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INVARIANT SUBSPACES FOR A CLASS OF $C_{1\cdot}$-CONTRACTIONS *

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Abstract. Let U be the unitary extension of a $\mathcal{C}_{1\cdot}$-contraction T. It is given a simple new proof for the following theorem. *If U is nonreductive, then T has a nontrivial invariant subspace.* Moreover, it is also shown that, *if U is reductive but does not have an invariant dense linear manifold made up entirely of cyclic vectors, then T has a nontrivial invariant subspace.*

1. Introduction

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ will stand for infinite-dimensional complex separable Hilbert spaces, and $\mathcal{B}[\mathcal{H},\mathcal{K}]$ will stand for the Banach space of all bounded linear transformations from $\mathcal{H}$ into $\mathcal{K}$. Let $\mathcal{N}(X) \subseteq \mathcal{H}$ and $\mathcal{R}(X) \subseteq \mathcal{K}$ denote the null space (i.e. the kernel) and range of $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, respectively. Set $\mathcal{B}[\mathcal{H}] = \mathcal{B}[\mathcal{H},\mathcal{H}]$ for short. If $T \in \mathcal{B}[\mathcal{H}]$ we shall say that T is an operator on $\mathcal{H}$, and $T^* \in \mathcal{B}[\mathcal{H}]$ will stand for the adjoint of T. By a subspace we mean a closed linear manifold, so that the closure $\mathcal{R}^-$ of a linear manifold $\mathcal{R}$ is a subspace. A set $\mathcal{S} \subseteq \mathcal{H}$ is invariant for an operator T on $\mathcal{H}$ if $T(\mathcal{S}) \subseteq \mathcal{S}$. A subspace $\mathcal{M}$ of $\mathcal{H}$ is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$. $\mathcal{M}$ is a reducing subspace for $T \in \mathcal{B}[\mathcal{H}]$ if it is invariant for both T and T^* (equivalently, if both $\mathcal{M}$ and its orthogonal complement, $\mathcal{M}^\perp$, are invariant for T). An operator is reducible if it has a nontrivial reducing subspace, and reductive if all its invariant subspaces are reducing.

A contraction is an operator T such that $||T|| \leq 1$. If T is a contraction on a Hilbert space $\mathcal{H}$, then $\{T^{*n}T^n;\ n \geq 1\}$ is a monotone bounded sequence of self-adjoint operators (in fact a nonincreasing sequence of nonnegative contractions) so that it converges strongly. Hence, associated with each contraction T on $\mathcal{H}$, there exists an operator A on $\mathcal{H}$ which is the strong limit of $\{T^{*n}T^n;\ n \geq 1\}$. That is, such that

$$T^{*n}T^n \xrightarrow{\ s\ } A.$$

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A few well-known properties of the strong limit A, that will be required in the sequel, are displayed below (for these and further properties on the operator A see e.g. [6, 12, 13, 14] and [11, Ch.3]).

(1) $\quad O \leq A \leq I \quad$ (i.e. A is a nonnegative contraction),

(2) $\quad T^*AT = A \quad$ (i.e. $||A^{\frac{1}{2}}Tx|| = ||A^{\frac{1}{2}}x||$ for every $x \in \mathcal{H}$),

(3) $\quad \mathcal{N}(A) = \{x \in \mathcal{H} : \; T^n x \to 0 \;\text{ as }\; n \to \infty\}.$

As usual (cf. [15, p.72]), a contraction T is said to be of class $\mathcal{C}_{0\cdot}$ if it is strongly stable (i.e. if $T^n \xrightarrow{s} O$, which means that $T^n x \to 0$ as $n \to \infty$ for every $x \in \mathcal{H}$), and of class $\mathcal{C}_{\cdot 0}$ if its adjoint T^* is strongly stable. A contraction T is said to be of class $\mathcal{C}_{1\cdot}$ if $T^n x \not\to 0$ as $n \to \infty$ for every nonzero $x \in \mathcal{H}$, and of class $\mathcal{C}_{\cdot 1}$ if T^* is of class $\mathcal{C}_{1\cdot}$. All combinations are possible and this leads to classes $\mathcal{C}_{00}$, $\mathcal{C}_{01}$, $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$. Property (3) says that $T \in \mathcal{C}_{0\cdot}$ if and only if $A = O$ (i.e. if and only if $\mathcal{N}(A) = \mathcal{H}$), and $T \in \mathcal{C}_{1\cdot}$ if and only if $\mathcal{N}(A) = \{0\}$. Now consider the linear transformation $V_0 \colon \mathcal{R}(A^{\frac{1}{2}}) \to \mathcal{R}(A^{\frac{1}{2}})$ defined by $V_0 A^{\frac{1}{2}} x = A^{\frac{1}{2}} T x$ for every $x \in \mathcal{H}$. The definiton of V_0 makes sense because A is self-adjoint (cf. property (1)) so that $A|_{\mathcal{R}(A)} \colon \mathcal{R}(A) \to \mathcal{R}(A)$ is injective. V_0 acts isometrically on $\mathcal{R}(A^{\frac{1}{2}})$. This is ensured by property (2): $||V_0 A^{\frac{1}{2}} x|| = ||A^{\frac{1}{2}} T x|| = ||A^{\frac{1}{2}} x||$ for every $x \in \mathcal{H}$. Extend V_0 to $\mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^-$ and get the isometry

$$V \colon \mathcal{R}(A)^- \;\to\; \mathcal{R}(A)^-.$$

Therefore each contraction T on $\mathcal{H}$ also induces an isometry V on $\mathcal{R}(A)^-$ such that

(4) $\quad A^{\frac{1}{2}} T = V A^{\frac{1}{2}} \quad$ (i.e. if $A \neq O$, then $A^{\frac{1}{2}}$ intertwines T to V).

For further properties on the isometry V see e.g. [6, 14] and [11, Ch.3]. Recall that an operator $T \in \mathcal{B}[\mathcal{H}]$ is intertwined to an operator $L \in \mathcal{B}[\mathcal{K}]$ if there exists a nonzero transformation $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ such that $XT = LX$. In such a case we say that X intertwines T to L. If $\mathcal{R}(X)^- = \mathcal{K}$, then T is said to be densely intertwined to L.

Suppose T is a $\mathcal{C}_{1\cdot}$-contraction (i.e. $\mathcal{N}(A) = \{0\}$ or, equivalently, $\mathcal{R}(A)^- = \mathcal{H}$ - reason: A is self-adjoint). In this case we can define a new inner product in $\mathcal{H}$, say $\langle\cdot;\cdot\rangle_A \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, given by

$$\langle x; y\rangle_A \;=\; \langle Ax; y\rangle$$

for all $x, y \in \mathcal{H}$, which generates the norm $||\cdot||_A$ in $\mathcal{H}$ such that

$$0 < ||A^{\frac{1}{2}} x|| = ||x||_A \leq ||x||$$

for every nonzero x in $\mathcal{H}$. The norms $||\cdot||$ and $||\cdot||_A$ are equivalent if and only if A is invertible (i.e. if and only if $\mathcal{R}(A) = \mathcal{H}$). Recall that A is invertible if and only if T is similar to an isometry (see e.g. [7, p.56]). Now consider the linear transformation $X_0 \colon (\mathcal{H}, \langle\cdot;\cdot\rangle_A) \to (\mathcal{R}(A^{\frac{1}{2}}), \langle\cdot;\cdot\rangle)$ defined by $X_0 x = A^{\frac{1}{2}} x$ for all $x \in \mathcal{H}$. Let $\mathcal{H}_A$ be the completion of $(\mathcal{H}, \langle\cdot;\cdot\rangle_A)$ and extend X_0 to the Hilbert space $\mathcal{H}_A$ to get the surjective isometry

$$X_A\colon \mathcal{H}_A \;\to\; \mathcal{H}.$$

Indeed, $\mathcal{R}(X_A) = \mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^- = \mathcal{H}$ and $||X_A x|| = ||A^{\frac{1}{2}}x|| = ||x||_A$ for every $x \in \mathcal{H}$. Next note that T acts isometrically on $(\mathcal{H}, \langle\cdot;\cdot\rangle_A)$. Actually, by property (2),

$$||Tx||_A = ||A^{\frac{1}{2}}Tx|| = ||A^{\frac{1}{2}}x|| = ||x||_A$$

for every $x \in \mathcal{H}$. Extend it to $\mathcal{H}_A$ and get the isometry

$$T_A\colon \mathcal{H}_A \;\to\; \mathcal{H}_A$$

such that $T_A x = Tx$ for every $x \in \mathcal{H}$. The unitary operator $X_A\colon\mathcal{H}_A \to \mathcal{H}$ intertwines the isometry $T_A\colon\mathcal{H}_A \to \mathcal{H}_A$ to the isometry $V\colon\mathcal{H} \to \mathcal{H}$, so that T_A and V are unitarily equivalent. In fact, property (4) leads to

$$X_A T_A x = A^{\frac{1}{2}}Tx = VA^{\frac{1}{2}}x = VX_A x$$

for every $x \in \mathcal{H}$. Let U be the minimal unitary extension of the isometry V. In light of the above unitary equivalence between the isometries T_A and V, U will be called *the unitary extension of* T.

The purpose of this paper is twofold. First we present in Section 2 a new and (almost) elementary proof for an important result which appeared in [8]: *if T is a $\mathcal{C}_{1\cdot}$-contraction whose unitary extension is nonreductive, then T has a nontrivial invariant subspace.* This in fact will come out as a corollary to the following theorem. *If a $\mathcal{C}_{1\cdot}$-contraction has no nontrivial invariant subspace, then the isometry V becomes a unitary operator on $\mathcal{H}$ whose spectrum is not the whole unit circle.* The main tools used in the proof are well-known results which are reprised below for the reader's convenience.

LEMMA 1. *If an operator T is a quasiaffine transform of a hyponormal operator L, then the spectrum of T contains the spectrum of L.*

PROOF. See [3] - also see [4, p.94]. □

Recall that an operator L is hyponormal if $LL^* \leq L^*L$. Clearly every isometry is hyponormal. $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is quasiinvertible if it is injective and has dense range (i.e. if $\mathcal{N}(X) = \{0\}$ and $\mathcal{R}(X)^- = \mathcal{K}$). $T \in \mathcal{B}[\mathcal{H}]$ is a quasiaffine transform of $L \in \mathcal{B}[\mathcal{K}]$ if there exists a quasiinvertible $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ intertwining T to L.

LEMMA 2. *A contraction whose spectrum contains the unit circle has a nontrivial invariant subspace.*

PROOF. See [2] - also see [1]. □

By '(almost) elementary' we mean that, except for the deep result in Lemma 2 (whose proof is not elementary), all proofs in the present paper (as well as the proof of Lemma 1) use only standard results of single operator theory.

Next we discuss in Section 3 the case where the unitary extension of a $\mathcal{C}_{1\cdot}$-contraction is reductive: *if a $\mathcal{C}_{1\cdot}$-contraction has no nontrivial invariant subspace, then its* (reductive) *unitary extension has an invariant dense linear manifold made up entirely of cyclic vectors.* Of course, what really is behind all this is the conjecture that a contraction not in $\mathcal{C}_{00}$ has a nontrivial invariant subspace. (Recall: if a contraction has no nontrivial invariant subspace, then it is either a $\mathcal{C}_{00}$, a $\mathcal{C}_{01}$ or a $\mathcal{C}_{10}$-contraction - see e.g. [10]).

2. Nonreductive

For a $\mathcal{C}_{1\cdot}$-contraction, the assumption on its spectrum in Lemma 2 can be replaced by the same assumption on the spectrum of its unitary extension.

PROPOSITION 1. *If an operator is densely intertwined to a nonunitary isometry, then it has a nontrivial invariant subspace.*

PROOF. The von Neumann-Wold decomposition says that any isometry J in $\mathcal{B}[\mathcal{K}]$ can be decomposed as $J = S_+ \oplus U$, where S_+ is a unilateral shift and U is unitary. Thus an isometry is nonunitary if and only if it has a unilateral shift as a direct summand. Recall that the point spectrum of S_+^*, which is contained in the point spectrum of J^*, is the open unit disc. If an operator T in $\mathcal{B}[\mathcal{H}]$ is such that $XT = JX$ for some X in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ with dense range (i.e. with $\mathcal{N}(X^*) = \{0\}$), then the point spectrum of T^* contains the point spectrum of J^* which in turn contains the open unit disc. Therefore T has a large supply of nontrivial invariant subspaces. □

THEOREM 1. *Let T be a $\mathcal{C}_{1\cdot}$-contraction. The spectrum of T contains the spectrum of the isometry V. Moreover, if the spectrum of the unitary extension of T is the unit circle, then T has a nontrivial invariant subspace.*

PROOF. Let T be a $\mathcal{C}_{1\cdot}$-contraction on a Hilbert space $\mathcal{H}$. Since $\mathcal{N}(A^{\frac{1}{2}}) = \mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^- = \mathcal{H}$, $A^{\frac{1}{2}}$ is a quasiinvertible operator which, according to (4), intertwines T to the isometry V on $\mathcal{H}$. Hence T is a quasiaffine transform of V. Thus, by Lemma 1, the spectrum of the isometry V is contained in the spectrum of T: $\sigma(V) \subseteq \sigma(T)$. From now on suppose that T has no nontrivial invariant subspace. In this case V is unitary (according to Proposition 1) so that V is the unitary extension of T. Moreover, Lemma 2 ensures that the unit circle $\mathit{\Gamma}$ is not contained in $\sigma(T)$. Therefore $\sigma(V) \neq \mathit{\Gamma}$, so that the spectrum of the unitary extension of T is not the whole unit circle. □

Recall that every unitary operator U can be decomposed as $U = S \oplus W$, where S is a bilateral shift and W is a reductive unitary operator (see e.g. [7, p.18]). Since a bilateral shift is nonreductive, a unitary operator is nonreductive if and only if it has a bilateral shift as a direct summand. Outcome: if a unitary U is nonreductive, then $\sigma(U) = \mathit{\Gamma}$ (reason:

$\Gamma = \sigma(S) \subseteq \sigma(U) \subseteq \Gamma$). The converse however does not hold: there exists a reductive unitary operator whose spectrum is the whole unit circle. The classical example comprises a unitary diagonal on ℓ^2_+. Set $W = diag(\gamma_k;\ k \geq 0)$ in $\mathcal{B}[\ell^2_+]$, where $\gamma_k = e^{2\pi i \alpha_k}$ for each k and $\{\alpha_k\}$ is a distinct enumeration of all rationals in $[0, 1)$. In such a case $\sigma(W) = \Gamma$ (for $\{\alpha_k\}$ is dense in $[0, 1]$) but the unitary W is reductive (see e.g. [5, p.243]). This shows that the assumptions in Theorem 1 and Corollary 1 below are not equivalent.

Corollary 1 [8]. *If the unitary extension of a $\mathcal{C}_{1\cdot}$-contraction T is nonreductive, then T has a nontrivial invariant subspace.*

3. Reductive

For any operator T on $\mathcal{H}$ let $\mathcal{L}at(T)$ denote the lattice of all invariant subspaces for T. The orbit of $x \in \mathcal{H}$ under T is the set $\{T^n x;\ n \geq 0\}$, whose span is the linear manifold $span\{T^n x\}_{n\geq 0} = \{p(T)x\colon p \text{ is a polynomial}\}$. Its closure, $\bigvee\{T^n x\}_{n\geq 0}$, clearly lies in $\mathcal{L}at(T)$. These are the cyclic subspaces in $\mathcal{L}at(T)$: $\mathcal{M} \in \mathcal{L}at(T)$ is cyclic if $\mathcal{M} = \bigvee\{T^n x\}_{n\geq 0}$ for some $x \in \mathcal{H}$. If $\bigvee\{T^n x\}_{n\geq 0} = \mathcal{H}$, then x is said to be a cyclic vector for T. We shall say that a linear manifold $\mathcal{R} \subseteq \mathcal{H}$ is *totally cyclic* for T if every nonzero vector in $\mathcal{R}$ is cyclic for T. Recall that T has no nontrivial invariant subspace (i.e. $\mathcal{L}at(T) = \{\{0\}, \mathcal{H}\}$) if and only if $\mathcal{H}$ is totally cyclic for T.

Proposition 2. *Suppose $T \in \mathcal{B}[\mathcal{H}]$ is densely intertwined to $L \in \mathcal{B}[\mathcal{K}]$. Let $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ be a transformation with dense range intertwining T to L. If $x \in \mathcal{H}$ is cyclic for T, then $Xx \in \mathcal{K}$ is cyclic for L. In particular, if a linear manifold $\mathcal{R} \subseteq \mathcal{H}$ is totally cyclic for T, then $X(\mathcal{R}) \subseteq \mathcal{K}$ is totally cyclic for L.*

Proof. Since $XT = LX$ it follows that $Xp(T) = p(L)X$ for every polynomial p. Therefore, for an arbitrary nonzero x in $\mathcal{H}$, $X(span\{T^n x\}_{n\geq 0}) = span\{L^n Xx\}_{n\geq 0}$. Thus $X(\bigvee\{T^n x\}_{n\geq 0}) \subseteq \bigvee\{L^n Xx\}_{n\geq 0}$, where the inclusion is ensured by the continuity of X (recall: if $X\colon \mathcal{H} \to \mathcal{K}$ is continuous, then $X(\mathcal{S}^-) \subseteq X(\mathcal{S})^-$ for every set $\mathcal{S} \subseteq \mathcal{H}$). Now suppose x is cyclic for T so that $\bigvee\{T^n x\}_{n\geq 0} = \mathcal{H}$. Hence $X(\mathcal{H}) \subseteq \bigvee\{L^n Xx\}_{n\geq 0}$ and Xx is cyclic for L because X has dense range. □

Theorem 2. *Let T be a $\mathcal{C}_{1\cdot}$-contraction. If T has no nontrivial invariant subspace, then T is a quasiaffine transform of its unitary extension V. Moreover, V is reductive and $\mathcal{R}(A^{\frac{1}{2}})$ is an invariant dense totally cyclic linear manifold for V.*

Proof. Let T be a $\mathcal{C}_{1\cdot}$-contraction on a Hilbert space $\mathcal{H}$. According to property (4), $\mathcal{R}(A^{\frac{1}{2}})$ is invariant for V, and T is a quasiaffine transform of the isometry V on $\mathcal{H}$ (for $\mathcal{N}(A^{\frac{1}{2}}) = \mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^- = \mathcal{H}$ - in particular, T is densely intertwined to V). From now on suppose that T has no nontrivial invariant subspace. Thus Proposition 1 ensures that V is unitary, and hence it is the unitary extension of T (which is reductive according to Corollary 1). Moreover, $\mathcal{H}$ is totally cyclic for T so that $\mathcal{R}(A^{\frac{1}{2}})$, which is dense in $\mathcal{H}$, is totally cyclic for V by Proposition 2. □

Corollary 2. *If the unitary extension of a $\mathcal{C}_{1\cdot}$-contraction T does not have a totally cyclic dense linear manifold, then T has a nontrivial invariant subspace.*

It is worth noticing that the converse of Theorem 2 does not hold. Indeed, it has been exhibited in [9] a subnormal $\mathcal{C}_{10}$-contraction $T:\mathcal{H}\to\mathcal{H}$ such that $\mathcal{H}$, which is dense in $\mathcal{H}_A$, is totally cyclic for the isometry $T_A:\mathcal{H}_A\to\mathcal{H}_A$. Moreover, T_A also was shown to be unitarily equivalent to a reductive unitary operator. Therefore the isometry $V:\mathcal{H}\to\mathcal{H}$ (which is unitarily equivalent to T_A) becomes a reductive unitary and $X_A(\mathcal{H})=\mathcal{R}(A^{\frac{1}{2}})$, which is dense in $\mathcal{H}$, is totally cyclic for V (the unitary extension of T).

4. Coda

The above example can also be applied to settle another related problem. In order to pose it properly, we first introduce a couple of well-known results.

Claim 1. *Suppose $T\in\mathcal{B}[\mathcal{H}]$ is densely intertwined to a reducible operator $L\in\mathcal{B}[\mathcal{K}]$. Let $\mathcal{M}\subset\mathcal{K}$ be an arbitrary nontrivial reducing subspace for L, and let $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ be a transformation with dense range intertwining T to L. T has no nontrivial invariant subspace only if the linear manifold $\mathcal{R}(X)$, which is dense in $\mathcal{K}=\mathcal{M}\oplus\mathcal{M}^\perp$, does not intercept both $\mathcal{M}\backslash\{0\}$ and $\mathcal{M}^\perp\backslash\{0\}$.*

Proof. See [11, pp.62,63]. □

This (i.e. the necessary condition $\mathcal{R}(X)\cap\mathcal{M}=\mathcal{R}(X)\cap\mathcal{M}^\perp=\{0\}$) cannot happen if $\mathcal{M}$ or $\mathcal{M}^\perp$ is finite-dimensional.

Claim 2. *Let $\mathcal{R}$ be a dense linear manifold of $\mathcal{K}$, and let $\mathcal{M}$ be a nontrivial subspace of $\mathcal{K}$. If $\mathcal{M}$ or $\mathcal{M}^\perp$ is finite-dimensional, then $\mathcal{R}\cap\mathcal{M}^\perp\neq\{0\}$ or $\mathcal{R}\cap\mathcal{M}\neq\{0\}$.*

Proof. See [11, pp.63-65]. □

It may be tempting to think that Claim 2 might hold if $\mathcal{M}$ and $\mathcal{M}^\perp$ were both infinte-dimensional. In such a case Claim 1 would provide a positive answer to a classical open question, namely, *does every operator which is densely intertwined to a reducible operator have a nontrivial invariant subspace?* In particular, *does every quasiaffine transform of a normal operator have a nontrivial invariant subspace?* (See Theorem 2). However, by using the example borrowed from [9], the temptation quickly fades away: Claim 2 does not hold without the finite-dimensional assumption. Indeed, the above example shows that there exists a dense linear manifold $\mathcal{R}\subset\mathcal{K}$, which is invariant and totally cyclic for a unitary operator U on $\mathcal{K}$. Take any nontrivial reducing subspace $\mathcal{M}$ for U (whose existence is ensured by the spectral theorem). Thus $\mathcal{K}=\mathcal{M}\oplus\mathcal{M}^\perp$ and both $\mathcal{M}$ and $\mathcal{M}^\perp$ are nontrivial invariant subspaces for U. Since $\mathcal{R}$ is totally cyclic for U, it follows that $\mathcal{R}\cap\mathcal{M}=\{0\}$. In fact, if there existed $x\neq 0$ in $\mathcal{R}\cap\mathcal{M}$, then $\bigvee\{U^n x\}_{n\geq 0}\subseteq\mathcal{M}\neq\mathcal{K}$ so that $x\in\mathcal{R}$ would not be cyclic for U, and hence $\mathcal{R}$ would not be totally cyclic for U. Similarly, $\mathcal{R}\cap\mathcal{M}^\perp=\{0\}$.

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