Progress in Nonlinear Diff. Equ. and their Appl. **42** (2000) 167–173

INVARIANT SUBSPACES AND QUASIAFFINE TRANSFORMS OF UNITARY OPERATORS

Carlos S. Kubrusly

Abstract. A classical conjecture on nontrivial invariant subspaces for Hilbert-space contractions reads as follows. "A $\mathcal{C}_1$.-contraction has a nontrivial invariant subspace". This turns out to be equivalent to a second conjecture, namely, "if a contraction is a quasiaffine transform of a unitary operator, then it has a nontrivial invariant subspace". Although these are still unsolved problems, it can be proved that *if a $\mathcal{C}_1$.-contraction has no nontrivial invariant subspace, then it is a quasiaffine transform of its own unitary extension, which is reductive and has an invariant dense and totally cyclic linear manifold.* This paper presents a brief review, based on [7] and [9], on the equivalence between the above conjectures.

1. Preliminaries

Throughout the paper $\mathcal{H}$ and $\mathcal{K}$ stand for infinite-dimensional complex separable Hilbert spaces, and $\mathcal{B}[\mathcal{H},\mathcal{K}]$ stands for the Banach space of all bounded linear transformations of $\mathcal{H}$ into $\mathcal{K}$. Let $\mathcal{N}(X) \subseteq \mathcal{H}$ and $\mathcal{R}(X) \subseteq \mathcal{K}$ denote the null space (i.e. the kernel) and range of $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, respectively. Set $\mathcal{B}[\mathcal{H}] = \mathcal{B}[\mathcal{H},\mathcal{H}]$ for short. If T lies in $\mathcal{B}[\mathcal{H}]$, then we say that T is an operator on $\mathcal{H}$, and T^* in $\mathcal{B}[\mathcal{H}]$ denotes the adjoint of T. By a subspace of $\mathcal{H}$ we mean a closed linear manifold of $\mathcal{H}$, so that the closure $\mathcal{R}^-$ of a linear manifold $\mathcal{R}$ of $\mathcal{H}$ is a subspace of $\mathcal{H}$. A set $\mathcal{S} \subseteq \mathcal{H}$ is invariant for an operator T on $\mathcal{H}$ if $T(\mathcal{S}) \subseteq \mathcal{S}$. A subspace $\mathcal{M}$ of $\mathcal{H}$ is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$. $\mathcal{M}$ is a reducing subspace for $T \in \mathcal{B}[\mathcal{H}]$ if it is invariant for both T and T^* (equivalently, if both $\mathcal{M}$ and its orthogonal complement, $\mathcal{M}^\perp$, are invariant for T).

Key words: *$\mathcal{C}_1$.-contractions, quasiaffine transforms, invariant subspaces.*
AMS Subject Classification: Primary 47A15; Secondary 47A45.

An operator is reducible if it has a nontrivial reducing subspace, and reductive if all its invariant subspaces are reducing.

An operator T on $\mathcal{H}$ is strongly stable (notation: $T^n \xrightarrow{s} O$) if the power sequence $\{T^n\}_{n\geq 0}$ converges strongly to the null operator (i.e. if $T^n x \to 0$ as $n\to\infty$ for every $x\in\mathcal{H}$). By a contraction we mean an operator T such that $\|T\|\leq 1$. As usual (cf. [11, p.72]), a contraction T is of class $\mathcal{C}_{0\cdot}$ if it is strongly stable, and of class $\mathcal{C}_{\cdot 0}$ if its adjoint T^* is strongly stable. Let $\mathcal{C}_{1\cdot}$ and $\mathcal{C}_{\cdot 1}$ be the classes of all contractions for which $T^n x \not\to 0$ and $T^{*n}x\not\to 0$, respectively, for every nonzero x in $\mathcal{H}$.

If T is a contraction on $\mathcal{H}$, then $\{T^{*n}T^n\}_{n\geq 0}$ is a monotone bounded sequence of self-adjoint operators (in fact a nonincreasing sequence of nonnegative contractions) so that it converges strongly. Since T^* is a contraction whenever T is, the sequence $\{T^nT^{*n}\}_{n\geq 0}$ also converges strongly. Hence, associated with each contraction T on $\mathcal{H}$, there exist operators A and A_* on $\mathcal{H}$ which are the strong limits of $\{T^{*n}T^n\}_{n\geq 0}$ and $\{T^nT^{*n}\}_{n\geq 0}$, respectively. That is,

$$T^{*n}T^n \xrightarrow{s} A \quad\text{and}\quad T^nT^{*n} \xrightarrow{s} A_*.$$

A few well-known properties of these strong limits, that will be required in the sequel, are displayed below (see e.g. [5] and [8, Ch.3]).

$$O\leq A\leq I \quad\text{and}\quad \|A\|=1 \text{ whenever } A\neq O, \tag{1}$$

$$T^*AT = A \quad (\text{i.e. } \|A^{\frac{1}{2}}Tx\| = \|A^{\frac{1}{2}}x\| \text{ for every } x\in\mathcal{H}), \tag{2}$$

$$\mathcal{N}(A) = \big\{x\in\mathcal{H}:\ T^nx\to 0 \text{ as } n\to\infty\big\}, \tag{3}$$

$$\mathcal{N}(I-A) = \big\{x\in\mathcal{H}:\ \|T^nx\|=\|x\| \text{ for all } n\geq 0\big\}. \tag{4}$$

Clearly, replacing T with T^* (and vice versa) we get similar properties for A_*. Property (3) says that $T\in\mathcal{C}_{0\cdot}$ if and only if $A=O$ (i.e. if and only if $\mathcal{N}(A)=\mathcal{H}$), and $T\in\mathcal{C}_{1\cdot}$ if and only if $\mathcal{N}(A)=\{0\}$. Therefore,

$$\begin{aligned}
T\in\mathcal{C}_{00} &\iff A=A_*=O,\\
T\in\mathcal{C}_{01} &\iff A=O \text{ and } \mathcal{N}(A_*)=\{0\},\\
T\in\mathcal{C}_{10} &\iff \mathcal{N}(A)=\{0\} \text{ and } A_*=O,\\
T\in\mathcal{C}_{11} &\iff \mathcal{N}(A)=\mathcal{N}(A_*)=\{0\}.
\end{aligned}$$

Now consider a linear transformation $V_0\colon\mathcal{R}(A^{\frac{1}{2}})\to\mathcal{R}(A^{\frac{1}{2}})$ defined by $V_0A^{\frac{1}{2}}x = A^{\frac{1}{2}}Tx$ for every $x\in\mathcal{H}$. The definition of V_0 makes sense because A is self-adjoint (property (1)) so that $A|_{\mathcal{R}(A)}\colon\mathcal{R}(A)\to\mathcal{R}(A)$ is injective. Observe that V_0 acts isometrically on $\mathcal{R}(A^{\frac{1}{2}})$. This is ensured by property (2). Indeed, $\|V_0 A^{\frac{1}{2}}x\| = \|A^{\frac{1}{2}}Tx\| = \|A^{\frac{1}{2}}x\|$ for every $x \in \mathcal{H}$. Extend V_0 over $\mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^-$ and get the isometry

$$V\colon \mathcal{R}(A)^- \to \mathcal{R}(A)^-.$$

Thus each contraction T on $\mathcal{H}$ also induces an isometry V on $\mathcal{R}(A)^-$ such that (see e.g. [5] and [8, Ch.3]).

(5) $\quad A^{\frac{1}{2}}T = VA^{\frac{1}{2}} \quad$ (i.e. if $A \neq O$, then $A^{\frac{1}{2}}$ intertwines T to V).

Recall that an operator $T \in \mathcal{B}[\mathcal{H}]$ is intertwined to an operator $L \in \mathcal{B}[\mathcal{K}]$ if there exists a nonzero transformation $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ such that $XT = LX$. In such a case we say that X intertwines T to L. If $\mathcal{R}(X)^- = \mathcal{K}$, then T is said to be densely intertwined to L. A transformation $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is quasiinvertible if it is injective and has a dense range (i.e. if $\mathcal{N}(X) = \{0\}$ and $\mathcal{R}(X)^- = \mathcal{K}$). $T \in \mathcal{B}[\mathcal{H}]$ is a quasiaffine transform of $L \in \mathcal{B}[\mathcal{K}]$ if there exists a quasiinvertible $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ intertwining T to L.

Suppose T is a $\mathcal{C}_1.$-contraction so that $\mathcal{N}(A) = \{0\}$ or, equivalently, $\mathcal{R}(A)^- = \mathcal{H}$ (reason: A is self-adjoint). In this case we can define a new inner product on $\mathcal{H}$, say $\langle\ ;\ \rangle_A\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, given by

$$\langle x; y\rangle_A = \langle Ax; y\rangle$$

for every $x, y \in \mathcal{H}$, which induces the norm $\|\ \|_A$ on $\mathcal{H}$ such that

$$0 < \|A^{\frac{1}{2}}x\| = \|x\|_A \leq \|x\|$$

for every nonzero vector x in $\mathcal{H}$. Consider the linear transformation $X_0\colon (\mathcal{H}, \langle\ ;\ \rangle_A) \to (\mathcal{R}(A^{\frac{1}{2}}), \langle\ ;\ \rangle)$ defined by $X_0 x = A^{\frac{1}{2}}x$ for every $x \in \mathcal{H}$. Let $\mathcal{H}_A$ be the completion of $(\mathcal{H}, \langle\ ;\ \rangle_A)$ and extend X_0 over the Hilbert space $\mathcal{H}_A$ to get the surjective isometry (i.e. the unitary operator)

$$X_A\colon \mathcal{H}_A \to \mathcal{H}.$$

In fact, $\mathcal{R}(X_A) = \mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^- = \mathcal{H}$ and $\|X_A x\| = \|A^{\frac{1}{2}}x\| = \|x\|_A$ for every $x \in \mathcal{H}$. Note that T acts isometrically on $(\mathcal{H}, \langle\ ;\ \rangle_A)$. Actually, by property (2),

$$\|Tx\|_A = \|A^{\frac{1}{2}}Tx\| = \|A^{\frac{1}{2}}x\| = \|x\|_A$$

for every $x \in \mathcal{H}$. Extend T over $\mathcal{H}_A$ and get the isometry

$$T_A \colon \mathcal{H}_A \to \mathcal{H}_A$$

such that $T_A x = Tx$ for each $x \in \mathcal{H}$. The unitary operator $X_A \colon \mathcal{H}_A \to \mathcal{H}$ intertwines the isometry $T_A \colon \mathcal{H}_A \to \mathcal{H}_A$ to the isometry $V \colon \mathcal{H} \to \mathcal{H}$, so that T_A and V are unitarily equivalent. Indeed, property (5) leads to

$$X_A T_A x \;=\; A^{\frac{1}{2}} T x \;= V A^{\frac{1}{2}} x \;=\; V X_A x$$

for every $x \in \mathcal{H}$. Let U be the minimal unitary extension of the isometry V. In light of the above unitary equivalence between the isometries T_A and V it is usual to refer to U as *the unitary extension of* T.

2. Introduction

Does every operator on a (separable) Hilbert space of dimension greater than one have a nontrivial invariant subspace? This is perhaps the most celebrated open question in operator theory. Its relevance is linked to the Spectral Theorem (normal operators do have nontrivial invariant subspaces) and also to the canonical Jordan form (every operator on a finite-dimensional space has a nontrivial invariant subspace). This makes the search for nontrivial invariant subspaces a natural, although challenging and recalcitrant one. The collection of all invariant subspaces for a given operator T is invariant under scalar multiplication. Therefore, if $T \neq O$ and $0 < \alpha \leq \|T\|^{-1}$, then T and αT (which is a contraction) share exactly the same lattice of invariant subspaces. Thus the invariant subspace problem is reduced to the class of all contractions: does every contraction have a nontrivial invariant subspace?

Isometries comprise a rather special class of $\mathcal{C}_{1\cdot}$-contractions. They are those contractions T on $\mathcal{H}$ for which $\|Tx\| = \|x\|$ for every $x \in \mathcal{H}$. Equivalently, those contractions T for which $A = I$ (see property (4), or simply recall that T is an isometry if and only if $T^{*n}T^n = I$ for all $n \geq 0$). It is easy to show that every isometry has a nontrivial invariant subspace. Actually, the next proposition says much more than this.

PROPOSITION 1. *If a contraction T on $\mathcal{H}$ has no nontrivial invariant subspace, then it is either a $\mathcal{C}_{00}$-contraction, a $\mathcal{C}_{01}$-contraction such that $\|A_* x\| < \|x\|$ for every nonzero x in $\mathcal{H}$, or a $\mathcal{C}_{10}$-contraction such that $\|Ax\| < \|x\|$ for every nonzero x in $\mathcal{H}$.*

PROOF. See [8, Ch.5]. □

Since a unitary operator is precisely a normal isometry, it follows that the unitaries comprise a set of particularly well-known operators. There are, however, some interesting open questions even for this special set of Hilbert-space operators. Sample: is a unitary operator weakly stable if and only if it is a bilateral shift or a direct summand of a bilateral shift? (An operator T is weakly stable if the power sequence $\{T^n\}_{n\geq 0}$ converges weakly to the null operator).

Some invariant subspace problems for contractions are equivalent to open questions on unitary operators. A class of such equivalent problems was considered in [7] and [9]. The purpose of this paper is to present a brief and unified survey on the results of [7] and [9].

3. Equivalent Invariant Subspace Problems

A classical open question in operator theory reads as follows.

(Q_0) *Does a contraction not in $\mathcal{C}_{00}$ have a nontrivial invariant subspace*?

Observe that this question asks whether the conclusion in Proposition 1 can be sharpened to $T \in \mathcal{C}_{00}$. That is, whether a contraction without a nontrivial invariant subspace is of class $\mathcal{C}_{00}$. In other words, whether a contraction without a nontrivial invariant subspace is strongly stable with a strongly stable adjoint. Recall that T has a nontrivial invariant subspace if and only if T^* has. Thus Proposition 1 leads to the following reformulation of question Q_0.

(Q_1) *Does a $\mathcal{C}_{1\cdot}$-contraction have a nontrivial invariant subspace*?

Another classical open question in operator theory is: does a quasiaffine transform of a normal operator have a nontrivial invariant subspace? (See e.g. [10, p.194].) A particular case of it, referring to contractions, reads as follows.

(Q_2) *Does a contraction, which is a quasiaffine transform of a unitary operator, have a nontrivial invariant subspace*?

Since a unitary operator is a normal isometry, and an isometry is a $\mathcal{C}_{1\cdot}$-contraction, questions (Q_1) and (Q_2) can be generalized as follows.

(Q_3) *Does a contraction, which is intertwined to a $\mathcal{C}_{1\cdot}$-contraction, have a nontrivial invariant subspace*?

It was shown in [7] that these questions are all pairwise equivalent, and hence the invariant subspace problems stated on them are reduced to the open question (Q_2) on unitary operators.

Theorem 1. *Every $\mathcal{C}_{1\cdot}$-contraction has a nontrivial invariant subspace if and only if every contraction which is a quasiaffine transform of a unitary operator has a nontrivial invariant subspace.*

4. Invariant Subspaces for $\mathcal{C}_{1\cdot}$-contractions

Every $\mathcal{C}_{1\cdot}$-contraction T is a quasiaffine transform of the isometry V, which turns out to be its unitary extension when T has no nontrivial invariant subspace. To prove this, we first state a couple of propositions whose proofs are based on standard results of single operator theory.

Proposition 2. *If T is a quasiaffine transform of a hyponormal operator L, then the spectrum of T includes the spectrum of L.*

Proof. See [3] — also see [4, p.94]. □

Recall that an operator L is hyponormal if $LL^* \leq L^*L$. Clearly, every isometry is a hyponormal contraction.

Proposition 3. *If an operator is densely intertwined to a nonunitary isometry, then it has a nontrivial invariant subspace.*

Proof. See e.g. [9] or [12]. □

The next lemma is a deep result whose proof is not elementary at all.

Lemma 1. *A contraction whose spectrum includes the unit circle has a nontrivial invariant subspace.*

Proof. See [2] — also see [1]. □

For a $\mathcal{C}_{1\cdot}$-contraction, the assumption on its spectrum in Lemma 1 can be replaced by the same assumption on the spectrum of its unitary extension.

Theorem 2. *Let T be a $\mathcal{C}_{1\cdot}$-contraction. The spectrum of T includes the spectrum of the isometry V. Moreover, if the spectrum of the unitary extension of T is the unit circle, then T has a nontrivial invariant subspace.*

Proof. Let T be a $\mathcal{C}_{1\cdot}$-contraction on $\mathcal{H}$. Since $\mathcal{N}(A^{\frac{1}{2}}) = \mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^- = \mathcal{H}$, it follows that $A^{\frac{1}{2}}$ is a quasiinvertible operator which, according to (5), intertwines T to the isometry V. Hence T is a quasiaffine transform of V. Thus, by Proposition 2, the spectrum of the isometry V is included in the spectrum of T: $\sigma(V) \subseteq \sigma(T)$. From now on suppose T has no nontrivial invariant subspace. In this case V is unitary (Proposition 3) so that V is the unitary extension of T. Moreover, Lemma 1 ensures that the unit circle Γ is not included in $\sigma(T)$. Therefore $\sigma(V) \neq \Gamma$, so that the spectrum of the unitary extension of T is not the whole unit circle. □

Recall: if a unitary operator U is nonreductive, then $\sigma(U) = \Gamma$. This shows that the next result, which is quite important in its own right and whose original proof in [6] precedes Lemma 1, may be obtained as an immediate corollary of Theorem 2.

Corollary 1. *If the unitary extension of a $\mathcal{C}_{1\cdot}$-contraction T is nonreductive, then T has a nontrivial invariant subspace.*

Let $\mathrm{Lat}(T)$ denote the lattice of all invariant subspaces for an arbitrary operator T on $\mathcal{H}$. The orbit of $x \in \mathcal{H}$ under T is the set $\{T^n x\}_{n\geq 0}$, whose (linear) span is the linear manifold $\mathrm{span}\{T^n x\}_{n\geq 0} = \{p(T)x\colon\, p \text{ is a polynomial}\}$ of $\mathcal{H}$. Its closure, $\bigvee\{T^n x\}_{n\geq 0}$, clearly lies in $\mathrm{Lat}(T)$. These are the cyclic subspaces in $\mathrm{Lat}(T)$: $\mathcal{M} \in \mathrm{Lat}(T)$ is cyclic if $\mathcal{M} = \bigvee\{T^n x\}_{n\geq 0}$ for some $x \in \mathcal{H}$. If $\bigvee\{T^n x\}_{n\geq 0} = \mathcal{H}$, then x is said to be a cyclic vector for T. We shall say that a linear manifold $\mathcal{R}$ of $\mathcal{H}$ is *totally cyclic* for T if every nonzero vector in $\mathcal{R}$ is cyclic for T. Recall that T has no nontrivial invariant subspace $\big($i.e. $\mathrm{Lat}(T) = \big\{\{0\}, \mathcal{H}\big\}\big)$ if and only if $\mathcal{H}$ is totally cyclic for T.

Proposition 4. *Suppose $T \in \mathcal{B}[\mathcal{H}]$ is densely intertwined to $L \in \mathcal{B}[\mathcal{K}]$. Let $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ be a transformation with dense range intertwining T to L. If $x \in \mathcal{H}$ is cyclic for T, then $Xx \in \mathcal{K}$ is cyclic for L. If a linear manifold $\mathcal{R}$ of $\mathcal{H}$ is totally cyclic for T, then $X(\mathcal{R}) \subseteq \mathcal{K}$ is totally cyclic for L.*

Proof. See [9]. □

Corollary 1 points the investigation to $\mathcal{C}_{1\cdot}$-contractions that have a reductive unitary extension, while Theorem 1 brings it back to question (Q_2).

Theorem 3. *If a $\mathcal{C}_{1\cdot}$-contraction has no nontrivial invariant subspace, then it is a quasiaffine transform of its unitary extension, which is reductive and has an invariant dense and totally cyclic linear manifold.*

Proof. Let T be a $\mathcal{C}_{1\cdot}$-contraction on $\mathcal{H}$. Observe by property (5) that $\mathcal{R}(A^{\frac{1}{2}})$ is invariant for V and that T is a quasiaffine transform of the isometry V (for $\mathcal{N}(A^{\frac{1}{2}}) = \mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^- = \mathcal{H}$ — in particular, T is densely intertwined to V). From now on suppose T has no nontrivial invariant subspace. Thus Proposition 3 ensures that V is unitary, and hence it is the unitary extension of T (which is reductive according to Corollary 1). Moreover, $\mathcal{H}$ is totally cyclic for T so that $\mathcal{R}(A^{\frac{1}{2}})$, which is dense in $\mathcal{H}$, is totally cyclic for V by Proposition 4. □

References

1. H. Bercovici, Notes on invariant subspaces, *Bull. Amer. Math. Soc.* **23** (1990) 1–36.
2. S.W. Brown, B. Chevreau and C. Pearcy, On the structure of contractions operators II, *J. Functional Anal.* **76** (1988) 30–55.
3. W.S. Clary, Equality of spectra of quasi-similar hyponormal operators, *Proc. Amer. Math. Soc.* **53** (1975) 88–90.
4. J.B. Conway, *The Theory of Subnormal Operators* (Mathematical Surveys and Monographs Vol. 36, Amer. Math. Soc., Providence, 1991).
5. E. Durszt, Contractions as restricted shifts, *Acta Sci. Math.* (*Szeged*) **48** (1985) 129–134.
6. L. Kérchy, Invariant subspaces of $C_{1\cdot}$-contractions with non-reductive unitary extensions, *Bull. London Math. Soc.* **19** (1987) 161–166.
7. C.S. Kubrusly, Equivalent invariant subspace problems, *J. Operator Theory* **38** (1997) 323–328.
8. C.S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory* (Birkhäuser, Boston, 1997).
9. C.S. Kubrusly, Invariant subspaces for a class of $C_{1\cdot}$-contractions, *Adv. Math. Sci. Appl.* **9** (1999) 129–135.
10. H. Radjavi and P. Rosenthal, *Invariant Subspaces* (Springer, New York, 1973).
11. B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Spaces* (North-Holland, Amsterdam, 1970).
12. K. Takahashi, The reflexivity of contractions with nonreductive *-residual parts, *Michigan Math. J.* **34** (1987) 153–159.

Catholic University of Rio de Janeiro
22543-900 Rio de Janeiro, RJ, Brazil
E-mail: *carlos@ele.puc-rio.br*

Supported in part by CNPq and FAPERJ