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EQUIVALENT INVARIANT SUBSPACE PROBLEMS

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Abstract. An elementary proof for the equivalence of two open questions for Hilbert-space operators is established: every contraction that does not belong to the class $\mathcal{C}_{00}$ has a nontrivial invariant subspace if and only if every contraction which is a quasiaffine transform of a unitary operator has a nontrivial invariant subspace.

Key words: *Hilbert-space operators, invariant subspaces, $\mathcal{C}_{00}$-contractions, quasiaffine transforms.*

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1. INTRODUCTION

Throughout the paper $\mathcal{H}$ and $\mathcal{K}$ will stand for infinite-dimensional complex separable Hilbert spaces. Let $\mathcal{B}[\mathcal{H},\mathcal{K}]$ denote the Banach space of all bounded linear transformations from $\mathcal{H}$ into $\mathcal{K}$. Set $\mathcal{B}[\mathcal{H}] = \mathcal{B}[\mathcal{H},\mathcal{H}]$ for short. If $T \in \mathcal{B}[\mathcal{H}]$ we shall say that T is an operator on $\mathcal{H}$. An operator on $\mathcal{H}$ is scalar if it is a complex multiple of the identity on $\mathcal{H}$. By a subspace we mean a closed linear manifold, so that the closure $\mathcal{R}^-$ of a linear manifold $\mathcal{R}$ is a subspace. A subspace $\mathcal{M}$ of $\mathcal{H}$ is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$, and invariant for $T \in \mathcal{B}[\mathcal{H}]$ if $T(\mathcal{M}) \subseteq \mathcal{M}$. It is hyperinvariant for T if it is invariant for every operator on $\mathcal{H}$ that commutes with T. Let the subspace $\mathcal{N}(X) \subseteq \mathcal{H}$ denote the null space (i.e. the kernel) of $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, and let the linear manifold $\mathcal{R}(X) \subseteq \mathcal{K}$ denote the range of X. The following elementary result on invariant subspace will be needed in the sequel.

Proposition 0. *Let T and L be nonzero operators on a Hilbert space $\mathcal{H}$. If $LT = O$, then $\mathcal{N}(L)$ and $\mathcal{R}(T)^-$ are nontrivial invariant subspaces for both T and L, which are also nontrivial hyperinvariant subspaces for L and T, respectively.*

Proof. If $LT = O$, then $\mathcal{R}(T) \subseteq \mathcal{N}(L)$. Thus $T(\mathcal{N}(L)) \subseteq T(\mathcal{H}) = \mathcal{R}(T) \subseteq \mathcal{N}(L)$. Since $T \neq O$, it follows that $\mathcal{R}(T) \neq \{0\}$, and hence $\mathcal{N}(L) \neq \{0\}$. Since $L \neq O$, $\mathcal{N}(L) \neq \mathcal{H}$. Therefore $\mathcal{N}(L)$ is a nontrivial invariant subspace for T. Now take the adjoint. Since $T^*L^* = (LT)^* = O$, $L^* \neq O$ and $T^* \neq O$, it follows that $\mathcal{N}(T^*)$ is a nontrivial invariant subspace for L^*, and hence $\mathcal{R}(T)^- = \mathcal{N}(T^*)^\perp$ (the orthogonal complement of $\mathcal{N}(T^*)$) is a nontrivial invariant subspace for L. Finally recall that $\mathcal{N}(L)$ and $\mathcal{R}(T)^-$ are trivially hyperinvariant for L and T, respectively. □

Let T be an operator on $\mathcal{H}$. We shall say that T is strongly stable (denoted by $T^n \xrightarrow{s} O$) if the power sequence $\{T^n;\ n \geq 1\}$ converges strongly to the null operator (i.e. if $T^n x \to 0$ as $n \to \infty$ for all $x \in \mathcal{H}$). By a contraction we mean an operator T such that $\|T\| \leq 1$. A contraction T is of class $\mathcal{C}_{0\cdot}$ if it is strongly stable, and of class $\mathcal{C}_{\cdot 0}$ if its adjoint T^* is strongly stable. Let $\mathcal{C}_{1\cdot}$ and $\mathcal{C}_{\cdot 1}$ be the classes of all contractions for which $T^n x \not\to 0$ and $T^{*n} x \not\to 0$, respectively, for every nonzero $x \in \mathcal{H}$. A classical open question in operator theory is:

Question 1. *Does a contraction not in $\mathcal{C}_{00}$ have a nontrivial invariant subspace*?

An operator $T \in \mathcal{B}[\mathcal{H}]$ is interwined to an operator $L \in \mathcal{B}[\mathcal{K}]$ if there exists a nonzero $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ such that $XT = LX$. In such a case we say that X interwines T to L. $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is quasiinvertible if it is injective and has dense range (i.e. if $\mathcal{N}(X) = \{0\}$ and $\mathcal{R}(X)^- = \mathcal{K}$). $T \in \mathcal{B}[\mathcal{H}]$ is a quasiaffine transform of $L \in \mathcal{B}[\mathcal{K}]$ if there exists a quasiinvertible $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ interwining T to L.

Question 2. *Does a contraction, which is a quasiaffine transform of a unitary operator, have a nontrivial invariant subspace*?

The purpose of this paper is to show that the above questions are equivalent, and to give an elementary proof for such an equivalence. By 'elementary' we mean that all proofs in this paper use only standard results of single operator theory

2. PRELIMINARIES

Let T be a contraction on a Hilbert space $\mathcal{H}$. Thus $\{T^{*n}T^n;\ n \geq 1\}$ is a monotone bounded sequence of self-adjoint operators (in fact a nonincreasing sequence of nonnegative contractions) so that it converges strongly. Since T^* is a contraction whenever T is, the sequence $\{T^n T^{*n};\ n \geq 1\}$ also converges strongly. Hence, associated with each contraction T on $\mathcal{H}$, there exist operators A and A_* on $\mathcal{H}$ which are the strong limits of $\{T^{*n}T^n;\ n \geq 1\}$ and $\{T^nT^{*n};\ n \geq 1\}$, respectively. A few well-known properties of the strong limit A, that will be required in the sequel, are stated below (for these and further properties on the operator A see e.g. [2, 5-7]).

(1) $\quad O \leq A \leq I,$

(2) $\quad T^*AT = A,$

(3) $\quad \mathcal{N}(A) \ = \ \{x \in \mathcal{H} : \ T^n x \to 0 \ \text{ as } \ n \to \infty\}.$

Moreover, associated with each contraction T on $\mathcal{H}$ there also exists an isometry V on $\mathcal{R}(A)^-$ such that (see e.g. [2, 7 and 9])

(4) $\quad A^{\frac{1}{2}}T = VA^{\frac{1}{2}}.$

According to property (3), it follows that $T \in \mathcal{C}_{0\cdot}$ if and only if $A = O$ (i.e. if and only if $\mathcal{N}(A) = \mathcal{H}$), and $T \in \mathcal{C}_{1\cdot}$ if and only if $\mathcal{N}(A) = \{0\}$. Therefore

$$
\begin{array}{lcl}
T \in \mathcal{C}_{00} & \Longleftrightarrow & A = A_* = O, \\
T \in \mathcal{C}_{01} & \Longleftrightarrow & A = O \quad \text{and} \quad \mathcal{N}(A_*) = \{0\}, \\
T \in \mathcal{C}_{10} & \Longleftrightarrow & \mathcal{N}(A) = \{0\} \quad \text{and} \quad A_* = O, \\
T \in \mathcal{C}_{11} & \Longleftrightarrow & \mathcal{N}(A) = \mathcal{N}(A_*) = \{0\}.
\end{array}
$$

Proposition 1. *If a nonscalar contraction has no nontrivial hyperinvariant subspace* (*in particular, if a contraction has no nontrivial invariant subspace*)*, then it is either a* $\mathcal{C}_{00}$*, a* $\mathcal{C}_{01}$*, or a* $\mathcal{C}_{10}$*-contraction.*

Proof. Let T be a contraction on $\mathcal{H}$. Property (3) ensures that the subspace $\mathcal{N}(A)$ is hyperinvariant for T. If $\{0\} \neq \mathcal{N}(A) \neq \mathcal{H}$, then $\mathcal{N}(A)$ is a nontrivial hyperinvariant subspace for T. Dually, if $\{0\} \neq \mathcal{N}(A_*) \neq \mathcal{H}$, then the subspace $\mathcal{N}(A_*)$ is nontrivial and hyperinvariant for T^* so that $\mathcal{N}(A_*)^\perp$ is a nontrivial hyperinvariant subspace for T. Thus, if a nonscalar contraction T has no nontrivial hyperinvariant subspace, then there are only three possibilities (since $\mathcal{N}(A) = \mathcal{N}(A_*) = \{0\}$ leads to a $\mathcal{C}_{11}$-contraction, and nonscalar $\mathcal{C}_{11}$-contractions are quasisimilar to nonscalar unitary operators so that they do have a nontrivial hyperinvariant subspace whenever $\dim(\mathcal{H}) > 1$ - see e.g. [3, pp.103-105] or [10, pp.78-80]). The three remaining cases are: $\mathcal{N}(A) = \mathcal{N}(A_*) = \mathcal{H}$, $\mathcal{N}(A) = \mathcal{H}$ and $\mathcal{N}(A_*) = \{0\}$, or $\mathcal{N}(A) = \{0\}$ and $\mathcal{N}(A_*) = \mathcal{H}$. Equivalently, $T \in \mathcal{C}_{00}$, $T \in \mathcal{C}_{01}$, or $T \in \mathcal{C}_{10}$. □

Question 1 asks whether the conclusion in Proposition 1 can be sharpened to $T \in \mathcal{C}_{00}$ if the contraction T has no nontrivial invariant subspace (i.e. *whether a contraction without a nontrivial invariant subspace is of class* $\mathcal{C}_{00}$). Since T has a nontrivial invariant (hyperinvariant) subspace if and only if T^* has, Proposition 1 leads to the following reformulation of Question 1.

Question 1′. *Does a $\mathcal{C}_{1\cdot}$-contraction have a nontrivial invariant subspace*?

This in fact is equivalent to asking whether a contraction without a nontrivial invariant subspace is of class $\mathcal{C}_{0\cdot}$. In other words, *whether a contraction without a nontrivial invariant subspace is strongly stable.* Thus Question 1 can be further reformulated as follows.

Question 1″. *Does a contraction T for which $A \neq O$ have a nontrivial invariant subspace*?

Questions 1′ and 1″ have been considered, for instance, in [4] and [1], respectively.

Lemma . *If a nonscalar contraction T on $\mathcal{H}$ has no nontrivial hyperinvariant subspace* (*in particular, if a contraction T on $\mathcal{H}$ has no nontrivial invariant subspace*) *and $A \neq O$, then $\mathcal{R}(A)^- = \mathcal{H}$ and the isometry $V : \mathcal{H} \to \mathcal{H}$ is unitary.*

Proof. Suppose a nonscalar contraction T on $\mathcal{H}$ has no nontrivial hyperinvariant subspace. If $A \neq O$, then Proposition 1 ensures that $T \in \mathcal{C}_{10}$. Hence $\mathcal{N}(A) = \{0\}$. Equivalently, $\mathcal{R}(A)^- = \mathcal{H}$ (recall: A is self-adjoint according to (1)). Now consider the isometry V on $\mathcal{H}$ such that (4) holds. Thus, since V is an isometry (i.e. $V^*V = I$, the identity on $\mathcal{H}$), it follows by properties (2) and (4) that

$$T^*A^{\frac{1}{2}}A^{\frac{1}{2}}T = A^{\frac{1}{2}}A^{\frac{1}{2}} = A^{\frac{1}{2}}V^*VV^*VA^{\frac{1}{2}} = T^*A^{\frac{1}{2}}VV^*A^{\frac{1}{2}}T.$$

Thus $T^*A^{\frac{1}{2}}(I - VV^*)A^{\frac{1}{2}}T = O$. Since $O \leq I - VV^*$ (for V^* is a contraction), $\|(I - VV^*)^{\frac{1}{2}}A^{\frac{1}{2}}Tx\|^2 = \langle T^*A^{\frac{1}{2}}(I - VV^*)A^{\frac{1}{2}}Tx; x\rangle = 0$ for all $x \in \mathcal{H}$, so that $(I - VV^*)^{\frac{1}{2}}A^{\frac{1}{2}}T = O$. Therefore

$$(I - VV^*)A^{\frac{1}{2}}T = O.$$

Since the nonzero operator T has no nontrivial hyperinvariant subspace, Proposition 0 ensures that $(I - VV^*)A^{\frac{1}{2}} = O$. Hence $(I - VV^*) = O$ because $\mathcal{R}(A^{\frac{1}{2}})^- = \mathcal{R}(A)^- = \mathcal{H}$. Outcome: $VV^* = I$, which means that the isometry V is also a coisometry. Equivalently, V is unitary. □

3. CONCLUSION

Another classical open question in operator theory is: *does a quasiaffine transform of a normal operator have a nontrivial invariant subspace*? (See e.g. [8, p.194].) Question 2 is a particular case of it. Recall that a unitary operator is precisely a normal isometry, and that an isometry is a $\mathcal{C}_{1\cdot}$-contraction. Thus Questions 1′ and 2 can be generalized as follows.

QUESTION 3. *Does a contraction, which is interwined to $\mathcal{C}_{1\cdot}$-contraction, have a nontrivial invariant subspace?*

THEOREM . *Questions* 1, 2 *and* 3 *are pairwise equivalent.*

Proof. ∘ If Question 3 has a negative answer, then there exists a contraction T in $\mathcal{B}[\mathcal{H}]$ without a nontrivial invariant subspace, a $\mathcal{C}_{1\cdot}$-contraction U in $\mathcal{B}[\mathcal{K}]$, and a nonzero X in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ such that $XT = UX$. A trivial induction shows that $XT^n = U^n X$ for every integer $n \geq 1$. Since $\mathcal{R}(X) \neq \{0\}$, there exists a nonzero $x \in \mathcal{H}$ such that $Xx \neq 0$. Hence $XT^n x = U^n Xx \not\to 0$ as $n \to \infty$ (for $U \in \mathcal{C}_{1\cdot}$), so that $T^n x \not\to 0$ as $n \to \infty$. Therefore T is not strongly stable, which means that $A \neq O$. Summing up: T is a contraction without a nontrivial invariant subspace for which $A \neq O$. Thus Question 1″ has a negative answer. Equivalently, a positive answer to Question 1 leads to a positive answer to Question 3.

∘ A positive answer to Question 3 tautologically leads to a positive answer to Question 2.

∘ If Question 1″ has a negative answer, then there exists a contraction T on $\mathcal{H}$, with $A \neq O$, which has no nontrivial invariant subspace. Therefore the above lemma ensures that $\mathcal{R}(A)^- = \mathcal{H}$ (i.e. $\mathcal{N}(A) = \{0\}$, for A is self-adjoint) and $A^{\frac{1}{2}}T = VA^{\frac{1}{2}}$, where V is a unitary operator on $\mathcal{H}$. Hence T is a contraction, without a nontrivial invariant subspace, which is a quasiaffine transform of a unitary operator. Thus Question 2 has a negative answer. Equivalently, a positive answer to Question 2 leads to a positive answer to Question 1. □

REMARK . Note that the above theorem still holds if "contraction" is replaced by "nonscalar contraction" and "invariant subspace" is changed to "hyperinvariant subspace" in Questions 1, 2 and 3.

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