent.*

(a) *T and T^* have property* PF.

(b) $C \in \mathcal{C}_{00}$.

(c) $A_* = A$.

Proof. Assertions (a) and (b) are equivalent by Theorem 1. It was shown in [9] that $A_* = A$ if and only if $T = U \oplus B$, where U is unitary and B is a contraction of class $\mathcal{C}_{00}$. This ensures that assertions (b) and (c) are equivalent too. □

Corollary 2. *$C \in \mathcal{C}_{10}$ if and only if T has property* PF *and is a quasiaffine transform of an isometry.*

Proof. $T = U \oplus C$ lies in $\mathcal{C}_{1\cdot}$ if and only if $C \in \mathcal{C}_{1\cdot}$ (since unitary operators lie in $\mathcal{C}_{11}$). Thus T is a quasiaffine transform of an isometry if and only if $C \in \mathcal{C}_{1\cdot}$ (Proposition 2), and T has property PF if and only if $C \in \mathcal{C}_{\cdot 0}$ (Theorem 1). □

Every scalar contraction has property PF. Indeed, if a contraction T is a multiple of the identity, then it is either unitary or a strict contraction. In the latter case it is uniformly stable (i.e. if $\|T\| < 1$, then $\|T^n\| \to 0$) and so of class $\mathcal{C}_{00}$. In both cases both T and T^* have property PF (Proposition 3 and Theorem 1). Therefore, if either T or T^* does not have property PF, then T is nonscalar.

Corollary 3. *If neither T nor T^* have property* PF*, then T has a nontrivial hyperinvariant subspace.*

Proof. If a nonscalar contraction T has no nontrivial hyperinvariant subspace, then it is either a $\mathcal{C}_{00}$, a $\mathcal{C}_{01}$ or a $\mathcal{C}_{10}$ contraction (cf. [6]). The claimed result thus follows by Theorem 1 (it is completely nonunitary and either T or T^* have property PF). □

"If a transformation intertwines a couple of normal operators, then it also intertwines their adjoints". In other words, if $N_1 \in \mathcal{B}[\mathcal{H}]$ and $N_2 \in \mathcal{B}[\mathcal{K}]$ are normal operators, and if $XN_1 = N_2X$ for some $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, then $XN_1^* = N_2^*X$. This is the Fuglede-Putnam Theorem. An important corollary of it reads as follows. *If $N_1 \in \mathcal{B}[\mathcal{H}]$ and $N_2 \in \mathcal{B}[\mathcal{K}]$ are normal operators, and if $XN_1 = N_2X$ for some $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, then $\mathcal{N}(X)$ reduces N_1, $\mathcal{R}(X)^-$ reduces N_2, and $N_1|_{\mathcal{N}(X)^\perp}$ and $N_2|_{\mathcal{R}(X)^-}$ are unitarily equivalent* [2] (see also [1, p.59]). Here is a couple of natural developments that fit the present context. Corollary 4 springs up as a counterpart of the above results that focuses on the operator equation $S^*XT = X$ of [2]. Corollary 5 mirrors the intertwinement-preserving property of the Fuglede-Putnam Theorem.

Corollary 4. *If $T_1 \in \mathcal{B}[\mathcal{H}]$ and $T_2 \in \mathcal{B}[\mathcal{K}]$ are contractions with property* PF*, and if $T_2XT_1^* = X$ for some $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, then $T_2^*XT_1 = X$, $\mathcal{N}(X)$ reduces T_1, $\mathcal{R}(X)^-$ reduces T_2, and $T_1|_{\mathcal{N}(X)^\perp}$ and $T_2|_{\mathcal{R}(X)^-}$ are unitarily equivalent unitary operators.*

Proof. Consider the Nagy-Foiaş-Langer decomposition for T_1 and T_2; that is,

$$T_1 = U_1 \oplus C_1 \qquad \text{and} \qquad T_2 = U_2 \oplus C_2$$

on $\mathcal{H} = \mathcal{U}_1 \oplus \mathcal{U}_1^\perp$ and $\mathcal{K} = \mathcal{U}_2 \oplus \mathcal{U}_2^\perp$, respectively. With respect to these decompositions write the transformation $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ as a 2×2 operator matrix, $X = \big(X_{ij}\big)_{i,j=1,2}$, where $X_{11} \in \mathcal{B}[\mathcal{U}_1, \mathcal{U}_2]$, $X_{12} \in \mathcal{B}[\mathcal{U}_1^\perp, \mathcal{U}_2]$, $X_{21} \in \mathcal{B}[\mathcal{U}_1, \mathcal{U}_2^\perp]$, and $X_{22} \in \mathcal{B}[\mathcal{U}_1^\perp, \mathcal{U}_2^\perp]$. Suppose $T_2XT_1^* = X$ so that

$$U_2X_{12}C_1^* = X_{12}, \qquad U_1X_{21}^*C_2^* = X_{21}^* \qquad \text{and} \qquad C_2X_{22}C_1^* = X_{22}.$$

If T_1 and T_2 have property PF, then C_1^* and C_2^* are strongly stable according to Theorem 1. Observe that $C_2^nX_{22}C_1^{*n} = X_{22}$ for every positive integer n. Since C_1^* is strongly stable and C_2 is a contraction (thus power bounded), $C_2^nX_{22}C_1^{*n} \xrightarrow{s} O$ so that X_{22} is null. Clearly, the same argument also shows that X_{12} and X_{21}^* (and hence X_{21}) are null as well (reason: U_2 and U_1 are contractions too). Outcome:

$$X \;=\; X_{11} \oplus O,$$

and therefore $\mathcal{N}(X) = \mathcal{N}(X_{11}) \oplus \mathcal{U}_1^\perp$ and $\mathcal{R}(X)^- = \mathcal{R}(X_{11})^- \oplus \{0\}$. The hypothesis $T_2XT_1^* = X$ also implies $U_2X_{11}U_1^* = X_{11}$. Since U_1 and U_2 are both unitary we get $U_2^*X_{11}U_1 = X_{11}$ (so that $T_2^*XT_1 = X$), and consequently

$$X_{11}U_1 \;=\; U_2X_{11}.$$

But U_1 and U_2 are normal operators. Thus the above italicized corollary of the Fuglede-Putnam Theorem says that $\mathcal{N}(X_{11})$ reduces U_1, $\mathcal{R}(X_{11})^-$ reduces U_2, and $U_1|_{\mathcal{N}(X_{11})^\perp} \cong U_2|_{\mathcal{R}(X_{11})^-}$ (i.e. $U_1|_{\mathcal{N}(X_{11})^\perp}$ and $U_2|_{\mathcal{R}(X_{11})^-}$ are unitarily equivalent). Hence $\mathcal{N}(X) = \mathcal{N}(X_{11}) \oplus \mathcal{U}_1^\perp$ reduces $T_1 = U_1 \oplus C_1$, $\mathcal{R}(X)^- = \mathcal{R}(X_{11})^- \oplus \{0\}$ reduces $T_2 = U_2 \oplus C_2$, and

$$T_1|_{\mathcal{N}(X)^\perp} = U_1|_{\mathcal{N}(X_{11})^\perp} \cong U_2|_{\mathcal{R}(X_{11})^-} = T_2|_{\mathcal{R}(X)^-}$$

(for $(\mathcal{N}(X_{11}) \oplus \mathcal{U}_1^\perp)^\perp = \mathcal{N}(X_{11})^\perp = \mathcal{U}_1 \ominus \mathcal{N}(X_{11})$), which are unitary (restriction of a unitary operator to a reducing subspace is again unitary). □

Corollary 5. *If $T_1 \in \mathcal{B}[\mathcal{H}]$ and $T_2 \in \mathcal{B}[\mathcal{K}]$ are contractions with property* PF, *one of them being a quasiaffine transform of an isometry, and if $XT_1^* = T_2X$ for some $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$, then $XT_1 = T_2^*X$, $\mathcal{N}(X)$ reduces T_1, $\mathcal{R}(X)^-$ reduces T_2^*, and $T_1|_{\mathcal{N}(X)^\perp}$ and $T_2^*|_{\mathcal{R}(X)^-}$ are unitarily equivalent unitary operators.*

Proof. Consider the setup of the previous proof. If $XT_1^* = T_2X$, then

$$X_{12}C_1^* = U_2X_{12}, \qquad X_{21}^*C_2^* = U_1X_{21}^* \qquad \text{and} \qquad X_{22}C_1^* = C_2X_{22}\ .$$

Note that $X_{22}C_1^{*n} = C_2^nX_{22}$ and, dually, $X_{22}^*C_2^{*n} = C_1^nX_{22}^*$ for every positive integer n. Since T_1 and T_2 have property PF, C_1^* and C_2^* are strongly stable (cf. Theorem 1) so that $X_{22}C_1^{*n} \xrightarrow{s} O$ and $X_{22}^*C_2^{*n} \xrightarrow{s} O$. Hence $C_2^nX_{22}v_1 \to 0$ for every $v_1 \in \mathcal{U}_1^\perp$ and $C_1^nX_{22}^*v_2 \to 0$ for every $v_2 \in \mathcal{U}_2^\perp$. If T_2 or T_1 is a quasiaffine transform of an isometry, then Corollary 2 says that C_2 lies in $\mathcal{C}_{10}$ or C_1 lies in $\mathcal{C}_{10}$, respectively. In the former case $X_{22}v_1 = 0$ for every v_1 in $\mathcal{U}_1^\perp$. In the latter case $X_{22}^*v_2 = 0$ for every v_2 in $\mathcal{U}_2^\perp$. In both cases X_{22} is the null transformation. Clearly, the same argument also shows that X_{12} and X_{21}^* (and so X_{21}) are null as well (reason: U_2 and U_1 lie in $\mathcal{C}_{11}$). This leads to

$$X = X_{11} \oplus O.$$

Moreover, the hypothesis $XT_1^* = T_2X$ also implies $X_{11}U_1^* = U_2X_{11}$. Since U_1^* and U_2 are normal operators, it follows by the Fuglede-Putnam Theorem that

$$X_{11}U_1 = U_2^*X_{11},$$

and therefore $XT_1 = T_2^*X$. But U_1 and U_2^* are normal operators too. Thus, proceeding as in the proof of the previous corollary, $\mathcal{N}(X) = \mathcal{N}(X_{11}) \oplus \mathcal{U}_1^\perp$ reduces $T_1 = U_1 \oplus C_1$, $\mathcal{R}(X)^- = \mathcal{R}(X_{11})^- \oplus \{0\}$ reduces $T_2^* = U_2^* \oplus C_2^*$, and

$$T_1|_{\mathcal{N}(X)^\perp} = U_1|_{\mathcal{N}(X_{11})^\perp} \cong U_2^*|_{\mathcal{R}(X_{11})^-} = T_2^*|_{\mathcal{R}(X)^-},$$

which are unitary operators. □

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