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A DECOMPOSITION FOR A CLASS OF CONTRACTIONS*

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Abstract. A Hilbert-space contraction T is the direct sum of a strongly stable contraction and an isometry whenever the strong limit of $\{T^{*n}T^n;\ n \geq 1\}$ is a projection. This is the case for any compact, quasinormal or cohyponormal contraction. Such a decomposition leads to a simple proof that a contraction with no proper invariant subspace is of class $\mathcal{C}_{00} \cup \mathcal{C}_{01} \cup \mathcal{C}_{10}$.

1. INTRODUCTION

Let T be an operator on a Hilbert space $\mathcal{H}$ (i.e. a bounded linear transformation of a separable complex Hilbert space $\mathcal{H}$ into itself). By a subspace of $\mathcal{H}$ we mean a closed linear manifold of it. $\mathcal{N}(T)$ will denote the null subspace (i.e. the kernel) of T. A strongly stable operator T is one whose power sequence $\{T^n;\ n \geq 1\}$ converges strongly to O (i.e. $T^n \xrightarrow{s} O$). Recall that a strongly stable operator is not necessarily a contraction (indeed not even necessarily similar to a contraction - cf. [4]). Let $\mathcal{C}_{00}$ be the class of all strongly stable contractions whose adjoint also is strongly stable. The class of all strongly stable contractions T for which the adjoint T^* is such that $T^{*n}x \not\to 0$ for every nonzero $x \in \mathcal{H}$ is denoted by $\mathcal{C}_{01}$. Similarly $\mathcal{C}_{10}$ denotes the class of all contractions T whose adjoint is strongly stable and $T^n x \not\to 0$ for every nonzero $x \in \mathcal{H}$. If T is a contraction (i.e. $||T|| \leq 1$) then $\{T^{*n}T^n;\ n \geq 1\}$ is a nonincreasing nonnegative sequence of contractions, thus strongly convergent. Let the operator A on $\mathcal{H}$ be its strong limit. Since T^* is a contraction whenever T is, let the operator A_* on $\mathcal{H}$ be the strong limit of $\{T^nT^{*n};\ n \geq 1\}$.

Such strong limits A and A_* have played an important role in the theory of completely nonunitary contractions (i.e. those contractions that have no unitary direct summand; equivalently, those contraction T for which $\mathcal{N}(I - A) \cap \mathcal{N}(I - A_*) = \{0\}$ – see e.g. [6]). For instance, these have been investigated by Durszt [3] and Pták and Vrbová [7] towards

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a generalization of Rota's model technique [10] (which in fact is applied to strongly stable contractions) to completely nonunitary contractions. The purpose of the present paper is to push forward such an investigation on the operators A and A_*. In particular, we shall establish a decomposition for contractions T for which A and A_* are projections. First we recall the following few well-known properties that will be required in the sequel (cf. [3], [6] and [7]).

(1) $O \leq A \leq I$.

(2) $||T^n x|| \to ||A^{\frac{1}{2}} x||$ as $n \to \infty$ for all $x \in \mathcal{H}$ (i.e. $T^{*n}T^n \xrightarrow{s} A \geq O$).

(3) $||A^{\frac{1}{2}} T^n x|| = ||A^{\frac{1}{2}} x||$ for all $x \in \mathcal{H}$ and every $n \geq 1$ (i.e. $T^{*n} A T^n = A \geq O$).

(4) $||AT^n x|| \to ||A^{\frac{1}{2}} x||$ as $n \to \infty$ for all $x \in \mathcal{H}$ (since $(I - A)T^n \xrightarrow{s} O$).

(5) $\mathcal{N}(A) = \{x \in \mathcal{H} : \quad T^n x \to 0\}$.

(6) $\mathcal{N}(I - A) = \{x \in \mathcal{H} : \quad ||T^n x|| = ||x|| \quad \forall n \geq 1\}$.

Properties (5) and (6) ensure that the subspaces $\mathcal{N}(A)$ and $\mathcal{N}(I-A)$ are invariant under T. Also note from (5) that a contraction T is strongly stable if and only if $A = O$, so that a contraction T is of class $\mathcal{C}_{00}$ if and only if $A = A_* = O$. Similarly, the classes $\mathcal{C}_{01}$ and $\mathcal{C}_{10}$ comprise those contractions T for which either $A = O$ and $\mathcal{N}(A_*) = \{0\}$, or $A_* = O$ and $\mathcal{N}(A) = \{0\}$, respectively. According to (1) A is idempotent (i.e. $A = A^2$) if and only if it is a projection (i.e. an orthogonal projection). Moreover it is readly verified from (2) and (3) that $||A|| = 1$ whenever $A \neq O$, which is a property of projections. However A may not be idempotent for an arbitrary contraction T (e.g. for $\mathcal{H} = \ell_2$ and $T = \text{shift}(k^{\frac{1}{2}}(k+2)^{\frac{1}{2}}/(k+1); \quad k \geq 1)$ it follows that $A = \text{diag}(k/k+1; \quad k \geq 1)$, so that $\mathcal{N}(A - A^2)$ may even be null). Indeed A will be a projection if and only if it commutes with T. Such an equivalence is in fact an immediate corollary to the following lemma.

Lemma 1. *$\mathcal{N}(A - A^2)$ is the largest subspace of $\mathcal{H}$ that is contained in $\mathcal{N}(AT - TA)$ and is invariant for T.*

Proof. Take an arbitrary contraction T. First note that

$$(||AT^n x|| - ||Ax||)^2 \leq ||AT^n x - T^n Ax||^2 \leq ||AT^n x||^2 - ||Ax||^2$$

for all $x \in \mathcal{H}$ and every $n \geq 1$ (the second inequality comes from (3)). Next recall that

$$||(A - A^2)^{\frac{1}{2}} x||^2 = ||A^{\frac{1}{2}} x||^2 - ||Ax||^2$$

and, by using property (3) again, for every $n \geq 1$

$$||Ax|| \leq ||AT^n x|| \leq ||A^{\frac{1}{2}} T^n x|| = ||A^{\frac{1}{2}} x||,$$

for all $x \in \mathcal{H}$. Since $\mathcal{N}(A - A^2) = \mathcal{N}((A - A^2)^{\frac{1}{2}})$ the above three results and property (4) ensure that

$$\mathcal{N}(A-A^2)=\{x\in\mathcal{H}: \quad \|A^{\frac{1}{2}}x\|=\|Ax\|\}=\{x\in\mathcal{H}: \quad \|AT^nx\|=\|Ax\| \;\; \forall n\geq 1\}$$
$$=\{x\in\mathcal{H}: \quad AT^nx=T^nAx \;\; \forall n\geq 1\} \;\;\subseteq\;\; \mathcal{N}(AT-TA);$$

and also that $\mathcal{N}(A-A^2)$ is invariant for T. Now let $\mathcal{M}$ be a subspace of $\mathcal{H}$, contained in $\mathcal{N}(AT-TA)$ and invariant for T. Take $x\in\mathcal{M}$ arbitrary. It is readily verified by induction that $T^nAx=AT^nx$ for every $n\geq 1$. Thus $x\in\mathcal{N}(A-A^2)$. Therefore $\mathcal{M}\subseteq\mathcal{N}(A-A^2)$. □

The next lemma exhibts an invariant subspace decomposition for the invariant subspace $\mathcal{N}(A-A^2)$. This will be our starting point for establishing a decomposition for contractions with $\mathcal{N}(A-A^2)=\mathcal{H}$.

Lemma 2. *$\mathcal{N}(A-A^2)=\mathcal{N}(A)\oplus\mathcal{N}(I-A)$.*

Proof. Since $\mathcal{N}(A)\cup\mathcal{N}(I-A)\subseteq\mathcal{N}(A-A^2)$, and since $\mathcal{N}(A)\perp\mathcal{N}(I-A)$ (for A is self-adjoint), it follows that $\mathcal{N}(A)\oplus\mathcal{N}(I-A)\subseteq\mathcal{N}(A-A^2)$. On the other hand, since $\mathcal{N}(A-A^2)$ is invariant under the self-adjoint adjoint operator A it reduces A. Thus $A=A_0\oplus A_1$, with $A_0=A|_{\mathcal{N}(A-A^2)}$ and $A_1=A|_{\mathcal{N}(A-A^2)^\perp}$. Note that A_0 is a projection on $\mathcal{N}(A-A^2)$ (for $O\leq A_0$ and $A_0=A_0^2$). Hence $\mathcal{N}(A-A^2)=\mathcal{N}(A_0)\oplus\mathcal{N}(A_0)^\perp=\mathcal{N}(A_0)\oplus\mathcal{N}(I-A_0)$. However $\mathcal{N}(A_0)\subseteq\mathcal{N}(A)$ and $\mathcal{N}(I-A_0)\subseteq\mathcal{N}(I-A)$, so that $\mathcal{N}(A_0)\oplus\mathcal{N}(I-A_0)\subseteq\mathcal{N}(A)\oplus\mathcal{N}(I-A)$. Therefore $\mathcal{N}(A-A^2)\subseteq\mathcal{N}(A)\oplus\mathcal{N}(I-A)$. □

2. A DECOMPOSITION FOR CONTRACTIONS WITH $A=A^2$

Recall that, for an arbitrary contraction T, the Nagy-Foias-Langer decomposition (cf. [12, p.9,52]) says that $T=C\oplus U$ where C is a completely nonunitary contraction and U is unitary. A particular case of it is obtained when T is an isometry leading to the von Neumann-Wold decomposition for isometries, which says that $T=S_+\oplus U$ where S_+ is a unilateral shift. We shall now focus on an intermediate situation. A contraction is an isometry if and only if $A=I$ (cf. property (3)). Let us relax this a little by asking only that A (and/or A_*) be a projection.

Let T be a contraction and let A and A_* be such that $T^{*n}T^n\stackrel{s}{\longrightarrow}A$ and $T^nT^{*n}\stackrel{s}{\longrightarrow}A_*$. The following implications are trivially verified. (i) If $T=G\oplus S_+\oplus U$, where G is a strongly stable contraction, S_+ is a unilateral shift and U is unitary, then $A=A^2$. Moreover, (ii) if $G=B\oplus S_-$, where B is a $\mathcal{C}_{00}$ contraction and S_- is a backward unilateral shift (i.e. the adjoint of a unilateral shift), then $A=A^2$ and $A_*=A_*^2$. Furthermore, (iii) if $T=B\oplus U$ then $A=A_*$. The next theorem and the following corollary establish the converses (it is of course understood that any of the direct summands in the decompositions below may be missing).

Theorem. *Let T be a contraction on $\mathcal{H}$. If $A=A^2$ then*

$$T=G\oplus V,$$

where $G := T|_{\mathcal{N}(A)}$ *is a strongly stable contraction and* $V := T|_{\mathcal{N}(I-A)}$ *is an isometry. Moreover, the von Neumann-Wold decomposition for the isometry* V*,*

$$V = S_+ \oplus U,$$

is such that the unilateral shift S_+ *is equal to* $T|_{\mathcal{N}(I-A)\cap\mathcal{N}(A_*)}$ *and the unitary operator* U *is equal to* $T|_{\mathcal{N}(I-A)\cap\mathcal{N}(I-A_*)}$*. Furthermore, if* $A = A^2$ *and* $A_* = A_*^2$ *then*

$$G = B \oplus S_-,$$

where $B := T|_{\mathcal{N}(A)\cap\mathcal{N}(A_*)}$ *is a* $\mathcal{C}_{00}$ *contraction and* $S_- := T|_{\mathcal{N}(A)\cap\mathcal{N}(I-A_*)}$ *is a backward unilateral shift.*

Proof. Suppose $\mathcal{H} \neq \{0\}$ to avoid trivialities. If $A = A^2$ then Lemma 2 ensures that $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{N}(I-A)$. Since $\mathcal{N}(A)$ and $\mathcal{N}(I-A)$ are clearly invariant under T (cf. (5) and (6)) they reduce T. Thus we get the (orthogonal direct sum) decomposition

$$T = G \oplus V$$

on $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{N}(I-A)$, where $G := T|_{\mathcal{N}(A)}$ is a strongly stable contraction and $V := T|_{\mathcal{N}(I-A)}$ is an isometry, according to (5) and (6), respectively. However, by Nagy-Foias-Langer decomposition for contractions (cf. [12, p.9,52]),

$$V = S_+ \oplus U$$

on $\mathcal{N}(I-A) = \mathcal{K}^\perp \oplus \mathcal{K}$, where $S_+ := V|_{\mathcal{K}^\perp}$ is a completely nonunitary isometry (which means a pure isometry, or equivalently a unilateral shift - see e.g. [2, p.15]) and $U := V|_{\mathcal{K}}$ is unitary; with

$$\begin{aligned}
\mathcal{K} :&= \{x \in \mathcal{N}(I-A) : \quad ||V^n x|| = ||V^{*n} x|| = ||x|| \qquad \forall n \geq 1\} \\
&= \{x \in \mathcal{N}(I-A) : \quad ||V^{*n} x|| = ||x|| \qquad \forall n \geq 1\} \\
&= \{x \in \mathcal{N}(I-A) : \quad ||T^{*n} x|| = ||x|| \qquad \forall n \geq 1\} \\
&= \mathcal{N}(I-A) \cap \mathcal{N}(I-A_*)
\end{aligned}$$

(since $T = G \oplus V$ on $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{N}(I-A)$ - cf. (6)). Thus $V|_{\mathcal{K}} = T|_{\mathcal{N}(I-A)\cap\mathcal{N}(I-A_*)}$. Now note that $\mathcal{K}^\perp = \mathcal{N}(I-A) \ominus \mathcal{K} \subseteq \mathcal{N}(A_*) \subseteq \mathcal{N}(A) \oplus \mathcal{K}^\perp$ (since $T^* = G^* \oplus S_+^* \oplus U^*$ on $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{K}^\perp \oplus \mathcal{K}$ and $S_+^{*n} \xrightarrow{s} O$ - cf. (5)). Hence $\mathcal{N}(I-A) \cap \mathcal{N}(A_*) \subseteq \mathcal{N}(I-A) \cap (\mathcal{N}(A) \oplus \mathcal{K}^\perp) = \mathcal{K}^\perp \subseteq \mathcal{N}(I-A) \cap \mathcal{N}(A_*)$. Therefore

$$\mathcal{K}^\perp = \mathcal{N}(I-A) \cap \mathcal{N}(A_*)$$

and hence $V|_{\mathcal{K}^\perp} = T|_{\mathcal{N}(I-A)\cap\mathcal{N}(A_*)}$. Next suppose $\mathcal{N}(A) \neq \{0\}$ (otherwise the remaing results are trivial) and let A'_* on $\mathcal{N}(A)$ be the strong limit of $\{G^n G^{*n};\ n \geq 1\}$ so that $A_* = A'_* \oplus O \oplus I$ and $(I - A_*) = (I - A'_*) \oplus I \oplus O$ on $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{K}^\perp \oplus \mathcal{K}$. Thus $\mathcal{N}(A_*) = \mathcal{N}(A'_*) \oplus \mathcal{K}^\perp \subseteq \mathcal{N}(A) \oplus \mathcal{K}^\perp$ so that

$$\mathcal{N}(A'_*) = \mathcal{N}(A_*) \cap \mathcal{N}(A),$$

and $\mathcal{N}(I-A_*)=\mathcal{N}(I-A'_*)\oplus\mathcal{K}\subseteq\mathcal{N}(A)\oplus\mathcal{K}$ so that

$$\mathcal{N}(I-A'_*)=\mathcal{N}(I-A_*)\cap\mathcal{N}(A).$$

If (in addition to $A=A^2$) $A_*=A_*^2$ then $A'_*=A'^2_*$. Hence $\mathcal{N}(A)=\mathcal{N}(A'_*)\oplus\mathcal{N}(I-A'_*)$ by Lemma 2. Since $\mathcal{N}(A'_*)$ and $\mathcal{N}(I-A'_*)$ are invariant under G^* (cf. (5) and (6)) they are reducing subspaces for G^*, and so are they for G. Thus we get the decomposition

$$G=B\oplus S_-$$

on $\mathcal{N}(A)=\mathcal{N}(A'_*)\oplus\mathcal{N}(I-A'_*)$, where $B:=G|_{\mathcal{N}(A'_*)}=T|_{\mathcal{N}(A_*)\cap\mathcal{N}(A)}$ and $S_-:=G|_{\mathcal{N}(I-A'_*)}=T|_{\mathcal{N}(I-A_*)\cap\mathcal{N}(A)}$, so that B is a $\mathcal{C}_{00}$ contraction and S_- is a strongly stable coisometry (cf. (5) and (6) once again). Thus S_- is a completely nonunitary coisometry, and hence its adjoint is a completely nonunitary isometry (i.e. a unilateral shift). □

Proposition. *If $A=A_*$ then $A=A^2$.*

Proof. If $A=A_*$ then (cf. property (3)) $A=T^{*n}AT^n=T^{*n}A_*T^n=T^{*n}T^nA_*T^{*n}T^n$ for every $n\geq 1$. Thus $A=A^3$ (since $T^{*n}T^n\xrightarrow{s}A$) so that $A^2=A^4$. Therefore A^2 is a projection (for $O\leq A^2$) and hence $A=(A^2)^{\frac{1}{2}}=(A^4)^{\frac{1}{2}}=A^2$. □

Corollary 1. *If $A=A_*$ then*

$$T=B\oplus U,$$

where $B:=T|_{\mathcal{N}(A)}$ is a $\mathcal{C}_{00}$ contraction and $U:=T|_{\mathcal{N}(I-A)}$ is unitary.

Proof. Apply the above proposition to the preceding theorem. □

3. EXAMPLES

In this section we shall exhibit some classes of contractions for which the strong limit A is a projection. These include compact, quasinormal and cohyponormal contractions.

Example 1. *If a contraction T is compact then $A=A_*$.*

Proof. If T is a contraction then $T=B\oplus U$, where B is a completely nonunitary contraction and U is unitary (this is the well-known Nagy-Foias-Langer decomposition for contractions already used in the theorem's proof). However a completely nonunitary contraction is weakly stable (i.e. $B^n\xrightarrow{w}O$ - see e.g. [5]). Therefore, if T is compact, then B is compact and weakly stable so that it is in fact uniformly stable (i.e. $B^n\xrightarrow{u}O$ - see e.g. [5]). Thus B is a $\mathcal{C}_{00}$ contraction and hence $A=A_*$. □

Recall that an operator T is quasinormal if it commutes with T^*T, subnormal if it has a normal extension (i.e. if it is the restriction of a normal operator to an invariant subspace), hyponormal if $TT^* \leq T^*T$, and normaloid if its spectral radius is equal to its norm (i.e. $r(T) = ||T||$). These classes are related as follows (see e.g. [2, ch.II]).

$$\text{Normal} \subset \text{Quasinormal} \subset \text{Subnormal} \subset \text{Hyponormal} \subset \text{Normaloid}.$$

Example 2. *If T is a hyponormal contraction then $A_*^2 = A_* \leq A$.*

Proof. Let T be a contraction. Applying Nagy-Foias-Langer decomposition once again we get $T = C \oplus U$, where C is a completely nonunitary contraction and U is unitary. Recall that the restriction of a hyponormal operator to an invariant subspace is hyponormal as well (see e.g. [2, p.47]). Thus C is a completely nonunitary hyponormal contraction and hence C^* is strongly stable (cf. [6], [9]). Equivalently $C^n C^{*n} \xrightarrow{s} O$. Therefore $T^n T^{*n} = C^n C^{*n} \oplus U^n U^{*n} \xrightarrow{s} O \oplus I = A_*$ so that $A_* = A_*^2$ (for a direct proof of such idempotency see [6]). Moreover $T^{*n} T^n = C^{*n} C^n \oplus U^{*n} U^n \xrightarrow{s} A' \oplus I = A$, where A' is the strong limit of $\{C^{*n} C^n;\ n \geq 1\}$, so that $O \leq A' \oplus O = A - A_*$. □

Example 3. *If T is a quasinormal contraction then $A_*^2 = A_* \leq A = A^2$.*

Proof. It is readily verified by induction that $TT^{*n}T^n = T^{*n}T^nT$ for every $n \geq 1$ whenever T is a quasinormal operator. Thus, if a quasinormal operator T is a contraction then $TA = AT$ by the very definition of A. Hence $A = A^2$ according to Lemma 1. Since a quasinormal operator is hyponormal, $A_*^2 = A_* \leq A$ by the previous example. □

Example 4. *If T is a normal contraction then $A = A_*$.*

Proof. Recall that $T^{*n}T^n = T^nT^{*n}$ for every $n \geq 1$ whenever T is a normal operator.□

It is worth noticing that hyponormality, quasinormality and normality in Examples 2 to 4 cannot be weakened to normaloidness, subnormality and quasinormality, respectively. That normality cannot be relaxed to quasinormality in Example 4 is trivial: a unilateral shift is a quasinormal contraction for which $A = I$ and $A_* = O$. To verify that quasinormality cannot be weakened to subnormality in Example 3 consider the following unilateral weighted shift on ℓ_2

$$T = \text{shift}\,(\alpha, 1, 1, \cdots), \qquad\qquad \alpha \in (0, 1).$$

This is a subnormal (see e.g. [2, p.58]) contraction for which A is a diagonal on ℓ_2,

$$A = \text{diag}\,(\alpha^2, 1, 1, \cdots),$$

so that $A \neq A^2$ (indeed $O = A_* = A_*^2 < A^2 \leq A$). Now by setting $L = T \oplus T^*$ we get a normaloid contraction such that $L^{*n}L^n \xrightarrow{s} A' = A \oplus O$ and $L^nL^{*n} \xrightarrow{s} A'_* = O \oplus A$, and hence $A' \neq A'^2$ and $A'_* \neq A'^2_*$. Thus hyponormality cannot be weakened to normaloidness in Example 2. The diagram below summarizes the above discussion.

$$
\begin{array}{ccccccccc}
(A_* = A) & \Rightarrow & (A_*^2 = A_* \leq A = A^2) & \Rightarrow & \cdots & \Rightarrow & (A_*^2 = A_* \leq A) & & \\
\Uparrow & \not\Nwarrow & \Uparrow & \not\Nwarrow & & & \Uparrow & \not\Nwarrow & \\
\text{Normal} & \Rightarrow & \text{Quasinormal} & \Rightarrow & \text{Subnormal} & \Rightarrow & \text{Hyponormal} & \Rightarrow & \text{Normaloid}
\end{array}
$$

Question. Is there a property involving the operators A_* and A that holds for any subnormal contraction but not for every hyponormal contraction? In other words, is there a property involving the operators A_* and A that fills the gap in the above diagram?

4. A FINAL REMARK

It has been established by Sz.-Nagy and Foias [11] that a contraction T with $A \neq O$ and $A_* \neq O$ has a proper invariant subspace. Apostol pointed out in [1] that it was unknown whether T has a proper invariant subspace if $A \neq O$; and proved that this was the case for a noncoquasitriangular contraction. By the same time Putnam [8] proved that this also was the case for a cohyponormal contraction; and later [9] he showed that a cohyponormal contraction with $A \neq O$ has in fact a unitary direct summand. Apparently it still remains unknown whether a contraction with no proper invariant subspace is of class $\mathcal{C}_{00}$. However the preceding theorem leads to the following classification that yields an elementary new proof for the above mentioned result of [11].

Corollary 2. *If a contraction has no proper invariant subspace then it is either a $\mathcal{C}_{00}$, a $\mathcal{C}_{01}$ or a $\mathcal{C}_{10}$ contraction.*

Proof. Let T be a contraction on $\mathcal{H}$. If $\{0\} \neq \mathcal{N}(A - A^2) \neq \mathcal{H}$, then $\mathcal{N}(A - A^2)$ is a proper invariant subspace for T (cf. Lemma 1). Similarly, if $\{0\} \neq \mathcal{N}(A_* - A_*^2) \neq \mathcal{H}$, then $\mathcal{N}(A_* - A_*^2)$ is a proper invariant subspace for T^* so that $\mathcal{N}(A_* - A_*^2)^\perp$ is a proper invariant subspace for T. Thus, if T has no proper invariant subspace, then there are only four admissible cases:

(i) $\mathcal{N}(A - A^2) = \{0\}$ and $\mathcal{N}(A_* - A_*^2) = \{0\}$,

(ii) $\mathcal{N}(A - A^2) = \{0\}$ and $\mathcal{N}(A_* - A_*^2) = \mathcal{H}$,

(iii) $\mathcal{N}(A - A^2) = \mathcal{H}$ and $\mathcal{N}(A_* - A_*^2) = \{0\}$,

(iv) $\mathcal{N}(A - A^2) = \mathcal{H}$ and $\mathcal{N}(A_* - A_*^2) = \mathcal{H}$.

Case (i) in fact is impossible. It actually leads to $\mathcal{N}(A) = \mathcal{N}(A_*) = \{0\}$ according to Lemma 2. Equivalently, T is a $\mathcal{C}_{11}$ contraction; and $\mathcal{C}_{11}$ contractions are quasi-similar to unitary operators so that they have proper invariant subspaces (cf. [12, pp.78,79]). If $\mathcal{N}(A_* - A_*^2) = \mathcal{H}$ then $T^* = G$, a strongly stable contraction, by the preceding theorem (reason: T^* has no proper invariant subspace and hence no shift as a direct summand).

If $\mathcal{N}(A - A^2) = \{0\}$ then Lemma 2 says that $\mathcal{N}(A) = \{0\}$. Therefore case (ii) leads to a $\mathcal{C}_{10}$ contraction. Similarly case (iii) leads to a $\mathcal{C}_{01}$ contraction. Finally, if $\mathcal{N}(A - A^2) = \mathcal{N}(A_* - A_*^2) = \mathcal{H}$, then $T = B \oplus S_- \oplus S_+ \oplus U$ by the previous theorem. Hence $T = B$, which is of class $\mathcal{C}_{00}$, since S_-, S_+ and U clearly have proper invariant subspaces. Thus case (iv) leads to a $\mathcal{C}_{00}$ contraction. □

REFERENCES

1. C. Apostol, Quasitriangularity in Hilbert space, *Indiana Univ. Math. J.* **22** (1973) 817–825.
2. J.B. Conway, *The Theory of Subnormal Operators* (American Mathematical Society, Providence, 1991).
3. E. Durszt, Contraction as restricted shifts, *Acta Sci. Math.* **48** (1985) 129–134.
4. C.S. Kubrusly, Strong stability does not imply similarity to a contraction, *Syst. Control Lett.* **14** (1990) 349–352.
5. C.S. Kubrusly, Similarity to contractions and weak stability, *Advan. Math. Sci. Appl.* **2** (1993) 335-343.
6. C.S. Kubrusly and P.C.M. Vieira, Strong stability for cohyponormal operators, *J. Operator Theory* **31** (1994) 123–127.
7. V. Pták and P. Vrbová, An abstract model for compressions, *Casopis Pest. Mate.* **113** (1988) 252–259.
8. C.R. Putnam, Resolvent vectors, invariant subspaces, and sets of zero capacity, *Math. Ann.* **205** (1973) 165–171.
9. C.R. Putnam, Hyponormal contractions and strong power convergence, *Pacific J. Math.* **57** (1975) 531–538.
10. G.-C. Rota, On models for linear operators, *Comm. Pure Appl. Math.* **13** (1960) 469–472.
11. B. Sz.-Nagy and C. Foias, Sur les contractions de l'espace de Hilbert IX. Factorisations de la fonction caracteristique. Sous-espaces invariants, *Acta Sci. Math.* **25** (1964) 283–316.
12. B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space* (North-Holland, Amsterdam, 1970).

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