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ON UNIFORM STABILITY*

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Abstract. Let T be an operator on a Hilbert space $\mathcal{H}$. If the sequence $\{\sum_{n=0}^{m}(\lambda T)^n;\ m \geq 0\}$ converges weakly for every λ in the unit circle, then T is uniformly stable. This leads to an elementary proof that uniform stability is equivalent to $\sum_{n=0}^{\infty}|\langle T^n x; y\rangle| < \infty$ for all $x, y \in \mathcal{H}$.

1. Introduction

Throughout the paper $\mathcal{H}$ will stand for a complex Hilbert space. By an operator on $\mathcal{H}$ we mean a bounded linear transformation from $\mathcal{H}$ into itself. Let $\mathcal{B}[\mathcal{H}]$ be the Banach algebra of all operators on $\mathcal{H}$. $r(T)$, $\sigma(T)$ and $\rho(T)$ will denote spectral radius, spectrum and resolvent set of $T \in \mathcal{B}[\mathcal{H}]$, respectively. As usual, the notation $T_n \xrightarrow{u} T$, $T_n \xrightarrow{s} T$ and $T_n \xrightarrow{w} T$ will mean that the sequence of operators $\{T_n \in \mathcal{B}[\mathcal{H}];\ n \geq 0\}$ converges to $T \in \mathcal{B}[\mathcal{H}]$ uniformly, strongly and weakly, respectively. By stability we mean power sequence convergence to the null operator. Thus an operator $T \in \mathcal{B}[\mathcal{H}]$ is said to be uniformly, strongly or weakly stable if $T^n \xrightarrow{u} O$, $T^n \xrightarrow{s} O$ or $T^n \xrightarrow{w} O$, respectively. The definitions of stability, as posed above, in fact refer to asymptotic stability for discrete linear systems.

The purpose of this paper is to present a new equivalent condition for uniform stability (cf. Lemma 1 and Corollary 1). Moreover, it will be shown that such an equivalent condition leads to an elementary proof for another one which has recently been established in [3] (cf. Corollary 2). Although the present paper is set up on a complex Hilbert space $\mathcal{H}$, the very same results do hold for a complex Banach space $\mathcal{X}$ as well (just replace the inner product $\langle x; y\rangle$ $x, y \in \mathcal{H}$ by the duality pair $[x; y]$ $x \in \mathcal{X}$ and $y \in \mathcal{X}^*$, the dual space of $\mathcal{X}$).

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2. Preliminaries

Uniform stability has been much investigated in the past, say, two decades. In fact there exists in current literature a huge collection of equivalent conditions for uniform stability. We shall give below a brief survey on uniform stability involving just a few of such equivalent conditions. They comprise a set of well-known results (see e.g. [1], [2] and the references therein) with a rather short and elementary proof which, as given below, will motivate the next result.

THEOREM 1. *Let $T \in \mathcal{B}[\mathcal{H}]$. The following assertions are equivalent.*

(a) $T^n \xrightarrow{u} O$ (i.e. $||T^n|| \to 0$ as $n \to \infty$).

(b) $r(T) < 1$.

(c) $||T^n|| \leq \beta\,\alpha^n$ *for every* $n \geq 0$, *for some* $\beta \geq 1$ *and* $\alpha \in (0,1)$.

(d) $\sum_{n=0}^{\infty} ||T^n||^p < \infty$ *for an arbitrary* $p > 0$.

(e) $\sum_{n=0}^{\infty} ||T^n x||^p < \infty$ *for all* $x \in \mathcal{H}$, *for an arbitrary* $p > 0$.

PROOF. Since $r(T)^n = r(T^n) \leq ||T^n||$ for every integer $n \geq 0$, it follows that (a) $\Rightarrow$ (b). The Gelfand-Beurling formula for the spectral radius $(r(T) = \lim_n ||T^n||^{1/n})$ ensures that, if (b) holds, then for any $\alpha \in (r(T), 1)$ there exists an integer $n_\alpha \geq 1$ such that $||T^n|| \leq \alpha^n$ for every $n \geq n_\alpha$. Thus (b) $\Rightarrow$ (c) with $\beta = \max_{0 \leq n \leq n_\alpha} ||T^n||\,\alpha^{-n_\alpha}$. It is trivially verified that (c) $\Rightarrow$ (d) $\Rightarrow$ (e) Finally note that $\sup_n ||T^n x|| < \infty$ for all $x \in \mathcal{H}$ whenever (e) holds true, and hence $\sup_n ||T^n|| < \infty$ by the Banach-Steinhaus Theorem. Also note that, for $m \geq 1$ and $p > 0$ arbitrary,

$$||m^{1/p} T^m x||^p = \sum_{n=0}^{m-1} ||T^{m-n} T^n x||^p \leq (\sup_n ||T^n||)^p \sum_{n=0}^{\infty} ||T^n x||^p.$$

Thus (e) implies that $\sup_m ||m^{1/p} T^m x|| < \infty$ for all $x \in \mathcal{H}$. Since $m^{1/p} T^m$ lies in $\mathcal{B}[\mathcal{H}]$ for each integer m, the Banach-Steinhaus Theorem ensures that $\sup_m ||m^{1/p} T^m|| < \infty$. Hence (e) $\Rightarrow$ (a). □

Another equivalent condition for uniform stability, which apparently might be thought of as a weak version of (e), has been recently established in [3]. It reads as follows.

(f) $\sum_{n=0}^{\infty} |\langle T^n x; y \rangle|^p < \infty$ *for all* $x, y \in \mathcal{H}$, *for an arbitrary* $p \geq 1$.

It has not been proved yet whether it holds for an arbitrary $p > 0$. That (e) $\Rightarrow$ (f) is trivial. On the other hand, the proof of (f) $\Rightarrow$ (a) does not mirror the proof of (e) $\Rightarrow$ (a) since the inequality $||T^{m-n} T^n x|| \leq ||T^{m-n}||\,||T^n x||$, which holds for norms, has no counterpart for absolute value of inner products. We shall not present a detailed proof of (f) $\Rightarrow$ (a). It is somewhat lengthy (for such a detailed proof the reader is referred to [3]). However the clever idea behind it, as proposed in [3], goes as follows. Suppose (f) holds true and, for each $y \in \mathcal{H}$, consider the map $W_y : \mathcal{H} \to \ell^p$ defined as $W_y(x) = (\langle T^0 x; y\rangle, \langle T^1 x; y\rangle, \langle T^2 x; y\rangle, \cdots)$ for all $x \in \mathcal{H}$, which clearly is linear. By using the Banach-Steinhaus Theorem it can be shown that it is bounded also. A second application of the Banach-Steinhaus Theorem ensures that

$$||W_y(x)||^p = \sum_{n=0}^{\infty} |\langle T^n x; y\rangle|^p \leq \omega^p ||x||^p ||y||^p$$

for all $x, y \in \mathcal{H}$, for some real constant $\omega > 1$. From now on the proof is split into two parts. One considering the case of $p = 1$ and the other the case of $p > 1$. Here we follow the case of $p > 1$ only, since we shall give below a simpler proof for the case of $p = 1$. Thus, by using the above inequality (with a little help from Hölder inequality) and noticing that (f) implies $r(T) \leq 1$ (for $T^n \xrightarrow{w} O \;\Rightarrow\; r(T) \leq 1$), it follows that

$$\begin{aligned} ||(\lambda I - T)^{-1}|| &= \sup_{||x||=||y||=1} |\langle(\lambda I - T)^{-1} x; y\rangle| \\ &= \sup_{||x||=||y||=1} \left|\left\langle \lim_{m\to\infty} \sum_{n=0}^{m} \lambda^{-(n+1)} T^n x; y \right\rangle\right| \\ &\leq \omega \left(|\lambda|^{p/(p-1)} - 1\right)^{-(p-1)/p} \end{aligned}$$

whenever $|\lambda| > 1$. Now, recalling that the distance of a point λ in the resolvent set to the spectrum $\sigma(T)$, $d(\lambda, \sigma(T))$, is an upper bound for $||(\lambda I - T)^{-1}||^{-1}$ (i.e. $||(\lambda I - T)^{-1}||^{-1} \leq d(\lambda, \sigma(T))$), it then follows that

$$\omega^{-1}(|\lambda|^{p/(p-1)} - 1)^{(p-1)/p} \leq d(\lambda, \sigma(T))$$

whenever $|\lambda| > 1$. Suppose $r(T) \neq 0$ (otherwise (b) holds tautologically). Since $\sigma(T)$ is closed, there exists $\mu \in \sigma(T)$ such that $|\mu| = r(T)$. Take $\gamma > 1$ arbitrary and set $\lambda = \gamma\mu/r(T)$. Note that $0 < r(T) \leq 1 < \gamma = |\lambda|$, and

$$d(\lambda, \sigma(T)) = \gamma - r(T).$$

This leads to an upper bound for the spectral radius: for every real number $\gamma > 1$,

$$r(T) \leq \gamma - \omega^{-1}(\gamma^{p/(p-1)} - 1)^{(p-1)/p}.$$

Thus, by setting $\gamma = (1 - \omega^{-p})^{-(p-1)/p}$ (which is greater than one),

$$r(T) \leq (1 - \omega^{-p})^{1/p} < 1.$$

Therefore (f) $\Rightarrow$ (b). Equivalently (cf. Theorem 1), (f) $\Rightarrow$ (a).

3. Main Results

In this section we shall give a new necessary and sufficient condition for uniform stability. This will lead to a rather elementary proof for the case of $p = 1$ that has been skipped in the previous section's discussion. Throughout the remaining text $\varGamma$ will denote the unit circle in the complex plane centered at the origin.

Lemma 1. *Let $T \in \mathcal{B}[\mathcal{H}]$. If $\{\sum_{n=0}^{m}(\lambda T)^n;\ m \geq 0\}$ converges weakly for every $\lambda \in \varGamma$, then $r(T) < 1$.*

Proof. First recall that, for each complex number $\lambda \neq 0$ and any operator $T \in \mathcal{B}[\mathcal{H}]$,

$$(\lambda I - T)\sum_{n=0}^{m}\lambda^{-(n+1)}T^n = \sum_{n=0}^{m}\lambda^{-(n+1)}T^n(\lambda I - T) = I - (\lambda^{-1}T)^{m+1}$$

for every integer $m \geq 0$. Suppose $\{\sum_{n=0}^{m}(\lambda T)^n;\ m \geq 0\}$ converges weakly, which means that there exists $L \in \mathcal{B}[\mathcal{H}]$ such that $\sum_{n=0}^{m}(\lambda T)^n \xrightarrow{w} L$ as $m \to \infty$. Thus $(\lambda^{-1}I - T)\lambda L = \lambda L(\lambda^{-1}I - T) = I$ (for $(\lambda T)^n \xrightarrow{w} O$) so that $(\lambda^{-1}I - T)^{-1} = \lambda L$, which lies in $\mathcal{B}[\mathcal{H}]$. Hence $\lambda^{-1} \in \rho(T)$. Therefore, if $\{\sum_{n=0}^{m}(\lambda T)^n;\ m \geq 0\}$ converges weakly for every $\lambda \in \varGamma$, then $\varGamma \subset \rho(T)$ and $T^n \xrightarrow{w} O$ (for $(\lambda T)^n \xrightarrow{w} O$ for every $\lambda \in \varGamma$). However $T^n \xrightarrow{w} O \Rightarrow r(T) \leq 1$ (reason: weak stability implies power boundedness which in turn implies $r(T) \leq 1$). Thus $r(T) < 1$ because $\varGamma \subset \rho(T)$. □

Corollary 1. *Let $T \in \mathcal{B}[\mathcal{H}]$. Each of the following assertions is equivalent to uniform stability.*

(g) $\{\sum_{n=0}^{m}(\lambda T)^n;\ m \geq 0\}$ *converges uniformly for an arbitrary* $\lambda \in \varGamma$.

(h) $\{\sum_{n=0}^{m}(\lambda T)^n;\ m \geq 0\}$ *converges strongly for every* $\lambda \in \varGamma$.

(i) $\{\sum_{n=0}^{m}(\lambda T)^n;\ m \geq 0\}$ *converges weakly for every* $\lambda \in \varGamma$.

Proof. Note that (h) $\Rightarrow$ (i), and (i) $\Rightarrow$ (b) by the previous lemma. On the other hand, $\varGamma \subset \rho(T)$ whenever (b) holds. Equivalently, (b) ensures the existence of $(\lambda I - T)^{-1} \in \mathcal{B}[\mathcal{H}]$ for every $\lambda \in \varGamma$. Thus

$$\sum_{n=0}^{m}(\lambda T)^n = \lambda^{-1}\sum_{n=0}^{m}(\lambda^{-1})^{-(n+1)}T^n = \lambda^{-1}(\lambda^{-1}I - T)^{-1}(I - (\lambda T)^{m+1})$$

for every $\lambda \in \varGamma$. Since (b) $\Rightarrow$ (a), it follows that $\sum_{n=0}^{m}(\lambda T)^n \xrightarrow{u} \lambda^{-1}(\lambda^{-1}I - T)^{-1} = (I - \lambda T)^{-1}$ as $m \to \infty$ for every $\lambda \in \varGamma$. Conclusion: (b) implies

(g′) $\{\sum_{n=0}^{m}(\lambda T)^n;\ m \geq 0\}$ *converges uniformly for every* $\lambda \in \varGamma$, which trivially leads to (h) and (g). Finally note that (g) $\Rightarrow$ (a). Equivalently (cf. Theorem 1), (g) $\Rightarrow$ (b). $\square$

Corollary 2. *Let $T \in \mathcal{B}[\mathcal{H}]$. T is uniformly stable if and only if*

(f′) $\sum_{n=0}^{\infty} |\langle T^n x; y\rangle| < \infty$ *for all $x, y \in \mathcal{H}$.*

Proof. Take $\lambda \in \Gamma$ and $x, y \in \mathcal{H}$ arbitrary. If assertion (f′) holds true, then the real sequence $\{\sum_{n=0}^{m} |\langle(\lambda T)^n x; y\rangle|;\; m \geq 0\}$ converges in $\mathbb{R}$, so that the complex sequence $\{\sum_{n=0}^{m}\langle(\lambda T)^n x; y\rangle;\; m \geq 0\}$ converges in $\mathbb{C}$ (recall: absolute convergence implies convergence for series in a Banach space). Therefore, since $\langle\sum_{n=0}^{m}(\lambda T)^n x; y\rangle = \sum_{n=0}^{m}\langle(\lambda T)^n x; y\rangle$, it follows that $\{\sum_{n=0}^{m}(\lambda T)^n;\; m \geq 0\}$ converges weakly for every $\lambda \in \Gamma$. Conclusion: (f′) implies (i). Note that assertion (e), with $p = 1$, trivially implies (f′). $\square$

4. Two Counterexamples

We close the paper with two remarks on the equivalent assertions of Corollary 1.

Remark 1. A weak version of assertion (h), namely

(j) $\{\sum_{n=0}^{m}(\lambda T)^n;\; m \geq 0\}$ *converges strongly for some $\lambda \in \Gamma$,*

does not imply uniform stability (and hence (j) does not imply (g)). For instance, the operator $T = -\mathrm{diag}(k/(k+1);\; k \geq 1)$ in $\mathcal{B}[\ell^2]$ is strongly stable (i.e. $T^n \xrightarrow{s} O$) and $1 \in \rho(T)$. Thus

$$\sum_{n=0}^{m} T^n = (I - T)^{-1}(I - T^{m+1}) \quad \xrightarrow{s} \quad (I - T)^{-1} \in \mathcal{B}[\ell^2].$$

However T is not uniformly stable. Indeed $r(T) = 1$ so that $(\lambda T)^n \overset{u}{\not\longrightarrow} O$, and hence $\{\sum_{n=0}^{m}(\lambda T)^n;\; m \geq 0\}$ does not converges uniformly, for every $\lambda \in \Gamma$.

Remark 2. Since strong convergence implies weak convergence, the above counterexample also ensures that the weak version of (i), viz.

(k) $\{\sum_{n=0}^{m}(\lambda T)^n;\; m \geq 0\}$ *converges weakly for some $\lambda \in \Gamma$,*

does not imply uniform stability. In fact, (k) does not even imply (j). For instance, let S be a bilateral shift on $\mathcal{H}$, and let S_1 be a direct summand of it for which $1 \in \rho(S_1)$ (so that there exists $(I - S_1)^{-1}$ continuous). Note that S_1 is weakly stable (i.e. $S_1^n \xrightarrow{w} O$ because $S^n \xrightarrow{w} O$). Thus

$$\sum_{n=0}^{m} S_1^n = (I - S_1)^{-1}(I - S_1^{m+1}) \quad \xrightarrow{w} \quad (I - S_1)^{-1}.$$

However, since S_1 is unitary, $(\lambda S_1)^n \overset{s}{\not\longrightarrow} O$ for every $\lambda \in \varGamma$ (indeed $||(\lambda S_1)^n x|| = ||x||$ for every $\lambda \in \varGamma$, every integer $n \geq 0$, and all $x \in \mathcal{H}$). Therefore $\{\sum_{n=0}^{m}(\lambda S_1)^n;\ m \geq 0\}$ does not converge strongly for every $\lambda \in \varGamma$.

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