International Journal of Mathematics and Mathematical Sciences **28** (2001) 223–230

PROPER CONTRACTIONS AND INVARIANT SUBSPACES

C.S. Kubrusly and N. Levan

Abstract. Let T be a contraction and let A be the strong limit of $\{T^{*n}T^n\}_{n\geq 1}$. We prove the following theorem. *If a hyponormal contraction T does not have a nontrivial invariant subspace, then T is either a proper contraction of class $\mathcal{C}_{00}$ or a nonstrict proper contraction of class $\mathcal{C}_{10}$ for which A is a completely nonprojective nonstrict proper contraction. Moreover, its self-commutator $[T^*,T]$ is a strict contraction.*

Keywords: *Hyponormal operators, invariant subspaces, proper contractions.*

AMS Subject Classification: 47A15; 47B20.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. By an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T^*\| = \|T\|$. The null space (kernel) of T, which is the subspace of $\mathcal{H}$, will be denoted by $\mathcal{N}(T)$. A contraction is an operator T such that $\|T\| \leq 1$ (i.e. $\|Tx\| \leq \|x\|$ for every x in $\mathcal{H}$ or, equivalently, $T^*T \leq I$). A strict contraction is an operator T such that $\|T\| < 1$ (i.e. $\sup_{0\neq x}(\|Tx\|/\|x\|) < 1$; equivalently, $T^*T \prec I$, which means that $T^*T \leq \gamma I$ for some $\gamma \in (0,1)$). An isometry is a contraction for which $\|Tx\| = \|x\|$ for every x in $\mathcal{H}$ (i.e. $T^*T = I$ so that $\|T\| = 1$).

We summarize below some well-known results on contractions that will be applied throughout the paper (see e.g. [16, p.40], [11], [5], [13], [9], [10] and [8, Ch.3]). If T is a contraction, then $T^{*n}T^n \xrightarrow{s} A$. That is, the sequence $\{T^{*n}T^n\}_{n\geq 1}$ of operators on $\mathcal{H}$ converges strongly to an operator A on $\mathcal{H}$, which means that $\|(T^{*n}T^n - A)x\| \to 0$ for every x in $\mathcal{H}$. Moreover, A is a nonnegative contraction (i.e. $O \leq A \leq I$), $\|A\| = 1$ whenever $A \neq O$, $T^{*n}AT^n = A$ for every integer $n \geq 1$ (so that T is an isometry if and only if $A = I$), $\|T^nx\| \to \|A^{1/2}x\|$ for every x in $\mathcal{H}$, and the null spaces of A and $I - A$, viz. $\mathcal{N}(A) = \{x \in \mathcal{H}\colon\ Ax = 0\}$ and $\mathcal{N}(I-A) = \{x \in \mathcal{H}\colon\ Ax = x\}$, are given by

$$\mathcal{N}(A) = \big\{x \in \mathcal{H}\colon\ T^nx \to 0\big\},$$

$$\mathcal{N}(I-A) \,=\, \big\{x\in\mathcal{H}\colon\; \|T^n x\| = \|x\| \text{ for every } n\geq 1\big\} \,=\, \big\{x\in\mathcal{H}\colon\; \|Ax\| = \|x\|\big\}.$$

Recall that T is a contraction if and only if T^* is. Thus $T^n T^{*n} \xrightarrow{\;s\;} A_*$, where $O \leq A_* \leq I$, $\|A_*\| = 1$ whenever $A_* \neq O$, $T^n A_* T^{*n} = A_*$ for every $n \geq 1$ (so that T is a coisometry — i.e. T^* is an isometry — if and only if $A_* = I$), $\|T^{*n}x\| \to \|A_*^{1/2}x\|$ for every x in $\mathcal{H}$, and

$$\mathcal{N}(A_*) \,=\, \big\{x\in\mathcal{H}\colon\; T^{*n}x \to 0\big\},$$

$$\mathcal{N}(I-A_*) \,=\, \big\{x\in\mathcal{H}\colon\; \|T^{*n} x\| = \|x\| \text{ for every } n\geq 1\big\} \,=\, \big\{x\in\mathcal{H}\colon\; \|A_*x\| = \|x\|\big\}.$$

An operator T on $\mathcal{H}$ is uniformly stable if the power sequence $\{T^n\}_{n\geq 1}$ converges uniformly to the null operator (i.e. $\|T^n\| \to 0$). It is strongly stable if $\{T^n\}_{n\geq 1}$ converges strongly to the null operator (i.e. $\|T^n x\| \to 0$ for every x in $\mathcal{H}$), and weakly stable if $\{T^n\}_{n\geq 1}$ converges weakly to the null operator (i.e. $\langle T^n x\,;y\rangle \to 0$ for every $x, y \in \mathcal{H}$ or, equivalently, $\langle T^n x\,;x\rangle \to 0$ for every $x \in \mathcal{H}$). It is clear that uniform stability implies strong stability, which implies weak stability. The converses fail (a unilateral shift is a weakly stable isometry and its adjoint is a strongly stable coisometry) but hold for compact operators. T is uniformly stable if and only if T^* is uniformly stable, and T is weakly stable if and only if T^* is weakly stable. However, strong convergence is not preserved under the adjoint operation so that strong stability for T does not imply strong stability for T^* (and vice-versa). If T is a strongly stable contraction (i.e. if $\mathcal{N}(A) = \mathcal{H}$, which means that $A = O$), then it is usual to say that T is a $\mathcal{C}_{0\cdot}$-contraction. If T^* is a strongly stable contraction (i.e. if $\mathcal{N}(A_*) = \mathcal{H}$, which means that $A_* = O$), then T is a $\mathcal{C}_{\cdot 0}$-contraction. On the other extreme, if a contraction T is such that $T^n x \not\to 0$ for every nonzero vector x in $\mathcal{H}$ (i.e. if $\mathcal{N}(A) = \{0\}$), then it is said to be a $\mathcal{C}_{1\cdot}$-contraction. Dually, if a contraction T is such that $T^{*n}x \not\to 0$ for every nonzero vector x in $\mathcal{H}$ (i.e. if $\mathcal{N}(A_*) = \{0\}$), then it is a $\mathcal{C}_{\cdot 1}$-contraction. These are the Nagy-Foiaş classes of contractions (see [16, p.72]). All combinations are possible leading to classes $\mathcal{C}_{00}$, $\mathcal{C}_{01}$, $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$. In particular, T and T^* are both strongly stable contractions if and only if T is of class $\mathcal{C}_{00}$. Generally,

$$\begin{aligned}
T \in \mathcal{C}_{00} \quad &\iff \quad A = A_* = O,\\
T \in \mathcal{C}_{01} \quad &\iff \quad A = O \;\text{ and }\; \mathcal{N}(A_*) = \{0\},\\
T \in \mathcal{C}_{10} \quad &\iff \quad \mathcal{N}(A) = \{0\} \;\text{ and }\; A_* = O,\\
T \in \mathcal{C}_{11} \quad &\iff \quad \mathcal{N}(A) = \mathcal{N}(A_*) = \{0\}.
\end{aligned}$$

If T is a strict contraction, then it is uniformly stable, and hence of class $\mathcal{C}_{00}$. Thus a contraction not in $\mathcal{C}_{00}$ is necessarily nonstrict (i.e. if $T \notin \mathcal{C}_{00}$, then $\|T\| = 1$). In particular, contractions in $\mathcal{C}_{1\cdot}$ or in $\mathcal{C}_{\cdot 1}$ are nonstrict.

2. Proper Contractions

An operator T is a proper contraction if $\|Tx\| < \|x\|$ for every nonzero x in $\mathcal{H}$ or, equivalently, if $T^*T < I$. The terms "strict" and "proper" contractions are sometimes interchanged in current literature. We adopt the terminology of [7, p.82] for strict contraction. Obviously, every strict contraction is a proper contraction, every proper contraction is a contraction, and the converses fail: any isometry is a contraction but not a proper contraction, and the diagonal operator $T = \mathrm{diag}\{(k+1)(k+2)^{-1}\}_{k=0}^{\infty}$ is a proper contraction on ℓ^2_+ but not a strict contraction. Thus proper contractions comprise a class of operators that is properly included in the class of all contractions and properly includes the class of all strict contractions. If T is a proper contraction, then so is T^*T (reason: ST is a proper contraction whenever S is a contraction and T is a proper contraction). Thus the point spectrum $\sigma_P(T^*T)$ lies in the open unit disc. If, in addition, T is compact, then so is T^*T and hence its spectrum $\sigma(T^*T)$, which is always closed, also lies in the open unit disc (for $\sigma(K)\backslash\{0\} = \sigma_P(K)\backslash\{0\}$ whenever K is compact). This implies that the spectral radius $r(T^*T)$ is less than one. Therefore, $\|T\|^2 = r(T^*T) < 1$. Conclusion: *the concepts of proper and strict contraction coincide for compact operators.*

Proper contractions have been investigated in connection with unitary dilations (*the minimal unitary dilation of a proper contraction is a bilateral shift whose multiplicity does not exceed the dimension of $\mathcal{H}$* — see [16, p.91]), and also with strong stability of contractive semigroups (cf. [1]). They were further investigated in [15] by considering different topologies in $\mathcal{H}$. Here are three basic properties of proper contractions that will be needed in the sequel.

Proposition 1. *T is a proper contraction if and only if T^* is a proper contraction.*

Proof. Recall that $\|T^*x\|^2 = \langle T^*x\,; T^*x\rangle = \langle TT^*x\,; x\rangle \leq \|TT^*x\|\,\|x\|$ for every x in $\mathcal{H}$, for all operators T on $\mathcal{H}$. Take an arbitrary nonzero vector x in $\mathcal{H}$. If $T^*x = 0$, then $\|T^*x\| < \|x\|$ trivially. On the other hand, if $T^*x \neq 0$ and T is a proper contraction, then $\|TT^*x\| < \|T^*x\| \neq 0$ so that $\|T^*x\|^2 < \|T^*x\|\,\|x\|$, and hence $\|T^*x\| < \|x\|$. That is, T^* is a proper contraction. Dually, since $T^{**} = T$, it follows that T is a proper contraction whenever T^* is. □

If S is a contraction and T is a proper contraction, then ST is a proper contraction (as we have already seen above) and so is S^*T^* by Proposition 1. Another application of Proposition 1 ensures that $TS = (S^*T^*)^*$ still is a proper contraction. Summing up: *left or right product of a contraction and a proper contraction is again a proper contraction.*

Proposition 2. *Every proper contraction is weakly stable.*

Proof. If $\|Tx\| < \|x\|$ for every nonzero x in $\mathcal{H}$, then T is completely nonisometric (i.e. there is no nonzero reducing subspace $\mathcal{M}$ for T such that $\|T^nx\| = \|x\|$ for every $x \in \mathcal{M}$ and every $n \geq 1$), and therefore completely nonunitary. But a completely nonunitary contraction is weakly stable. In fact, the Foguel decomposition for contractions says that every contraction is the direct sum of a weakly stable contraction and a unitary operator (see e.g. [6, p.55] or [8, p.106]). □

The converse of Proposition 2 fails: shifts are weakly stable isometries. However, as it was raised in [1], *a proper contraction is not necessarily strongly stable.* Indeed, if T is the weighted unilateral shift $T = \mathrm{shift}\{(k+1)^{1/2}(k+2)^{-1}(k+3)^{1/2}\}_{k=0}^{\infty}$ on ℓ^2_+, which is a proper contraction because $(k+1)(k+2)^{-2}(k+3) < 1$ for every $k \geq 0$, then A is the diagonal operator $A = \mathrm{diag}\{(k+1)(k+2)^{-1}\}_{k=0}^{\infty} \neq O$ (cf. [10] or [8, pp.51,52]) so that T is not strongly stable. As a matter of fact, $\mathcal{N}(A) = \{0\}$ and (as it is readily verified) $A_* = O$. Hence T is a proper contraction of class $\mathcal{C}_{10}$. The converse is much simpler: *strongly stable contractions are not necessarily proper contractions.* For instance, a backward unilateral shift S_+^* is a strongly stable coisometry (in fact, an operator is a strongly stable coisometry if and only if it is a backward unilateral shift). Thus S_+^* is a strongly stable contraction but not a proper contraction (it is a nonproper contraction of class $\mathcal{C}_{01}$). Actually, even *a $\mathcal{C}_{00}$-contraction is not necessarily a proper contraction.* For example, the weighted bilateral shift $T = \mathrm{shift}\{(|k|+1)^{-1}\}_{k=-\infty}^{\infty}$ on ℓ^2 is a contraction of class $\mathcal{C}_{00}$ (reason: $\prod_{k=0}^{n}(|k|+1)^{-1} = (n!)^{-1} \to 0$ as $n \to \infty$, which means that both products $\prod_{k=0}^{\infty}(|k|+1)^{-1}$ and $\prod_{k=-\infty}^{0}(|k|+1)^{-1}$ diverge to 0 — see [3, p.181]) but not a proper contraction because $(|k|+1)^{-1} = 1$ for $k = 0$. It is worth noticing that the weighted bilateral shift $T = \mathrm{shift}\{1-(|k|+2)^{-2}\}_{k=-\infty}^{\infty}$ on ℓ^2 is a proper contraction of class $\mathcal{C}_{11}$. Indeed, $0 < 1-(|k|+2)^{-2} < 1$ for each integer k, and both products $\prod_{k=0}^{\infty}(1-(|k|+2)^{-2})$ and $\prod_{k=-\infty}^{0}(1-(|k|+2)^{-2})$ do not diverge to 0 (cf. [3, p.181] again) — these products converge once the series $\sum_{k=0}^{\infty}(|k|+2)^{-2}$ converges.

PROPOSITION 3. *If T is a proper contraction, then A is a proper contraction.*

PROOF. Let T be a proper contraction and take an arbitrary nonzero vector x in $\mathcal{H}$. If $T^m x = 0$ for some $m \geq 1$, then $T^n x = 0$ for every integer $n \geq m$. If $T^n x \neq 0$ for every integer $n \geq 1$, then $\|T^{n+1}x\| = \|TT^n x\| < \|T^n x\| < \|x\|$ so that $\{\|T^n x\|\}_{n\geq 1}$ is a strictly decreasing sequence of positive numbers. In the former case T is trivially strongly stable so that $A = O$, a trivial proper contraction. In the latter case $\{\|T^n x\|\}_{n\geq 1}$ converges in the real line to $\|A^{1/2}x\|$ so that $\|A^{1/2}x\| < \|x\|$. Thus $\|Ax\| \leq \|A^{1/2}x\| < \|x\|$. □

A backward unilateral shift shows that the converse of Proposition 3 does not hold true as well (i.e. *there exist nonproper contractions T for which A is a proper contraction*).

3. Invariant Subspaces

A subspace $\mathcal{M}$ of $\mathcal{H}$ is a closed linear manifold of $\mathcal{H}$. $\mathcal{M}$ is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$. If T is an operator on $\mathcal{H}$ and $T(\mathcal{M}) \subseteq \mathcal{M}$, then $\mathcal{M}$ is invariant for T (or $\mathcal{M}$ is T-invariant). If $\mathcal{M}$ is a nontrivial invariant subspace for T, then its orthogonal complement $\mathcal{M}^\perp$ is a nontrivial invariant subspace for T^*. If $\mathcal{M}$ is invariant for both T and T^* (equivalently, if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are T-invariant), then $\mathcal{M}$ reduces T. A classical open question in operator theory is: *does a contraction not in $\mathcal{C}_{00}$ have a nontrivial invariant subspace*? Although this is still an unsolved problem we know that the following result holds true.

LEMMA 1. *If a contraction has no nontrivial invariant subspace, then it is either a $\mathcal{C}_{00}$, a $\mathcal{C}_{01}$, or a $\mathcal{C}_{10}$-contraction.*

PROOF. See, for instance, [8, p.71]. □

The class of contractions T for which A is a projection was investigated in [4] and [10]. It coincides with the class of all contractions T that commute with A; that is, $A = A^2$ if and only if $AT = TA$ (cf. [4]). Equivalently, $\mathcal{N}(A - A^2) = \mathcal{H}$ if and only if $\mathcal{N}(AT - TA) = \mathcal{H}$. The next proposition extends this equivalence.

PROPOSITION 4. *$\mathcal{N}(A - A^2)$ is the largest subspace of $\mathcal{H}$ that is included in $\mathcal{N}(AT - TA)$ and is T-invariant.*

PROOF. See [10] (or [8, p.52]). □

We shall say that A is completely nonprojective if $Ax \neq A^2x$ for every nonzero x in $\mathcal{H}$ (i.e. if $\mathcal{N}(A - A^2) = \{0\}$). Since $\mathcal{N}(A - A^2)$ reduces the self-adjoint operator A, this means that no nonzero direct summand of A is a projection. If A is completely nonprojective, then T is a $\mathcal{C}_{1\cdot}$-contraction (for $\mathcal{N}(A) \subseteq \mathcal{N}(A - A^2)$).

LEMMA 2. *If a contraction T has no nontrivial invariant subspace, then either T is strongly stable or A is a completely nonprojective nonstrict proper contraction.*

PROOF. Suppose T is a contraction without a nontrivial invariant subspace. Since $\mathcal{N}(A - A^2)$ is an invariant subspace for T (by Proposition 4), it follows that either $\mathcal{N}(A - A^2) = \mathcal{H}$ or $\mathcal{N}(A - A^2) = \{0\}$. In the former case A is a projection (i.e. $A = A^2$). However, as it was shown in [10], if A is a projection then T is the direct sum of a strongly stable contraction G, a unilateral shift S_+, and a unitary operator U, where any of the direct summands of the decomposition

$$T = G \oplus S_+ \oplus U$$

may be missing (see also [8, p.83]). But T has no nontrivial invariant subspace so that $T = G$. That is, T is a strongly stable contraction, for S_+ and U clearly have nontrivial invariant subspaces (isometries have nontrivial invariant subspaces). In the latter case A is a completely nonprojective proper contraction. Indeed, $\{x \in \mathcal{H}\colon \|Ax\| = \|x\|\} = \mathcal{N}(I - A) \subseteq \mathcal{N}(A - A^2) = \{0\}$. Finally, the contraction A is not strict (i.e. $\|A\| = 1$) whenever T is not strongly stable (i.e. whenever $A \neq O$). □

Another classical open question in operator theory is: *does a hyponormal operator have a nontrivial invariant subspace*? Recall that an operator T on $\mathcal{H}$ is hyponormal if $TT^* \leq T^*T$ (equivalently, if $\|T^*x\| \leq \|Tx\|$ for every x in $\mathcal{H}$), and T is cohyponormal if T^* is hyponormal. Here is a consequence of Lemmas 1 and 2 for hyponormal contractions. It uses the fact that a cohyponormal contraction T is such that A is a projection. This implies that a completely nonunitary cohyponormal contraction is strongly stable (cf. [14], [12] and [9]).

THEOREM 1. *If a hyponormal contraction T has no nontrivial invariant subspace, then it is either a $\mathcal{C}_{00}$-contraction or a $\mathcal{C}_{10}$-contraction for which A is a completely nonprojective nonstrict proper contraction.*

PROOF. If T has no nontrivial invariant subspace, then T^* has no nontrivial invariant subspace. If T is a contraction, then Lemmas 1 and 2 ensure that either $A = A_* = O$, $A = O$ and A_* is a completely nonprojective nonstrict proper contraction, or A is a completely nonprojective nonstrict proper contraction and $A_* = O$. However, if T is hyponormal, then A_* is a projection [9] so that $A_* = O$ (see also [8, p.78]). $\square$

Can the conclusion in Theorem 1 be sharpened to $T \in \mathcal{C}_{00}$? In other words, *does a hyponormal contraction not in $\mathcal{C}_{00}$ have a nontrivial invariant subspace*? The question has an affirmative answer if we replace "$\mathcal{C}_{00}$-contraction" with "proper contraction". That is, *if a hyponormal contraction is not a proper contraction, then it has a nontrivial invariant subspace*. This will be proved in Theorem 2 below, but first we consider the following auxiliary result. Let D denote the self-commutator of T; that is,

$$D \,=\, [T^*, T] \,=\, T^*T - TT^*.$$

Thus a hyponormal is precisely an operator T for which D is nonnegative (i.e. $D \geq O$).

Proposition 5. *If T is a hyponormal contraction, then D is a contraction whose power sequence converges strongly. If P is the strong limit of $\{D^n\}_{n\geq 1}$, then $PT = O$.*

Proof. Take an arbitrary x in $\mathcal{H}$ and an arbitrary nonnegative integer n. Suppose T is hyponormal and let $R = D^{1/2} \geq O$ be the unique nonnegative square root of $D \geq O$. If, in addition, T is a contraction, then

$$\begin{aligned} \langle D^{n+1}x\,;x\rangle \,=\, \|R^{n+1}x\|^2 \,&=\, \langle DR^n x\,;R^n x\rangle \,=\, \|TR^n x\|^2 - \|T^*R^n x\|^2 \\ &\leq\, \|R^n x\|^2 - \|T^*R^n x\|^2 \,\leq\, \|R^n x\|^2 \,=\, \langle D^n x\,;x\rangle. \end{aligned}$$

This shows that R (and so D) is a contraction: set $n = 0$ above. It also shows that $\{D^n\}_{n\geq 1}$ is a decreasing sequence of nonnegative contractions. Since a bounded monotone sequence of self-adjoint operators converges strongly,

$$D^n \xrightarrow{\;s\;} P \geq O.$$

Indeed, the strong limit P of $\{D^n\}_{n\geq 1}$ is nonnegative, for the set of all nonnegative operators on $\mathcal{H}$ is weakly (thus strongly) closed. As a matter of fact, $P = P^2$ (the weak limit of any weakly convergent power sequence is idempotent) and so $P \geq O$ is a projection. Moreover,

$$\sum_{n=0}^{m} \|T^*R^n x\|^2 \,\leq\, \sum_{n=0}^{m} \big(\|R^n x\|^2 - \|R^{n+1}x\|^2\big) \,=\, \|x\|^2 - \|R^{m+1}x\|^2 \,\leq\, \|x\|^2$$

for all $m \geq 0$ so that $\|T^*R^n x\| \to 0$ as $n \to \infty$. Hence

$$T^*Px \,=\, T^* \lim_n D^n x \,=\, \lim_n T^*R^{2n}x \,=\, 0$$

for every x in $\mathcal{H}$, and therefore $PT = O$ (since P is self-adjoint). $\square$

Theorem 2. *If a hyponormal contraction has no nontrivial invariant subspace, then it is a proper contraction and its self-commutator is a strict contraction.*

Proof. (a) Take an arbitrary operator T on $\mathcal{H}$ and arbitrary x in $\mathcal{H}$. Note that

$$T^*Tx = \|T\|^2 x \quad \text{if and only if} \quad \|Tx\| = \|T\|\,\|x\|.$$

Indeed, if $T^*Tx = \|T\|^2 x$, then $\|Tx\|^2 = \langle T^*Tx\,;x\rangle = \|T\|^2\|x\|^2$. Conversely, if $\|Tx\| = \|T\|\,\|x\|$, then $\langle T^*Tx\,;\|T\|^2x\rangle = \|T\|^4\|x\|^2$ and hence

$$\begin{aligned}\|T^*Tx - \|T\|^2x\|^2 \;&=\; \|T^*Tx\|^2 - 2\mathrm{Re}\,\langle T^*Tx\,;\|T\|^2x\rangle + \|T\|^4\|x\|^2\\ &=\; \|T^*Tx\|^2 - \|T\|^4\|x\|^2 \;\le\; \big(\|T^*T\|^2 - \|T\|^4\big)\|x\|^2 \;=\; 0.\end{aligned}$$

Put $\mathcal{M} = \{x \in \mathcal{H}\colon\, \|Tx\| = \|T\|\,\|x\|\} = \mathcal{N}(\|T\|^2 I - T^*T)$, which is a subspace of $\mathcal{H}$. If T is hyponormal, then $\mathcal{M}$ is T-invariant. In fact, if T is hyponormal and $x \in \mathcal{M}$, then

$$\|T(Tx)\| \;\le\; \|T\|\|Tx\| \;=\; \|\,\|T\|^2x\| \;=\; \|T^*Tx\| \;\le\; \|T(Tx)\|$$

and so $Tx \in \mathcal{M}$ (see also [6, p.9]). Now let T be a hyponormal contraction. If $\|T\| < 1$, then it is trivially a proper contraction. If $\|T\| = 1$ and T has no nontrivial invariant subspace, then $\mathcal{M} = \{x \in \mathcal{H}\colon\, \|Tx\| = \|x\|\} = \{0\}$ (actually, if $\mathcal{M} = \mathcal{H}$, then T is an isometry and isometries have invariant subspaces). Hence T is a proper contraction.

(b) Let $D \ge O$ be the self-commutator of a hyponormal contraction T and let P be the strong limit of $\{D^n\}_{n\ge1}$ so that $PT = O$ (cf. Proposition 5). Suppose T has no nontrivial invariant subspace. Since $\mathcal{N}(P)$ is a nonzero invariant subspace for T whenever $PT = O$ and $T \ne O$, it follows that $\mathcal{N}(P) = \mathcal{H}$. Hence $P = O$ and so D is strongly stable ($D^n \xrightarrow{\;s\;} O$). Moreover, since $\bigvee\{T^n x\}_{n\ge0}$ is a nonzero invariant subspace for T whenever $x \ne 0$, it follows that $\bigvee\{T^n x\}_{n\ge0} = \mathcal{H}$ for each $x \ne 0$ (every nonzero vector in $\mathcal{H}$ is a cyclic vector for T). Thus the Berger-Shaw Theorem (see, for instance, [2, p.152]) ensures that D is a trace-class operator so that D is compact (i.e. T is essentially normal). But for compact operators strong stability coincides with uniform stability, and uniform stability always means spectral radius less than one. Hence the nonnegative D is a strict contraction because it is clearly normaloid (i.e. $\|D\| = r(D) < 1$). □

Remark: According to the Berger-Shaw Theorem a hyponormal contraction without a nontrivial invariant subspace has a trace-class self-commutator D with trace-norm $\|D\|_1 \le 1$. If $D \ne O$ is not a rank-one operator, then $\|D\| < \|D\|_1 \le 1$. The above argument ensures the inequality $\|D\| < 1$ whenever a hyponormal contraction has no nontrivial invariant subspace, including the case of a hyponormal contraction with a rank-one self-commutator.

An operator is seminormal if it is hyponormal or cohyponormal. Recall that T^* has a nontrivial invariant subspace if and only if T has, T^* is a proper contraction if and only if T is (Proposition 1), and $[T,T^*] = -[T^*,T]$. Thus the above theorem also holds for cohyponormal contractions. *If a seminormal contraction has no nontrivial invariant subspace, then it is a proper contraction and its self-commutator is a strict contraction.* This prompts the question: can we drop “hyponormal” from the theorem statement? In particular, *is it true that every nonproper contraction has a nontrivial invariant subspace*? Theorems 1 and 2 yield the following result.

Corollary 1. *If a hyponormal contraction T has no nontrivial invariant subspace, then it is either a proper contraction of class $\mathcal{C}_{00}$ or a nonstrict proper contraction of class $\mathcal{C}_{10}$ for which A is a completely nonprojective nonstrict proper contraction. Moreover, its self-commutator $[T^*, T]$ is a strict contraction.*

References

1. K.N. Boyadzhiev and N. Levan, Strong stability of Hilbert space contraction semigroups, *Studia Sci. Math. Hungar.* **30** (1995) 165–182.
2. J.B. Conway, *The Theory of Subnormal Operators* (Mathematical Surveys and Monographs Vol. 36, Amer. Math. Soc., Providence, 1991).
3. R.G. Douglas, Canonical models, *Topics in Operator Theory* (Mathematical Surveys No. 13, Amer. Math. Soc., Providence, 2nd pr. 1979) 161–218.
4. B.P. Dugall, On unitary parts of contractions, *Indian J. Pure Appl. Math.* **25** (1994) 1243–1247.
5. E. Durszt, Contractions as restricted shifts, *Acta Sci. Math. (Szeged)* **48** (1985) 129–134.
6. P.A. Fillmore, *Notes in Operator Theory* (Van Nostrand, New York, 1970).
7. P.R. Halmos, *A Hilbert Space Problem Book*, 2nd edn. (Springer, New York, 1982).
8. C.S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory* (Birkhäuser, Boston, 1997).
9. C.S. Kubrusly and P.C.M. Vieira, Strong stability for cohyponormal operators, *J. Operator Theory* **31** (1994) 123–127.
10. C.S. Kubrusly, P.C.M. Vieira and D.O. Pinto, A decomposition for a class of contractions, *Adv. Math. Sci. Appl.* **6** (1996) 523–530.
11. N. Levan, Canonical decompositions of completely nonunitary contractions, *J. Math. Anal. Appl.* **101** (1984) 514–526.
12. K. Okubo, The unitary part of paranormal operators, *Hokkaido Math. J.* **6** (1977) 273–275.
13. V. Pták and P. Vrbová, An abstract model for compressions, *Časopis Pěst. Mat.* **113** (1988) 252–266.
14. C.R. Putnam, Hyponormal contractions and strong power convergence, *Pacific J. Math.* **57** (1975) 531–538.
15. M. Shih, P. Tam and K.-K. Tan, Renorms and topological linear contractions on Hilbert spaces, *Sci. China Ser. A* **42** (1999) 246–254.
16. B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space* (North-Holland, Amsterdam, 1970).

Catholic University of Rio de Janeiro, 22453-900 Rio de Janeiro, RJ, Brazil
E-mail: *carlos@ele.puc-rio.br*
Supported in part by CNPq

University of California at Los Angeles, Los Angeles, CA 90024-1594, USA
E-mail: *levan@ee.ucla.edu*