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State Feedback H_∞-Control for Discrete-Time Infinite-Dimensional Stochastic Bilinear Systems *

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Abstract - In this paper we consider the class of infinite-dimensional discrete-time linear systems with multiplicative random disturbances (i.e. with states multiplied by a random sequence), also known as stochastic bilinear systems. We obtain necessary and sufficient conditions, in terms of an algebraic Riccati-like operator equation, for existence of a state-feedback controller that stabilizes the system and ensures that the influence of the additive disturbance on the output is smaller than some pre specified bound. In a deterministic framework this problem is equivalent to the H_∞-control problem in a state-space formulation. Our results, when specialized to the case with no multiplicative random disturbance, reduces to the ones known for the deterministic case. Due to the intrinsic probabilistic nature of the stochastic bilinear model (the multiplicative noise acting on the state of the system makes it a stochastic process, regardless the additive disturbance) a probabilistic framework for the aforementioned problem had to be developed, leading to a stochastic H_∞-control problem.

Key Words - discrete stochastic bilinear systems, infinite-dimensional systems, operator theory, H_∞-control.

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1. Introduction

A great deal of attention has been given over the past two decades to the analysis of linear systems containing multiplicative random disturbances (i.e. with states multiplied by a random sequence), also called stochastic bilinear systems, motivated, at least partly, by various areas of application. For example, population models, nuclear fission and heat transfer, immunology, etc ([15]). Several aspects regarding structural properties of such models, in discrete and continuous time, finite and infinite dimensional, have been investigated in current literature where fundamental questions as well as practical and theoretical motivations for considering such a special class of systems can be found (e.g. see [2],[7],[14],[15],[17], [22],[23],[25] - for further references see [10]).

During the past decade a great number of papers have been published on H_∞-(sub)optimal control, since the pioneering work by Zames [26]. An early account of the developments in H_∞- control theory can be found in [6]. Although the H_∞-control problem was originally formulated in the frequency domain, a great deal of attention has recently been given to time-domain methods based on algebraic Riccati equations ([1],[4],[18],[19]). The development of the dynamic-game theoretic approach to worst-case design, as an alternative to frequency-domain H_∞-techniques [19], provided a solution to the problem of disturbance attenuation for a broader class of systems. This approach has been used for H_∞-control in infinite horizon time-invariant linear systems of finite and infinite dimensions, finite horizon time-varying linear systems, and nonlinear systems ([1],[8],[9],[18],[19]). Thus the H_∞-control problem in its equivalent state-space formulation can be viewed as a minimax optimization problem where the controller is the minimizing player and the disturbance the maximizing player.

The problem we shall consider in this paper is to derive necessary and sufficient conditions for existence of a state feedback controller that stabilizes a discrete-time infinite-dimensional stochastic bilinear system and ensures that the influence of the additive disturbance on the output is smaller than some bound. If the model had no multiplicative noise, that is, if we considered the deterministic case, then this problem would be the usual H_∞-control problem for discrete-time linear systems in a state-space approach, and necessary and sufficient conditions for the aforementioned problem in terms of algebraic Riccati equations can be found, for instance, in [18]. Therefore, the reason for calling the problem considered here as a stochastic H_∞-control problem is twofold:

I) The multiplicative noise acting on the state of the system makes our problem intrinsically a probabilistic one, since the state of the system is a stochastic process, regardless the additive disturbance.

II) When restricted to the deterministic case, that is, if the model has no multiplicative random disturbance, our problem reduces to the usual H_∞-control problem in a state-space approach.

Of course, the stochastic problem posed above can only be defined in a state-space formulation, since it would not be possible to set a link between the state-space domain and frequency domain in this case.

We apply a game theoretic approach to analyze the underlying problem and obtain a solution in terms of an algebraic Riccati-like operator equation, which generalizes known results for the linear case (cf. [18]). Indeed, the algebraic Riccati-like operator presented in Section 5 contains some extra terms, which are function of the correlation (S) and expected value (s) of the multiplicative noise, not found in the deterministic case. When we specialize the result to the case with no multiplicative random disturbance ($s = 0$ and $S = 0$), these extra terms go to zero and the resulting algebraic Riccati-like operator reduces to the usual one for H_∞-control.

Due to the intrinsic probabilistic nature of the model state, additive stochastic inputs have to be considered. Indeed, the H_∞-control problem was originally defined for the linear case on a deterministic framework, since a worst (disturbance)-case controller was being designed and therefore there was nothing to gain in considering stochastic disturbances. In fact it has been shown for the linear case (cf. [1],[4],[18],[19]) that the "maximizing" disturbance of the minimax problem is in the form of a state-feedback, a result that will also hold for the bilinear stochastic case (see e.g., (8), Remark 2 and proof of Lemma 2 below). However, since the state of the bilinear model under consideration is a stochastic process, additive stochastic inputs are naturally included among the possible "maximizing" disturbance. This reasoning also explains the main difference between the results presented in this paper and those in [3], where an optimization criterion similar to the one in problem OP (Section 5 below) was considered but the additive input sequences were assumed to be zero-mean and independent of the past states, thus excluding state-feedback as possible "maximizing" additive input disturbance. These assumptions considerably simplified the problem and an exact solution to problem OP (instead of SOP) via the algebraic Riccati-like operator equation was obtained in [3].

The present work is organized in the following way. Section 2 presents the notation, borrowed from the discrete-time stochastic bilinear system literature. It is not necessarily standard, as far as the H_∞ literature is concerned, but necessary for pursuing the proposed approach. The model under consideration is described in Section 3, and some stability results that will be required in the sequel are presented in Section 4. The main theorem is stated in Section 5, where a necessary and sufficient condition for the solution to a stochastic H_∞-control problem is formulated in terms of a solution to an algebraic Riccati-like operator equation, and mirrors its linear counterpart (cf. [18]). Indeed the H_∞-control of discrete-time linear systems leads to some extra invertibility conditions not found in the continuous-time case and this is also the case for the discrete-time bilinear stochastic models through condition (i) of the Theorem (Section 5 below). Condition (ii) is the algebraic Riccati-like operator equation and condition

(iii) assures stability of the closed loop system as well as the strict inequality in problem SOP (see Proposition 3 and Remark 2 below). The proof of sufficiency is established in Section 6 while necessity is presented in Section 7. Our analysis relies on the properties of the operators $\mathcal{F}$ and $\mathcal{F}^{\#}$ (see Section 3) established in [3] and [13]. The structure of proofs is essentially based on the approach of [18],[19] but many techniques used in deterministic linear systems do not generalize to our case. In particular, the solutions to minimax problems in the proof of necessity require a result due to Yakubovich (Proposition 7 below) since, unlike the linear case, an explicit characterization of solutions to these problems are not easily obtained. Also a truncation of the minimax problem is required in the stochastic case. The construction of the probability spaces with the independence properties assumed in the paper is presented in the Appendix.

2. Notation

Let X and X' be Banach spaces and denote by $B[X, X']$ the Banach space of all bounded linear transformations of X into X'. For simplicity we set $B[X] = B[X, X]$ and denote by $G[X]$ the group of all invertible operators from $B[X]$. The norms in X, X' and the induced uniform norm in $B[X, X']$ will all be denoted by $\|\cdot\|$, and $r(\cdot)$ will stand for the spectral radius in the Banach algebra $B[X]$. For any nontrivial complex Hilbert space H_0 we shall denote by $\langle\cdot;\cdot\rangle$ the inner product in H_0 ($\langle\cdot;\cdot\rangle_{H_0}$ with norm $\|\cdot\|_{H_0}$ if H_0 is a probabilistic space) and an upper star * will stand for adjoint as usual. Let $B^+[H_0] = \{T \in B[H_0];\ T \geq 0\}$ be the weakly closed convex cone of all self-adjoint nonnegative operators in $B[H_0]$ and define $G^+[H_0] = B^+[H_0] \cap G[H_0]$. Let H_0 be separable and $B_\infty[H_0]$ the class of all compact operators from $B[H_0]$. If $T \in B_\infty[H_0]$, let $\{\lambda_k;\ k \geq 0\}$ be the nonincreasing nonnegative null sequence made up of all eigenvalues of $(T^*T)^{1/2} \in B_\infty[H_0]$, each of them counted according to its multiplicity and set $\|T\|_1 = \sum_{k=0}^{\infty} \lambda_k$. Let $B_1[H_0] = \{T \in B_\infty[H_0];\ \|T\|_1 < \infty\}$ denote the class of all nuclear operators from $B[H_0]$ and set $B_1^+[H_0] = B_1[H_0] \cap B^+[H_0]$. $\|\cdot\|_1$ is a norm in $B_1[H_0]$ and $(B_1[H_0], \|\cdot\|_1)$ is a Banach space. The trace of $T \in B_1[H_0]$ is defined as $\mathrm{tr}(T) = \sum_{k=0}^{\infty}\langle Te_k; e_k\rangle$, which does not depend on the choice of the orthonormal basis $\{e_k;\ k \geq 0\}$ for H_0. $|tr(T)| \leq tr(T^*T)^{1/2} = \|T\|_1$, so that $\mathrm{tr}(\cdot) : B_1[H_0] \to \mathbb{C}$ is a bounded linear functional. For $f, g \in H_0$, let $f \circ g \in B_1[H_0]$ be defined as $(f \circ g)h = \langle h; g\rangle f$ for all $h \in H_0$, so that $(f \circ f) \in B_1^+[H_0]$. Set $l_2(H_0) = \bigoplus_{k=0}^{\infty} H_0$, the direct sum of countably infinite copies of H_0, which is a Hilbert space made up of all sequences $\{x_k \in H_0,\ k \geq 0;\ \sum_{k=0}^{\infty}\|x_k\|^2 < \infty\}$. Let (Ω, Σ, μ) be a probability space, where Σ is a σ-field of subsets of a nonempty set Ω and μ a probability measure on Σ. Let $\mathcal{H}_0 = L_2(\Omega, \Sigma, \mu; H_0)$ denote the Hilbert space of all second order H_0-valued random variables with inner product given by $\langle x; y\rangle_{\mathcal{H}_0} = \mathcal{E}(\langle x; y\rangle)$ for all $x, y \in \mathcal{H}_0$ where $\mathcal{E}$ stands for the expectation of the underlying scalar valued random variables. Accordingly, the norm of $x \in \mathcal{H}_0$ is given by $\|x\|_{\mathcal{H}_0} = (\mathcal{E}(\|x\|^2))^{1/2}$. For any $x, y \in \mathcal{H}_0$, the expectation and correlation operators will be denoted by $Ex \in H_0$ and $E(x \circ y) \in B_1[H_0]$ respectively (cf. [12]). It is easy to verify that $E(x \circ x) \in B_1^+[H_0]$ and $\langle x; y\rangle_{\mathcal{H}_0} = \mathrm{tr}(E(x \circ y))$. Finally for any family $\{x_\iota \in \mathcal{H}_0;\ \iota \in \Phi \neq \emptyset\}$ set $\mathcal{I}_{\{x_\iota;\ \iota\in\Phi\}} = \{y \in \mathcal{H}_0;\ y$ is independent of $\{x_\iota \in \mathcal{H}_0;\ \iota \in \Phi\}\}$. In particular for any $x \in \mathcal{H}_0$, $\mathcal{I}_x = \{y \in \mathcal{H}_0;\ y$ is independent of $x\}$.

3. Description of the Model

Throughout this paper H, H', H'' and H''' will stand for separable complex Hilbert spaces. Set $\mathcal{H} = L_2(\Omega, \Sigma, \mu; H)$, $\mathcal{H}' = L_2(\Omega, \Sigma, \mu; H'')$, $\mathcal{H}'' = L_2(\Omega, \Sigma, \mu; H'')$ and $\mathcal{H}''' = L_2(\Omega, \Sigma, \mu; H''')$, where (Ω, Σ, μ) is the underlying probability space. We assume that $\{w_i \in \mathcal{H};\ i \geq 0\}$ is a stationary independent random sequence with expected value and correlation operator denoted by $s \in H$ and $S \in B_1^+[H]$ respectively, and set $C = (S - s \circ s) \in B_1^+[H]$. We assume that $\mathcal{X} \subset l_2(\mathcal{H})$, $\mathcal{X}_n \subset \bigoplus_{k=0}^n \mathcal{H}$, $\mathcal{V} \subset l_2(\mathcal{H}')$, $\mathcal{V}_n \subset \bigoplus_{k=0}^n \mathcal{H}'$, $\mathcal{U} \subset l_2(\mathcal{H}'')$, $\mathcal{U}_n \subset \bigoplus_{k=0}^n \mathcal{H}''$, $\mathcal{Z} \subset l_2(\mathcal{H}''')$, and $\mathcal{Z}_n \subset \bigoplus_{k=0}^n \mathcal{H}'''$ are Hilbert spaces with the following property. If $\mathbf{x} = (x_0, x_1, ...) \in \mathcal{X}$, $\mathbf{x}_n = (x_0, x_1, ..., x_n) \in \mathcal{X}_n$, $\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}$, $\mathbf{v}_n = (v_0, v_1, ..., v_n) \in \mathcal{V}_n$, $\mathbf{u} = (u_0, u_1, ...) \in \mathcal{U}$, $\mathbf{u}_n = (u_0, u_1, ..., u_n) \in \mathcal{U}_n$, $\mathbf{z} = (z_0, z_1, ...) \in \mathcal{Z}$, and $\mathbf{z}_n = (z_0, z_1, ..., z_n) \in \mathcal{Z}_n$; then $w_j \in \mathcal{I}_{\{x_0,...,x_i,v_0,...,v_i,,u_0,...,u_i,z_0,...,z_i,w_0,...,w_{i-1}\}}$ for all $j \geq i$. Notice that if $\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}$ then v_i may not be independent of past states x_k, $k \leq i$, and this is a crucial difference between the approach of this paper and the one in [3]. In the Appendix we show how one could construct a probability space (Ω, Σ, μ) and $\mathcal{X}$, $\mathcal{X}_n$, $\mathcal{V}$, $\mathcal{V}_n$, $\mathcal{U}$, $\mathcal{U}_n$, $\mathcal{Z}$, $\mathcal{Z}_n$, which lead to the above properties.

Consider a discrete time bilinear system in a stochastic environment, whose model is given by the following infinite-dimensional difference equation:

$$x_{i+1} = \left(A_0 + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle \right) x_i + Bu_i + Dv_i, \qquad x_0 \in \mathcal{X}_0 \tag{1}$$

where $\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}$, $\mathbf{u} = (u_0, u_1, ...) \in \mathcal{U}$, $A_0 \in B[H]$, $\{A_k \in B[H], k \geq 1\}$ is a bounded sequence, $D \in B[H', H]$, $B \in B[H'', H]$ and $\{e_k; k \geq 1\}$ is an orthonormal basis for H made up of the eigenvectors of $S \in B_1^+[H]$. Now suppose $u_i = -Kx_i$ for some $K \in B[H, H'']$. Since $\mathbf{x}_n = (x_0, ..., x_n) \in \mathcal{X}_n$ and $\mathbf{v}_n = (v_0, ..., v_n) \in \mathcal{V}_n$, it follows that

$$w_0 \in \mathcal{I}_{\{x_0,v_0\}} \text{ and } w_i \in \mathcal{I}_{\{x_0,v_0,...,v_i,w_0,...,w_{i-1}\}}.$$

Set $R_i = E(v_i \circ v_i) \in B_1^+[H']$ and $Q_i = E(x_i \circ x_i) \in B_1^+[H]$ for every $i \geq 0$. By a straightforward modification of Lemma 2 in [12] and the fact that $w_i \in \mathcal{I}_{\{x_i,v_i\}}$ we can show that the state correlation sequence evolve as follows.

$$Q_{i+1} = \mathcal{F}_{BK}(Q_i) + E(F_{BK}x_i \circ Dv_i) + E(F_{BK}x_i \circ Dv_i)^* + DR_iD^*. \quad (2)$$

Here $\mathcal{F}_{BK}$ and F_{BK} are operators in $B[B[H]]$ and $B[H]$, respectively, defined as

$$\mathcal{F}_{BK}(P) = F_{BK}PF_{BK}^* + \mathcal{T}(P), \qquad \forall P \in B[H],$$

$$F_{BK} = (A_0 - BK) + \sum_{k=1}^{\infty}\langle s; e_k\rangle A_k \in B[H],$$

with $\mathcal{T} \in B[B[H]]$ given by

$$\mathcal{T}(P) = \sum_{k,l=1}^{\infty}\langle Ce_k; e_l\rangle A_k P A_l^*, \qquad \forall P \in B[H].$$

Associated with $\mathcal{T}$ and $\mathcal{F}_{BK}$ set $\mathcal{T}^{\#} \in B[B[H]]$ and $\mathcal{F}_{BK}^{\#} \in B[B[H]]$ as follows: for all $P \in B[H]$,

$$\mathcal{T}^{\#}(P) = \sum_{k,l=1}^{\infty}\langle Ce_k; e_l\rangle A_l^* P A_k$$

$$\mathcal{F}_{BK}^{\#}(P) = F_{BK}^* P F_{BK} + \mathcal{T}^{\#}(P).$$

Set $\mathcal{F}^{\#} = \mathcal{F}_0^{\#}$, $\mathcal{F} = \mathcal{F}_0$, $F = F_0$, and $\Gamma_B = \{K \in B[H, H'']; \ r(\mathcal{F}_{BK}^{\#}) < 1\}$.

4. Some Stability Results

The following propositions will be required for proving the main results of the next sections.

Proposition 1: *Consider model* (1) *with* $B = 0$ *and* $D = 0$. *If* $\mathbf{x} = (x_0, x_1, ...) \in \mathcal{X}$ *for every* $x_0 \in \mathcal{X}_0$ *then* $r(\mathcal{F}^{\#}) < 1$.

Proof: This is a straightforward corollary to Lemma 2 in [13]. □

Proposition 2: *Consider model* (1) *with* $u_i = -Kx_i$ *for some* $K \in B[H, H'']$. *Then* $r(\mathcal{F}_{BK}^{\#}) < 1$ *if and only if* $\mathbf{x} = (x_0, x_1, ...) \in \mathcal{X}$ *for every* $\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}$ *and* $x_0 \in \mathcal{X}_0$.

Proof: Let us prove sufficiency. All we have to show is that $\mathbf{x} \in l_2(\mathcal{H})$ whenever $\mathbf{v} \in \mathcal{V} \subset l_2(\mathcal{H}')$ since $\mathbf{x}_n = (x_0, ..., x_n) \in \mathcal{X}_n$ for all $n \geq 0$, and therefore the independence condition required for $\mathbf{x}$ to belong to $\mathcal{X}$ will be satisfied (see Section 3). According to Lemma 1 in [12] it follows that, for $i \geq 1$

$$x_i = \left((\mathcal{A}_{BK})_{w_{i-1}} ... (\mathcal{A}_{BK})_{w_0}\right) x_0 + \sum_{j=1}^{i-1} \left((\mathcal{A}_{BK})_{w_{i-1}} ... (\mathcal{A}_{BK})_{w_j}\right) Dv_{j-1} + Dv_{i-1},$$

where

$$(\mathcal{A}_{BK})_{w_i} = (A_0 - BK) + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle.$$

Thus, by the triangle inequality in $\mathcal{H}$,

$$\|x_i\|_{\mathcal{H}} \leq \| \left((\mathcal{A}_{BK})_{w_{i-1}} ... (\mathcal{A}_{BK})_{w_0}\right) x_0 \|_{\mathcal{H}} + \sum_{j=1}^{i-1} \| \left((\mathcal{A}_{BK})_{w_{i-1}} ... (\mathcal{A}_{BK})_{w_j}\right) Dv_{j-1} \|_{\mathcal{H}} + \|Dv_{i-1}\|_{\mathcal{H}}. \tag{3}$$

Claim: $\| \left((\mathcal{A}_{BK})_{w_{i-1}} ... (\mathcal{A}_{BK})_{w_j}\right) Dv_{j-1} \|^2_{\mathcal{H}} \leq \|\mathcal{F}^{\#}_{BK}{}^{i-j}\| \, \|R_{j-1}\|_1 \|D\|^2,$

for every $1 \leq j \leq i$. Indeed, take $1 \leq j$ arbitrary and set $x'_0 = Dv_{j-1}$, $w'_{(i-j)} = w_i$ and $x'_{(i-j)+1} = (\mathcal{A}_{BK})_{w'_{(i-j)}} x'_{(i-j)}$ for every $i \geq j$, so that

$$x'_{(i-j)} = (\mathcal{A}_{BK})_{w'_{(i-j)-1}} ... (\mathcal{A}_{BK})_{w'_0} x'_0 = (\mathcal{A}_{BK})_{w_{i-1}} ... (\mathcal{A}_{BK})_{w_j} Dv_{j-1}$$

for every $i > j$. Since $w'_0 \in \mathcal{I}_{x'_0}$ and $w'_{(i-j)} \in \mathcal{I}_{\{x'_0, w'_0, ..., w'_{(i-j)-1}\}}$ for every $i > j$, it follows from Lemma 2 in [12] that $E(x'_{(i-j)+1} \circ x'_{(i-j)+1}) = \mathcal{F}_{BK}(E(x'_{(i-j)} \circ x'_{(i-j)}))$ for every $i \geq j$, so that

$$E(x'_{(i-j)} \circ x'_{(i-j)}) = \mathcal{F}_{BK}{}^{(i-j)}(E(x'_0 \circ x'_0)) = \mathcal{F}_{BK}{}^{(i-j)}(DR_{j-1}D^*),$$

and hence (cf. [13])

$$\|x'_{(i-j)}\|^2_{\mathcal{H}} = Vert E(x'_{(i-j)} \circ x'_{(i-j)})\|_1 \leq \|\mathcal{F}^{\#}_{BK}{}^{i-j}\| \, \|R_{j-1}\|_1 \|D\|^2$$

for every $i \geq j$, which proves the claimed result. Similarly we get

$$\| \left((\mathcal{A}_{BK})_{w_{i-1}} ... (\mathcal{A}_{BK})_{w_0}\right) x_0 \|^2_{\mathcal{H}} \leq \|\mathcal{F}^{\#}_{BK}{}^{i}\| \, \|Q_0\|_1$$

for every $i \geq 0$. Recall that $\|Dv_{i-1}\|^2_{\mathcal{H}} = \|E(Dv_{i-1} \circ Dv_{i-1})\|_1 \leq \|D\|^2 \|R_{i-1}\|_1$ for every $i \geq 1$, and set: $a = (\zeta_0, \zeta_1, ...)$ with $\zeta_i = \|\mathcal{F}^{\#}_{BK}{}^{i}\|^{1/2}$ for each $i \geq 0$, and $b = (\beta_0, \beta_1, ...)$ with $\beta_0 = \|Q_0\|_1^{1/2}$ and $\beta_j = \|R_{j-1}\|_1^{1/2} \|D\|$ for each $j \geq 1$. Therefore, from (3), we get

$$\|x_i\|_{\mathcal{H}} \leq \gamma_i := \sum_{j=0}^{i} \zeta_{i-j} \beta_j$$

for every $i \geq 0$. Since $a \in l_1$ (for $\|a\|_1 = \sum_{i=0}^{\infty} \|\mathcal{F}_{BK}^{\#}{}^i\|^{1/2} < \infty$, because $r(\mathcal{F}_{BK}^{\#}) < 1$, cf. [11]) and $b \in l_2$ (for $\mathbf{v} \in \mathcal{V} \subset l_2(\mathcal{H}')$ so that $\sum_{j=0}^{\infty} \|R_j\|_1 = \sum_{j=0}^{\infty} \|v_j\|_{\mathcal{H}'}^2 = \|\mathbf{v}\|_{\mathcal{V}}^2 < \infty$, and hence $\|b\|_2 = (\sum_{j=0}^{\infty} \beta_j^2)^{1/2} < \infty$), it follows that the convolution $c = a * b = \{\gamma_i = \sum_{j=0}^{i} \zeta_{i-j}\beta_j;\ i \geq 0\}$ lies itself in l_2 with $\|c\|_2 \leq \|a\|_1 \|b\|_2$ (see e.g., [5, p. 529]). Hence

$$\|\mathbf{x}\|_{\mathcal{X}} = \left(\sum_{i=0}^{\infty} \|x_i\|_{\mathcal{H}}^2 \right)^{1/2} \leq \left(\sum_{i=0}^{\infty} \gamma_i^2 \right)^{1/2} = \|c\|_2 < \infty$$

for every $\mathbf{v} \in \mathcal{V}$ and $x_0 \in \mathcal{X}_0$. From Proposition 1 and making $\mathbf{v} = 0$ we obtain the proof of necessity. $\square$

5. Main Theorem

For $(x_0, \mathbf{v}, \mathbf{q}) \in \mathcal{X}_0 \oplus \mathcal{V} \oplus \mathcal{U}$ and $K \in B[H, H'']$ define the linear operator X_{BK} from $\mathcal{X}_0 \oplus \mathcal{V} \oplus \mathcal{U}$ to $\mathcal{X}$ as

$$X_{BK}(x_0, \mathbf{v}, \mathbf{q}) = \mathbf{x} = (x_0, x_1, ...),$$

where $\mathbf{x}$ is generated by (1) with $u_i = -Kx_i + q_i$, $\mathbf{q} = (q_0, q_1, ...) \in \mathcal{U}$. An immediate consequence of the proof of Proposition 2 is that, if $r(\mathcal{F}_{BK}^{\#}) < 1$, then $\mathbf{x} \in \mathcal{X}$ and X_{BK} is bounded and therefore $X_{BK} \in B[\mathcal{X}_0 \oplus \mathcal{V} \oplus \mathcal{U}, \mathcal{X}]$. Let $H''' = H \oplus H''$ and consider the following H'''-valued random sequence $\{z_i;\ i \geq 0\}$:

$$z_i = \begin{bmatrix} M^{1/2} x_i \\ u_i \end{bmatrix} = \begin{bmatrix} M^{1/2} x_i \\ -Kx_i + q_i \end{bmatrix},$$

where $M \in B^+[H]$. Note that $\mathbf{z} = (z_0, z_1, ...) \in \mathcal{Z}$ and hence the bounded linear operator $Z_{BK} \in B[\mathcal{X}_0 \oplus \mathcal{V} \oplus \mathcal{U}, \mathcal{Z}]$ such that

$$Z_{BK}(x_0, \mathbf{v}, \mathbf{q}) = \mathbf{z} = (z_0, z_1, ...)$$

is well-defined. As usual in H_∞-control problems, we assume $x_0 = 0$ and $\mathbf{q} = \mathbf{0}$ in order to have linear operators on $\mathbf{v}$ for the state and output sequences. That is, we define $X_{BK}^0 \in B[\mathcal{V}, \mathcal{X}]$ and $Z_{BK}^0 \in B[\mathcal{V}, \mathcal{Z}]$ as

$$X_{BK}^0(\mathbf{v}) = X_{BK}(0, \mathbf{v}, \mathbf{0}) \qquad \text{and} \qquad Z_{BK}^0(\mathbf{v}) = Z_{BK}(0, \mathbf{v}, \mathbf{0}).$$

Given the above bilinear system, it is desired to find a state-feedback closed-loop controller that minimizes the impact of the disturbances $\mathbf{v}$ on the output $\mathbf{z}$ (cf [1],[4],[18],[19]). That is :

Problem OP: *Find $K \in \Gamma_B$ that minimizes*

$$\|Z_{BK}^0\| = \sup_{\mathbf{v} \in \mathcal{V}} \frac{\|Z_{BK}^0(\mathbf{v})\|_{\mathcal{Z}}}{\|\mathbf{v}\|_{\mathcal{V}}}.$$

For the linear deterministic case this is the so-called state-feedback H_∞-optimal control problem and, in general, is hard to be solved. herefore one poses a suboptimal problem which, in terms of the above bilinear system, would be:

Problem SOP: *Given $\delta > 0$ find $K \in \Gamma_B$ such that $\|Z^0_{BK}\| < \delta$. That is, find $K \in \Gamma_B$ such that for every $\mathbf{v} \in \mathcal{V}$, $\mathbf{v} \neq \mathbf{0}$,*

$$\|Z^0_{BK}(\mathbf{v})\|^2_{\mathcal{Z}} = \sum_{i=0}^{\infty} \left(\|M^{1/2}x_i\|^2_{\mathcal{H}} + \|u_i\|^2_{\mathcal{H}''} \right) < \delta^2 \sum_{i=0}^{\infty} \|v_i\|^2_{\mathcal{H}''}.$$

(Note that there is no loss of generality in assuming a cost in the form $\|u_i\|^2 = \langle u_i; u_i \rangle$ instead of $\langle N u_i; u_i \rangle$ for $N \in G^+[H]$).

In the next sections we shall prove the following theorem, which solves the problem SOP, and generalizes to stochastic bilinear systems the results previously established for the linear deterministic case (cf. [18]).

Theorem: *Suppose the pair $(M, \mathcal{F})$ is detectable* (cf. [3]) *and consider some $\delta > 0$ fixed. Then there exists $K \in \Gamma_B$ such that $\|Z^0_{BK}\| < \delta$ if and only if there exists $P \in B^+[H]$ satisfying the following conditions:*

(i) $I - \frac{1}{\delta^2} D^*PD \in G^+[H]$,

(ii) $P = M + (\mathcal{F}^{\#}_{BK_u})_{\frac{1}{\delta}D(-K_v)}(P) K_u^* K_u - K_v^* K_v$

where

$$K_u = (I + B^*PB)^{-1} B^*PF_{\frac{1}{\delta}D(-K_v)} \in B[H, H''],$$
$$K_v = (I - \frac{1}{\delta^2}\, D^*PD)^{-1} \frac{1}{\delta}\, D^*PF_{BK_u} \in B[H, H'].$$

That is,

$$K_u = \left(I + B^*PB + \frac{1}{\delta^2} B^*PD \left(I - \frac{1}{\delta^2} D^*PD \right)^{-1} D^*PB \right)^{-1} \left(B^* \left(I + \frac{1}{\delta^2} PD(I - \frac{1}{\delta^2} D^*PD)^{-1} D^* \right) PF \right)$$

$$K_v = \left(I - \frac{1}{\delta^2} D^*PD + \frac{1}{\delta^2} D^*PB \left(I + B^*PB \right)^{-1} B^*PD \right)^{-1} \left(\frac{1}{\delta} D^* \left(I - PB(I + B^*PB)^{-1} B^* \right) PF \right)$$

(iii) $r((\mathcal{F}^{\#}_{BK_u})_{\frac{1}{\delta}D(-K_v)}) < 1$.

Moreover, in such a case, $K_u \in \Gamma_B$ and $\|Z^0_{BK_u}\| < \delta$.

Remark 1: Note that $(\mathcal{F}^{\#}_{BK_u})_{\frac{1}{\delta}\ D(-K_v)}$ is obtained from $\mathcal{F}^{\#}_{BK_u}$ in the very same way as $\mathcal{F}^{\#}_{BK_u}$ was obtained from $\mathcal{F}^{\#}$, that is,

$$(\mathcal{F}^{\#}_{BK_u})_{\frac{1}{\delta}\ D(-K_v)} = (F_{BK_u} + \frac{1}{\delta}DK_v)^* P(F_{BK_u} + \frac{1}{\delta}DK_v) + \mathcal{T}^{\#}(P).$$

6. Sufficient Condition

In this section we prove the sufficiency part of the Theorem.

Lemma 1: *For $\delta > 0$ fixed, suppose that there exists $P \in B^+[H]$ satisfying conditions* (i), (ii) *and* (iii) *of Theorem. Then $K_u \in \Gamma_B$ and $\|Z^0_{BK_u}\| < \delta$.*

Note that $(M, \mathcal{F})$ detectability is not needed in this part. The following result, which is an immediate adaptation of Proposition 1 in [3], will be used in the sequel:

Proposition 3: *Let H_0 be an arbitrary Hilbert space. Take F, $Q \in B[H]$, $N = N^* \in B[H_0]$ and $B \in B[H_0, H]$ arbitrary. Suppose that, for some $P \in B^+[H]$, $(N + B^*PB)^{-1}$ exists in $B[H]$. Set $K_P = (N + B^*PB)^{-1}B^*PF$. The following assertions are equivalent:*

(a) $Q + K_P^* N K_P = P - \mathcal{F}^{\#}_{BK_P}(P)$.

(b) *For an arbitrary $K \in B[H, H_0]$,*

$$Q + K^*NK = P - \mathcal{F}^{\#}_{BK}(P) + (K - K_P)^*(B^*PB + N)(K - K_P).$$

(c) $Q = P - \mathcal{F}^{\#}(P) + K_P^*(B^*PB + N)K_P$.

The proof of Lemma 1 will follow from the next propositions.

Proposition 4: *Under the hypothesis of Lemma 1, $r((\mathcal{F}^{\#}_{BK_u})) < 1$.*

Proof: From condition (ii) of Theorem,

$$P - (\mathcal{F}^{\#}_{BK_u})_{\frac{1}{\delta}\ D(-K_v)}(P) = M + K_u^*K_u - K_v^*K_v$$

and from Proposition 3, (a) $\Rightarrow$ (c), we get

$$MK_u^*K_u = P - \mathcal{F}^{\#}_{BK_u}(P) + K_v^*\left(\frac{D^*PD}{\delta^2} - I\right)K_v.$$

That is,

$$M + K_u^*K_u + K_v^*\left(I - \frac{D^*PD}{\delta^2}\right)K_v = P - \mathcal{F}^{\#}_{BK_u}(P). \tag{4}$$

Define $\widehat{K} \in B[H, H \oplus H' \oplus H'']$ as follows:

$$\widehat{K} = \begin{bmatrix} M^{1/2} \\ \left(I - \frac{D^*PD}{\delta^2}\right)^{1/2} K_v \\ K_u \end{bmatrix}.$$

Therefore (4) can be re-written as

$$\widehat{K}^* \widehat{K} = P - \mathcal{F}^{\#}_{BK_u}(P). \tag{5}$$

Defining $\widehat{B} \in B[H \oplus H' \oplus H'', H]$ as

$$\widehat{B} = \begin{bmatrix} O & -\frac{1}{\delta} D \left(I - \frac{D^*PD}{\delta^2}\right)^{-1/2} & O \end{bmatrix}$$

we get

$$\widehat{B}\widehat{K} = -\frac{1}{\delta} D K_v,$$

so that

$$(\mathcal{F}^{\#}_{BK_u})_{\widehat{B}\widehat{K}} = (\mathcal{F}^{\#}_{BK_u})_{\frac{1}{\delta} D(-K_v)}$$

and from condition (iii) of Theorem,

$$r((\mathcal{F}^{\#}_{BK_u})_{\widehat{B}\widehat{K}}) = r((\mathcal{F}^{\#}_{BK_u})_{\frac{1}{\delta}\ D(-K_v)}) < 1. \tag{6}$$

From (5), (6) and Proposition 2 of [3] we get $r((\mathcal{F}^{\#}_{BK_u})) < 1$. □

In the reaming section we consider, for any $\mathbf{v} \in \mathcal{V}$, $\mathbf{x} = (0, x_1, ...) = X^0_{BK_u}(\mathbf{v})$ and $\mathbf{z} = (0, z_1, ...) = Z^0_{BK_u}(\mathbf{v})$. We have the following results.

Proposition 5: *Consider the hypothesis of Lemma* 1. *For every* $i \geq 0$,

$$\begin{aligned} \|P^{1/2}x_{i+1}\|^2_{\mathcal{H}} - \|P^{1/2}x_i\|^2_{\mathcal{H}} &= -\|z_i\|^2_{\mathcal{H}'''} + \delta^2 \|v_i\|^2_{\mathcal{H}'} \\ &- \delta^2 \left\| \left(I - \frac{D^*PD}{\delta^2}\right)^{1/2} \left(\frac{K_v}{\delta} x_i -_i\right) \right\|^2_{\mathcal{H}'}. \end{aligned} \tag{7}$$

Proof: Recalling that $\|P^{1/2}x_{i+1}\|^2_{\mathcal{H}} = \text{tr}(PQ_{i+1})$ we get from (2)

$$\begin{aligned} \|P^{1/2}x_{i+1}\|^2_{\mathcal{H}} &= \text{tr}\,(PQ_{i+1}) = \text{tr}\,(P\,(\mathcal{F}_{BK_u}(Q_i) \\ &+\ E(F_{BK_u}x_i \circ Dv_i) + E(F_{BK_u}x_i \circ Dv_i)^* + DR_iD^*))\,. \end{aligned}$$

From [13] we have

$$\mathrm{tr}\Big(P\mathcal{F}_{BK_u}(Q_i)\Big) = \mathrm{tr}\left(\mathcal{F}^{\#}_{BK_u}(P)Q_i\right).$$

Since

$$\begin{aligned} \|P^{1/2}x_i\|^2_{\mathcal{H}} &= \mathrm{tr}\,(PQ_i)\,, \\ \delta\left\langle\left(I-\frac{D^*PD}{\delta}\right)K_v x_i; v_i\right\rangle_{\mathcal{H}'} &= \langle D^*PF_{BK_u}x_i;\;\; v_i\rangle_{\mathcal{H}'} \\ &= \mathrm{tr}\,(PE(F_{BK_u}x_i\circ Dv_i))\,, \end{aligned}$$

and

$$\|v_i\|^2_{\mathcal{H}'} = \mathrm{tr}\,(R_i)\,,$$

(4) yields to

$$\begin{aligned} &\|P^{1/2}x_{i+1}\|^2_{\mathcal{H}} - \|P^{1/2}x_i\|^2_{\mathcal{H}} = \mathrm{tr}\Big(\Big(\mathcal{F}^{\#}_{BK_u}(P)-P\Big)Q_i \\ &+ P\Big(E(F_{BK_u}x_i\circ Dv_i) + E(F_{BK_u}x_i\circ Dv_i)^* + DR_iD^*\Big)\Big) \\ &= \mathrm{tr}\Big(-\Big(M+K_u^*K_u+K_v^*\Big(I-\frac{D^*PD}{\delta^2}\Big)K_v\Big)Q_i \\ &+ PE(F_{BK_u}x_i\circ Dv_i) + PE(F_{BK_u}x_i\circ Dv_i)^* \\ &- \delta^2\Big(I-\frac{D^*PD}{\delta^2}\Big)R_i\Big) + \delta^2\|v_i\|^2_{\mathcal{H}'} \\ &= -\Big(\|M^{1/2}x_i\|^2_{\mathcal{H}} + \|K_ux_i\|^2_{\mathcal{H}''} + \Big\|\Big(I-\frac{D^*PD}{\delta^2}\Big)^{1/2}K_vx_i\Big\|^2_{\mathcal{H}'}\Big) \\ &+ \delta\Big\langle\Big(I-\frac{D^*PD}{\delta}\Big)K_vx_i;v_i\Big\rangle_{\mathcal{H}'} + \delta\Big\langle v_i;\Big(I-\frac{D^*PD}{\delta}\Big)K_vx_i\Big\rangle_{\mathcal{H}'} \\ &- \delta^2\Big\|\Big(I-\frac{D^*PD}{\delta^2}\Big)^{1/2}v_i\Big\|^2_{\mathcal{H}'} + \delta^2\|v_i\|^2_{\mathcal{H}'} \\ &= -\|z_i\|^2_{\mathcal{H}'''} + \delta^2\|v_i\|^2_{\mathcal{H}'} - \delta^2\Big\|\Big(I-\frac{D^*PD}{\delta^2}\Big)^{1/2}\Big(\frac{K_v}{\delta}x_i - v_i\Big)\Big\|^2_{\mathcal{H}'}. \qquad \square \end{aligned}$$

Recalling that $x_0 = 0$ and $\|x_i\|_{\mathcal{H}} \to 0$ as $i \to \infty$ (indeed, $r((\mathcal{F}^{\#}_{BK_u})) < 1$ and from Proposition 2, $\|\mathbf{x}\|^2_{\mathcal{X}} = \sum_{i=0}^{\infty}\|x_i\|^2_{\mathcal{H}} < \infty$, so that $\|x_i\|_{\mathcal{H}} \to 0$ as $i \to \infty$), we get from (7) that

$$\begin{aligned} &\sum_{i=0}^{N}\left(\|P^{1/2}x_{i+1}\|^2_{\mathcal{H}} - \|P^{1/2}x_i\|^2_{\mathcal{H}}\right) = \\ &\qquad = \|P^{1/2}x_{N+1}\|^2_{\mathcal{H}} \le \|P\|\;\|x_{N+1}\|^2_{\mathcal{H}} \to 0 \;\text{ as }\; N\to\infty. \end{aligned}$$

Thus

$$0 = \sum_{i=0}^{\infty}\left[-\|z_i\|^2_{\mathcal{H}'''} + \delta^2\|v_i\|^2_{\mathcal{H}'} - \delta^2\left\|\left(I-\frac{D^*PD}{\delta^2}\right)^{1/2}\left(\frac{K_v}{\delta}x_i - v_i\right)\right\|^2_{\mathcal{H}'}\right];$$

that is,

$$\|\mathbf{z}\|_{\mathcal{Z}}^2 = \delta^2 \left(\|\mathbf{v}\|_{\mathcal{V}}^2 - \|\mathbf{r}\|_{\mathcal{V}}^2\right), \tag{8}$$

where $\mathbf{r} = (r_0, r_1, ...) \in \mathcal{V}$ is defined as

$$r_i = \left(I - \frac{D^* P D}{\delta^2}\right)^{1/2} \left(\frac{K_v}{\delta} x_i - v_i\right), \qquad i \geq 0. \tag{9}$$

Remark 2: From equation (8) we get $\|Z_{BK_u}^0\| \leq \delta$, since

$$\frac{\|\mathbf{z}\|_{\mathcal{Z}}^2}{\|\mathbf{v}\|_{\mathcal{V}}^2} = \delta^2 \left(1 - \frac{\|\underline{r}\|_{\mathcal{V}}^2}{\|\mathbf{v}\|_{\mathcal{V}}^2}\right) \leq \delta^2,$$

and the "maximizing" disturbance sequence would be in the feedback form $v_i = \frac{K_v}{\delta} x_i$ ($r_i = 0$). Since $x_0 = 0$, this feedback disturbance would be 0 and therefore it can only be optimal asymptotically. To obtain the strict inequality of Lemma 1, we shall need condition (iii) of Theorem to assure the invertibility of the operator $\widetilde{V}$ defined next.

Define the operator $\widetilde{V} \in B[\mathcal{V}]$ as $\widetilde{V} = \frac{1}{\delta} K_v X_{BK_u}^0 - I$. That is, for every $\mathbf{v} \in \mathcal{V}$, $\widetilde{V}(\mathbf{v}) = \tilde{\mathbf{v}} = (\tilde{v}_0, \tilde{v}_1, ...) = \frac{1}{\delta} K_v X_{BK_u}^0(\mathbf{v}) - \mathbf{v}$. We have the following result.

Proposition 6: *Consider the hypothesis of Lemma* 1. *Then* $\widetilde{V} \in G[\mathcal{V}]$.

Proof: Define the operators $\widetilde{Y} \in B[\mathcal{V}, \mathcal{X}]$ and $\tilde{V}_{\text{inv}} \in B[\mathcal{V}]$ as:

(a) for $\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}$, $\widetilde{Y}(\mathbf{v}) = (\tilde{y}_0, \tilde{y}_1, ...)$ is given by

$$\tilde{y}_{i+1} = \left(A_0 - BK_u + \frac{1}{\delta} DK_v + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle \right) \tilde{y}_i - Dv_i, \tag{10}$$

for $i \geq 0$ and $\tilde{y}_0 = 0$. Note that, from condition (iii) of the Theorem, $r((\mathcal{F}_{BK_u}^{\#})_{\frac{1}{\delta} D(-K_v)}) < 1$, and therefore from Proposition 2, $\widetilde{Y} \in B[\mathcal{V}, \mathcal{X}]$.

(b) $\tilde{V}_{\text{inv}} = \frac{1}{\delta} K_v \widetilde{Y} - I$. That is, for $\mathbf{v} \in \mathcal{V}$, $\tilde{V}_{\text{inv}}(\mathbf{v}) = \tilde{\mathbf{s}} = (\tilde{s}_0, \tilde{s}_1, ...) = \frac{1}{\delta} K_v \widetilde{Y}(\mathbf{v}) - \mathbf{v}$. Since $\widetilde{Y} \in B[\mathcal{V}, \mathcal{X}]$, it follows that $\tilde{V}_{\text{inv}} \in B[\mathcal{V}]$. Notice that from (10),

$$\widetilde{Y}(\mathbf{v}) = X_{BK_u}^0(\tilde{\mathbf{s}}) \tag{11}$$

and

$$\widetilde{Y}(\tilde{\mathbf{v}}) = X_{BK_u}^0(\mathbf{v}). \tag{12}$$

Let us show that $\widetilde{V}\tilde{V}_{\text{inv}} = \tilde{V}_{\text{inv}}\widetilde{V} = I$. Indeed from (11),

$$\begin{aligned}
\widetilde{V}\tilde{V}_{\text{inv}}(\mathbf{v}) = \widetilde{V}(\tilde{\mathbf{s}}) &= \frac{1}{\delta}K_v X^0_{BK_u}(\tilde{\mathbf{s}}) - \tilde{\mathbf{s}} \\
&= \frac{1}{\delta}K_v X^0_{BK_u}(\tilde{\mathbf{s}}) - \left(\frac{1}{\delta}K_v \widetilde{Y}(\mathbf{v}) - \mathbf{v}\right) \\
&= \mathbf{v} + \frac{1}{\delta}K_v \left(X^0_{BK_u}(\tilde{\mathbf{s}}) - \widetilde{Y}(\mathbf{v})\right) = \mathbf{v}
\end{aligned}$$

and from (12),

$$\begin{aligned}
\tilde{V}_{\text{inv}}\widetilde{V}(\mathbf{v}) = \tilde{V}_{\text{inv}}(\tilde{\mathbf{v}}) &= \frac{1}{\delta}K_v \widetilde{Y}(\tilde{\mathbf{v}}) - \tilde{\mathbf{v}} \\
&= \frac{1}{\delta}K_v \widetilde{Y}(\tilde{\mathbf{v}}) - \left(\frac{1}{\delta}K_v X^0_{BK_u}(\mathbf{v}) - \mathbf{v}\right) \\
&= \mathbf{v} + \frac{1}{\delta}K_v \left(\widetilde{Y}(\tilde{\mathbf{v}}) - X^0_{BK_u}(\mathbf{v})\right) = \mathbf{v}
\end{aligned}$$

which shows that $\widetilde{V}^{-1} = \tilde{V}_{\text{inv}} \in B[\mathcal{V}]$. □

We can now proceed to the proof of Lemma 1.

Proof of Lemma 1: Consider $\alpha_1 > 0$ such that $\|\widetilde{V}^{-1}\| < \alpha_1$, and $\alpha_2 > 0$ such that $I - \frac{D^*PD}{\delta^2} \geq \alpha_2^2 I$. Since $(1/\alpha_1)\|\mathbf{v}\|_{\mathcal{V}} \leq \|\widetilde{V}^{-1}\|^{-1}\|\mathbf{v}\|_{\mathcal{V}} \leq \|\widetilde{V}(\mathbf{v})\|_{\mathcal{V}}$ for any $\mathbf{v} \in \mathcal{V}$, we conclude from (9) that

$$\begin{aligned}
\|\mathbf{r}\|^2_{\mathcal{V}} &= \sum_{i=0}^{\infty} \left\langle \left(I - \frac{D^*PD}{\delta^2}\right)\left(\frac{1}{\delta}K_v x_i - v_i\right); \left(\frac{1}{\delta}K_v x_i - v_i\right)\right\rangle_{\mathcal{H}'} \\
&\geq \alpha_2^2 \sum_{i=0}^{\infty} \left\|\frac{1}{\delta}K_v x_i - v_i\right\|^2_{\mathcal{H}'} \\
&= \alpha_2^2 \sum_{i=0}^{\infty} \|[\widetilde{V}(\mathbf{v})]_i\|^2_{\mathcal{H}'} = \alpha_2^2\|\widetilde{V}(\mathbf{v})\|^2_{\mathcal{V}} \geq \frac{\alpha_2^2}{\alpha_1^2}\|\mathbf{v}\|^2_{\mathcal{V}}, \qquad (13)
\end{aligned}$$

and replacing (13) in (8) we obtain, for every $\mathbf{v} \neq \mathbf{0}$ in $\mathcal{V}$,

$$\frac{\|\mathbf{z}\|^2_{\mathcal{Z}}}{\|\mathbf{v}\|^2_{\mathcal{V}}} \leq \delta^2\left(1 - \frac{\alpha_2^2}{\alpha_1^2}\right) < \delta^2$$

or, in other words, $\|Z^0_{BK_u}\| < \delta$. □

7. Necessary Condition

In this section we shall prove the necessity part of the Theorem, which reads as follows.

Lemma 2: *For $\delta > 0$ fixed, suppose that $(M, \mathcal{F})$ is detectable* (cf [3]) *and there exists $K \in \Gamma_B$ such that $\|Z^0_{BK}\| < \delta$. Then there exists $P \in B^+[H]$ satisfying conditions* (i), (ii) *and* (iii) *of Theorem.*

From the Theorem in [3], there exists a unique $L \in B^+[H]$ such that

$$M = L - \mathcal{F}^{\#}(L) + F^*LB(I + B^*LB)^{-1}B^*LF = L - \mathcal{F}^{\#}_{BK_L}(L) - K_L^*K_L, \quad (14)$$

where

$$K_L = (I + B^*LB)^{-1}B^*LF \in \Gamma_B. \quad (15)$$

For any $\mathbf{q} = (q_0, q_1, ...) \in \mathcal{U}$, $x_0 \in \mathcal{X}_0$ and $\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}$, we set throughout this section

$$\mathbf{x} = (x_0, x_1, ...) = X_{BK_L}(x_0, \mathbf{v}, \mathbf{q}) \quad (16.a)$$

and

$$\mathbf{z} = (z_0, z_1, ...) = Z_{BK_L}(x_0, \mathbf{v}, \mathbf{q}). \quad (16.b)$$

Set $\mathcal{J} = \mathcal{X}_0 \oplus \mathcal{V} \oplus \mathcal{U}$ (as we shall see in the proof of Proposition 9 below, $\mathcal{J}$ will play the role of H_0 in Proposition 7) and, for $(x_0, \mathbf{v}, \mathbf{q}) \in \mathcal{J}$,

$$\begin{aligned} J(x_0, \mathbf{v}, \mathbf{q}) &= \sum_{i=0}^{\infty} \left(\|M^{1/2}x_i\|^2_{\mathcal{H}} + \|u_i\|^2_{\mathcal{H}''} - \delta^2 \|v_i\|^2_{\mathcal{H}'} \right) \\ &= \|Z_{BK_L}(x_0, \mathbf{v}, \mathbf{q})\|^2_{\mathcal{Z}} - \delta^2 \|\mathbf{v}\|^2_{\mathcal{V}}. \end{aligned}$$

In order to prove Lemma 2 we follow a dynamic game approach similar to the one used for the linear deterministic case, and solve the following minimax problem:

$$\widehat{J}(x_0) = \sup_{\mathbf{v} \in \mathcal{V}} \inf_{\mathbf{q} \in \mathcal{U}} J(x_0, \mathbf{v}, \mathbf{q}) \quad \text{for every} \quad x_0 \in \mathcal{X}_0.$$

This minimax problem will be solved by using twice the following result, due to Yakubovich [24], which was presented in [9, p.11].

Proposition 7: *Consider a Hilbert space H_0 and a quadratic form $J(\zeta) = \langle S\zeta; \zeta \rangle$, $\zeta \in H_0$ with $S \in B[H_0]$, $S = S^*$. Let $\mathcal{M}_0$ be a closed subspace of H_0 and $\mathcal{M}$ a translation of $\mathcal{M}_0$ by an element $m \in H_0$ (i.e., $\mathcal{M} = \mathcal{M}_0 + m$). Suppose that the following condition is satisfied:*

$$\inf_{\zeta \in \mathcal{M}_0} \frac{\langle S\zeta; \zeta \rangle}{\langle \zeta; \zeta \rangle} > 0.$$

Then there exists a unique element $\widetilde{\zeta} \in \mathcal{M}$ *such that* $J(\widetilde{\zeta}) = \inf_{\zeta\in\mathcal{M}} J(\zeta)$, *where* $\widetilde{\zeta}$ *is given by* $\widetilde{\zeta} = p + m$ *with* $p \in \mathcal{M}_0$ *and* $p = Gm$ *for some* $G \in B[H_0]$.

We shall now solve the first of the minimax problem, that is,

$$\widetilde{J}(x_0, \mathbf{v}) = \inf_{\mathbf{q}\in\mathcal{U}} J(x_0, \mathbf{v}, \mathbf{q}) \quad \text{for every} \quad x_0 \in \mathcal{X}_0, \mathbf{v} \in \mathcal{V}.$$

The solution to this problem will follow as an application of Proposition 7. But before using Proposition 7 we need to establish the next result which ensures that the hypothesis of Proposition 7 are satisfied.

Proposition 8: *Consider the hypothesis of Lemma* 2. *Let* L, K_L *be as in* (14) *and* (15) *respectively. For every* $x_0 \in \mathcal{X}_0$ *and* $\mathbf{q} \in \mathcal{U}$,

$$\begin{aligned}\|Z_{BK_L}(x_0, \mathbf{0}, \mathbf{q})\|^2_{\mathcal{Z}} &= \sum_{i=0}^{\infty} \left(\|M^{1/2}x_i\|^2_{\mathcal{H}} + \|u_i\|^2_{\mathcal{H}''}\right) \\ &= \|L^{1/2}x_0\|^2_{\mathcal{H}} + \sum_{i=0}^{\infty} \|\left(I + B^*LB\right)^{1/2} q_i\|^2_{\mathcal{H}''}. \quad (17)\end{aligned}$$

Proof: Set $V_i = E(q_i \circ q_i)$. By using similar arguments as those in Proposition 5 and from (15) we get

$$\begin{aligned}\operatorname{tr}\left(L\mathcal{F}_{BK_L}(Q_i)\right) &= \operatorname{tr}\left(\mathcal{F}^{\#}_{BK_L}(L)Q_i\right), \\ \|L^{1/2}x_i\|^2_{\mathcal{H}} &= \operatorname{tr}\left(LQ_i\right), \\ \|q_i\|^2_{\mathcal{H}} &= \operatorname{tr}\left(V_i\right),\end{aligned}$$

$$\begin{aligned}\langle K_L x_i; q_i\rangle_{\mathcal{H}''} &= \langle (I + B^*LB)K_L x_i; q_i\rangle_{\mathcal{H}''} - \langle B^*LBK_L x_i; q_i\rangle_{\mathcal{H}''} \\ &= \langle B^*LFx_i; q_i\rangle_{\mathcal{H}''} - \langle B^*LBK_L x_i; q_i\rangle_{\mathcal{H}''} \\ &= \langle B^*LF_{BK_L} x_i; q_i\rangle_{\mathcal{H}''} = \operatorname{tr}\left(LE(F_{BK_L} x_i \circ Bq_i)\right).\end{aligned}$$

Hence, from (2), (14) and recalling that $\|L^{1/2}x_{i+1}\|^2_{\mathcal{H}} = \operatorname{tr}\left(LQ_{i+1}\right)$, we conclude

$$\begin{aligned}
&\|L^{1/2}x_{i+1}\|^2_{\mathcal{H}} - \|L^{1/2}x_i\|^2_{\mathcal{H}} \\
&= \mathrm{tr}\Big(L\Big(\mathcal{F}_{BK_L}(Q_i) + E(F_{BK_L}x_i \circ Bq_i) + E(F_{BK_L}x_i \circ Bq_i)^* + BV_iB^*\Big)\Big) \\
&- \mathrm{tr}\Big(LQ_i\Big) = \mathrm{tr}\Big(\Big(\mathcal{F}^{\#}_{BK_L}(L) - L\Big)Q_i\Big) + \langle K_Lx_i; q_i\rangle_{\mathcal{H}''} \\
&+ \langle q_i; K_Lx_i\rangle_{\mathcal{H}''} + \mathrm{tr}\Big((I + B^*LB)V_i\Big) - \mathrm{tr}\Big(V_i\Big) \\
&= -\mathrm{tr}\Big(\Big(M + K_L^*K_L\Big)Q_i\Big) + \langle K_Lx_i; q_i\rangle_{\mathcal{H}''} + \langle q_i; K_Lx_i\rangle_{\mathcal{H}''} \\
&+ \|(I + B^*LB)^{1/2}q_i\|^2_{\mathcal{H}''} - \|q_i\|^2_{\mathcal{H}''} = -\Big(\|M^{1/2}x_i\|^2_{\mathcal{H}} + \|K_Lx_i\|^2_{\mathcal{H}''}\Big) \\
&+ \langle K_Lx_i; q_i\rangle_{\mathcal{H}''} + \langle q_i; K_Lx_i\rangle_{\mathcal{H}''} + \|(I + B^*LB)^{1/2}q_i\|^2_{\mathcal{H}''} \\
&- \|q_i\|^2_{\mathcal{H}''} = -\Big(\|M^{1/2}x_i\|^2_{\mathcal{H}} + \|q_i - K_Lx_i\|^2_{\mathcal{H}''}\Big) \\
&+ \|(I + B^*LB)^{1/2}q_i\|^2_{\mathcal{H}''} = -\Big(\|M^{1/2}x_i\|^2_{\mathcal{H}} + \|u_i\|^2_{\mathcal{H}''}\Big) \\
&+ \|(I + B^*LB)^{1/2}q_i\|^2_{\mathcal{H}''}.
\end{aligned}$$

Summing up over i from 0 to ∞ and recalling that $\|x_i\|_{\mathcal{H}} \to 0$ as $i \to \infty$ (since that $r(\mathcal{F}^{\#}_{BK_L}) < 1$, cf. Proposition 2) we get (17). □

We can now apply Proposition 7 and solve the first of the minimax problem.

Proposition 9: *Consider the hypothesis of Lemma 2. For each $x_0 \in \mathcal{X}_0$ and $\mathbf{v} \in \mathcal{V}$ there exists a unique element $\tilde{\mathbf{q}} \in \mathcal{U}$ such that $\widetilde{J}(x_0, \mathbf{v}) = J(x_0, \mathbf{v}, \tilde{\mathbf{q}}) = \inf_{\mathbf{q}\in\mathcal{U}} J(x_0, \mathbf{v}, \mathbf{q})$. Moreover, there exists $G \in B[\mathcal{V}, \mathcal{U}]$ such that $\tilde{\mathbf{q}} = G\mathbf{v}$.*

Proof: Initially note that

$$\begin{aligned}
J(x_0, \mathbf{v}, \mathbf{q}) &= \langle Z_{BK_L}(x_0, \mathbf{v}, \mathbf{q}); Z_{BK_L}(x_0, \mathbf{v}, \mathbf{q})\rangle_{\mathcal{Z}} - \langle(0, \delta^2\mathbf{v}, \mathbf{0}); (x_0, \mathbf{v}, \mathbf{q})\rangle_{\mathcal{J}} \\
&= \langle Z^*_{BK_L}Z_{BK_L}(x_0, \mathbf{v}, \mathbf{q}); (x_0, \mathbf{v}, \mathbf{q})\rangle_{\mathcal{J}} - \langle(0, \delta^2\mathbf{v}, \mathbf{0}); (x_0, \mathbf{v}, \mathbf{q})\rangle_{\mathcal{J}} \\
&= \langle S(x_0, \mathbf{v}, \mathbf{q}); (x_0, \mathbf{v}, \mathbf{q})\rangle_{\mathcal{J}},
\end{aligned}$$

where $S(x_0, \mathbf{v}, \mathbf{q}) = Z^*_{BK_L}Z_{BK_L}(x_0, \mathbf{v}, \mathbf{q}) - (0, \delta^2\mathbf{v}, \mathbf{0})$. Note that $\mathcal{J}$ is a Hilbert space, $S \in B[\mathcal{J}]$ and $S = S^*$. Set $\mathcal{M}_0 = \{(x_0, \mathbf{v}, \mathbf{q}) \in \mathcal{J};\ \ x_0 = 0, \mathbf{v} = 0\}$ and, for $x_0' \in \mathcal{X}_0$, $\mathbf{v}' \in \mathcal{V}$, $\mathcal{M} = \mathcal{M}_0 + m = \{(x_0, \mathbf{v}, \mathbf{q}) \in \mathcal{J};\ \ x_0 = x_0', \mathbf{v} = \mathbf{v}'\}$ where $m = (x_0', \mathbf{v}', \mathbf{0})$. $\mathcal{M}_0$ is a closed subspace of $\mathcal{J}$ and $\mathcal{M}$ a translation of $\mathcal{M}_0$ by an element m. From (17) we have

$$\begin{aligned}
\inf_{\zeta\in\mathcal{M}_0} \frac{\langle S\zeta; \zeta\rangle_{\mathcal{J}}}{\langle\zeta; \zeta\rangle_{\mathcal{J}}} = \inf_{\mathbf{q}\in\mathcal{U}} \frac{\|Z_{BK_L}(0, 0, \mathbf{q})\|^2_{\mathcal{Z}}}{\|\mathbf{q}\|^2_{\mathcal{U}}} &= \\
= \inf_{\mathbf{q}\in\mathcal{U}} \frac{\|\,(I + B^*LB)^{1/2}\,\mathbf{q}\|^2_{\mathcal{U}}}{\|\mathbf{q}\|^2_{\mathcal{U}}} &\geq 1
\end{aligned}$$

and, according to Proposition 7, there exists a unique element $\widetilde{\zeta} \in \mathcal{M}$ such that $J(\widetilde{\zeta}) = \inf_{\zeta\in\mathcal{M}} J(\zeta)$, where $\widetilde{\zeta} = p + m$, $p = (0, \mathbf{0}, \tilde{\mathbf{q}}) \in \mathcal{M}_0$ and $p = G'm$ for some $G' \in B[\mathcal{J}]$. Thus $G'm = G'(x_0', \mathbf{v}', \mathbf{0}) = (0, \mathbf{0}, \tilde{\mathbf{q}})$. Also note from (17) that we must have $G'(x_0', \mathbf{0}, \mathbf{0}) = (0, \mathbf{0}, \mathbf{0})$ and hence, for some $G \in B[\mathcal{V},\mathcal{U}]$, $\tilde{\mathbf{q}} = G\mathbf{v}$ where $J(x_0, \mathbf{v}, \tilde{\mathbf{q}}) = \inf_{\mathbf{q}\in\mathcal{U}} J(x_0, \mathbf{v}, \mathbf{q})$. $\square$

We shall now solve the second of the minimax problem. For that we introduce the following operators $\widetilde{X}$ and $\bar{X}$ in $B[\mathcal{X}_0 \oplus \mathcal{V}, \mathcal{X}]$ and $\widetilde{Z}$ and $\bar{Z}$ in $B[\mathcal{X}_0 \oplus \mathcal{V}, \mathcal{Z}]$: for $x_0 \in \mathcal{X}_0$ and $\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}$, $\widetilde{X}(x_0, \mathbf{v}) = X_{BK_L}(x_0, \mathbf{v}, G\mathbf{v})$, $\widetilde{Z}(x_0, \mathbf{v}) = Z_{BK_L}(x_0, \mathbf{v}, G\mathbf{v})$,

$$\bar{X}(x_0, \mathbf{v}) = (\bar{x}_0, \bar{x}_1, ...) = X_{BK}(x_0, \mathbf{v}, \mathbf{0}),$$

and

$$\bar{Z}(x_0, \mathbf{v}) = (\bar{z}_0, \bar{z}_1, ...) = Z_{BK}(x_0, \mathbf{v}, \mathbf{0}), \tag{18}$$

where K is as in Lemma 2. Set $\mathcal{D} = \mathcal{X}_0 \oplus \mathcal{V}$ which, in the proof of Proposition 11, will play the role of H_0 in Proposition 7. The following result is needed in order to use Proposition 7 to solve the second minimax problem.

Proposition 10: *Consider the hypothesis of Lemma* 2*. For* $(x_0, \mathbf{v}) \in \mathcal{D}$,

$$\|\widetilde{Z}(x_0, \mathbf{v})\|_{\mathcal{Z}}^2 - \delta^2\|\mathbf{v}\|_{\mathcal{V}}^2 \leq \|\bar{Z}(x_0, \mathbf{v})\|_{\mathcal{Z}}^2 - \delta^2\|\mathbf{v}\|_{\mathcal{V}}^2.$$

Proof: For any $\mathbf{q} \in \mathcal{U}$,

$$\|\widetilde{Z}(x_0, \mathbf{v})\|_{\mathcal{Z}}^2 - \delta^2\|\mathbf{v}\|_{\mathcal{V}}^2 \leq \|Z_{BK_L}(x_0, \mathbf{v}, \mathbf{q})\|_{\mathcal{Z}}^2 - \delta^2\|\mathbf{v}\|_{\mathcal{V}}^2$$

and, choosing $\mathbf{q} = (q_0, q_1, ...)$ as $q_i = K_L x_i - K x_i$, $i \geq 0$, where x_i is defined as in (16), we get $x_i = \bar{x}_i$, $i \geq 0$, where $\bar{x}_i$ is defined as in (18), and thus $Z_{BK_L}(x_0, \mathbf{v}, \mathbf{q}) = \bar{Z}(x_0, \mathbf{v})$. Note that indeed $\mathbf{q} \in \mathcal{U}$. $\square$

From the hypothesis of Lemma 2, $K \in \Gamma_B$ is such that for every $\mathbf{v} \neq \mathbf{0}$ in $\mathcal{V}$, $\|Z_{BK}^0(\mathbf{v})\|_{\mathcal{Z}} < \delta\|\mathbf{v}\|_{\mathcal{V}}$, and hence we can find $\alpha > 0$ such that

$$\frac{\|Z_{BK}^0(\mathbf{v})\|_{\mathcal{Z}}^2}{\|\mathbf{v}\|_{\mathcal{V}}^2} \; < \; \delta^2 - \alpha^2; \tag{19}$$

and from Proposition 10 it follows that, for every $\mathbf{v} \in \mathcal{V}$,

$$\alpha^2\|\mathbf{v}\|_{\mathcal{V}}^2 \leq \delta^2\|\mathbf{v}\|_{\mathcal{V}}^2 - \|\bar{Z}(0, \mathbf{v})\|_{\mathcal{Z}}^2 \leq \delta^2\|\mathbf{v}\|_{\mathcal{V}}^2 - \|\widetilde{Z}(0, \mathbf{v})\|_{\mathcal{Z}}^2.$$

Thus

$$\inf_{\mathbf{v}\in\mathcal{V}} \left(\frac{\delta^2\|\mathbf{v}\|_{\mathcal{V}}^2 - \|\widetilde{Z}(0, \mathbf{v})\|_{\mathcal{Z}}^2}{\|\mathbf{v}\|_{\mathcal{V}}^2} \right) \geq \alpha^2. \tag{20}$$

Next we shall solve the second minimax problem. For $(x_0, \mathbf{v}) \in \mathcal{D}$,

$$\begin{aligned}\widetilde{J}(x_0, \mathbf{v}) &= \|\widetilde{Z}(x_0, \mathbf{v})\|_{\mathcal{Z}}^2 - \delta^2 \|\mathbf{v}\|_{\mathcal{V}}^2 \\ &= \langle \widetilde{Z}^* \widetilde{Z}(x_0, \mathbf{v}); (x_0, \mathbf{v})\rangle_{\mathcal{D}} - \langle (0, \delta^2 \mathbf{v}); (x_0, \mathbf{v})\rangle_{\mathcal{D}} \\ &= \langle \widetilde{S}(x_0, \mathbf{v}); (x_0, \mathbf{v})\rangle_{\mathcal{D}},\end{aligned}$$

where $\widetilde{S}(x_0, \mathbf{v}) = \widetilde{Z}^* \widetilde{Z}(x_0, \mathbf{v}) - \delta^2(0, \mathbf{v})$. $\mathcal{D}$ is a Hilbert space, $\widetilde{S} \in B[\mathcal{D}]$ and $\widetilde{S} = \widetilde{S}^*$. The following result follows.

Proposition 11: *Consider the hypothesis of Lemma 2. For each $x_0 \in \mathcal{X}_0$ there exists a unique element $\widehat{\mathbf{v}} \in \mathcal{V}$ such that $-\widehat{J}(x_0) = (-\widetilde{J}(x_0, \widehat{\mathbf{v}})) = \inf_{\mathbf{v} \in \mathcal{V}}(-\widetilde{J}(x_0, \mathbf{v}))$. Moreover, for some $T \in B[\mathcal{X}_0,\ \mathcal{V}]$, $\widehat{\mathbf{v}} = Tx_0$.*

Proof: Set

$$\begin{aligned}\tilde{\mathcal{M}}_0 &= \{(x_0, \mathbf{v}) \in \mathcal{D}; \quad x_0 = 0\}, \\ \widetilde{\mathcal{M}} &= \tilde{\mathcal{M}}_0 + \widetilde{m} = \{(x_0, \mathbf{v}) \in \mathcal{D}; \quad x_0 = \tilde{x}_0\},\end{aligned}$$

where $\widetilde{m} = (\tilde{x}_0, 0) \in \mathcal{D}$. $\tilde{\mathcal{M}}_0$ is a closed subspace of $\mathcal{D}$ and $\widetilde{\mathcal{M}}$ a translation of $\tilde{\mathcal{M}}_0$ by the element $\widetilde{m}$. Note from (20) that

$$\inf_{\zeta \in \tilde{\mathcal{M}}_0} \left(-\frac{\langle \widetilde{S}\zeta; \zeta\rangle_{\mathcal{D}}}{\langle \zeta; \zeta\rangle_{\mathcal{D}}} \right) \ \geq\ \alpha^2 > 0$$

and, according to Proposition 7, there exists a unique element $\widehat{\zeta} \in \widetilde{\mathcal{M}}$ such that $(-\widetilde{J}(\widehat{\zeta})) = \inf_{\zeta \in \widetilde{\mathcal{M}}}(-\widetilde{J}(\zeta))$ where $\widehat{\zeta} = \widetilde{p} + \widetilde{m}$, $\widetilde{p} = (\mathbf{0}, \widehat{\mathbf{v}}) \in \tilde{\mathcal{M}}_0$ and $\widetilde{p} = \widetilde{T}\widetilde{m}$ for some $\widetilde{T} \in B[\mathcal{D}]$. Therefore, $\widetilde{T}\widetilde{m} = \widetilde{T}(\tilde{x}_0, \mathbf{0}) = (0, \widehat{\mathbf{v}})$ and, for some $T \in B[\mathcal{X}_0, \mathcal{V}]$, $\widehat{\mathbf{v}} = Tx_0$ where $(-\widetilde{J}(x_0, \widehat{\mathbf{v}})) = \inf_{\mathbf{v} \in \mathcal{V}}(-\widetilde{J}(x_0, \mathbf{v}))$. □

We have solved the minimax problem posed initially, with $G\mathbf{v} \in \mathcal{U}$ providing the minimizing control for the minimization problem defined by $\widetilde{J}(x_0, \mathbf{v})$ above and Tx_0 providing the maximizing disturbance for the maximization problem defined by $\widehat{J}(x_0)$ above. We shall now construct the operator $P \in B^+[H]$ which will satisfy conditions (i), (ii) and (iii) of Theorem. Define the operators $\widehat{X} \in B[\mathcal{X}_0, \mathcal{X}]$ and $\widehat{Z} \in B[\mathcal{X}_0, \mathcal{Z}]$ in the following way: for $x_0 \in \mathcal{X}_0$, $\widehat{X}(x_0) = \widetilde{X}(x_0, Tx_0) = X_{BK_L}(x_0, Tx_0, G(Tx_0))$, $\widehat{Z}(x_0) = \widetilde{Z}(x_0, Tx_0) = Z_{BK_L}(x_0, Tx_0, G(Tx_0))$. Therefore $\widehat{X}(x_0)$ and $\widehat{Z}(x_0)$ give the state sequence and output sequence when the maximizing disturbance Tx_0 for $\mathbf{v}$ and the minimizing control $G(Tx_0)$ for $\mathbf{q}$ are used. Note that $\widehat{J}(.) :\ \mathcal{X}_0 \to I\!\!R_+$ is given by

$$
\begin{aligned}
\widehat{J}(x_0) &= \sup_{\mathbf{v}\in\mathcal{V}} \inf_{\mathbf{q}\in\mathcal{U}} J(x_0,\mathbf{v},\mathbf{q}) \\
&= \sup_{\mathbf{v}\in\mathcal{V}} (\|\widetilde{Z}(x_0,\mathbf{v})\|^2_{\mathcal{Z}} - \delta^2\|\mathbf{v}\|^2_{\mathcal{V}}) \\
&= -\inf_{\mathbf{v}\in\mathcal{V}} (-\widetilde{J}(x_0,\mathbf{v})) = \widetilde{J}(x_0,Tx_0) \\
&= \|\widehat{Z}(x_0)\|^2_{\mathcal{Z}} - \delta^2\|Tx_0\|^2_{\mathcal{V}} \geq \|\widetilde{Z}(x_0,0)\|^2_{\mathcal{Z}} \geq 0
\end{aligned}
$$

and indeed $\widehat{J}(x_0) \geq 0$. Define the operators $\mathcal{P} \in B^+[\mathcal{X}_0]$ and $P \in B^+[H]$ as follows. Set

$$\mathcal{P} = \widehat{Z}^*\widehat{Z} - \delta^2 T^*T,$$

and let P be an operator from H into H given by

$$Px = E(\mathcal{P}x) \quad \text{for each} \quad x \in H \subset \mathcal{X}_0. \tag{21}$$

From the properties of the expected value operator E (cf. [12]) it is easy to verify that P in fact is a bounded linear operator. Note that for every $x_0 \in \mathcal{X}_0$,

$$
\begin{aligned}
0 \leq \widehat{J}(x_0) &= \langle \widehat{Z}(x_0); \widehat{Z}(x_0)\rangle_{\mathcal{Z}} - \delta^2\langle Tx_0; Tx_0\rangle_{\mathcal{V}} \\
&= \langle (\widehat{Z}^*\widehat{Z} - \delta^2 T^*T)(x_0); x_0\rangle_{\mathcal{X}_0} = \langle \mathcal{P}x_0; x_0\rangle_{\mathcal{X}_0}
\end{aligned}
$$

and, from the definition of the expected value operator, for every $x \in H$,

$$0 \leq \widehat{J}(x) = \langle \mathcal{P}x; x\rangle_{\mathcal{H}} = \mathcal{E}(\langle \mathcal{P}x; x\rangle) = \langle E(\mathcal{P}x); x\rangle = \langle Px; x\rangle$$

showing that indeed $\mathcal{P} \geq 0$ and $P \geq 0$. It remains to show that P satisfies conditions (i), (ii) and (iii) of Theorem. We shall do this by considering a truncated minimax problem, the truncation being on the "disturbance" space $\mathcal{V}$, and then proving that the truncated problem converges to the original one. Set

$$\mathcal{S}_n = \{\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}; \quad v_{n+i} = 0 \quad \text{for} \quad i \geq 0\},$$

and

$$
\begin{aligned}
\widehat{J}_n(x_0) &= \sup_{\mathbf{v}\in\mathcal{S}_n} \left(\|\widetilde{Z}(x_0,\mathbf{v})\|^2_{\mathcal{Z}} - \delta^2\|\mathbf{v}\|^2_{\mathcal{V}}\right) \\
&= \sup_{\mathbf{v}\in\mathcal{S}_n} \left(\|Z_{BK_L}(x_0,\mathbf{v},G\mathbf{v})\|^2_{\mathcal{Z}} - \delta^2\|\mathbf{v}\|^2_{\mathcal{V}}\right) \geq \|\widetilde{Z}(x_0,\mathbf{0})\|^2_{\mathcal{Z}} \geq 0,
\end{aligned}
$$

with $x_0 \in \mathcal{X}_0$ and, for every $\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}$, set $g_n \in B[\mathcal{V}, \mathcal{S}_n]$ as

$$g_n(\mathbf{v}) = (v_0, ..., v_{n-1}, 0, 0, ...).$$

Proposition 12: *Consider the hypothesis of Lemma 2. For every* $x_0 \in \mathcal{X}_0$, $0 \le \widehat{J}_n(x_0) \uparrow \widehat{J}(x_0)$ *as* $n \to \infty$.

Proof: Since $\mathcal{S}_n \subset \mathcal{S}_{n+1} \subset \mathcal{V}$ it is clear that $\widehat{J}_n(x_0) \le \widehat{J}_{n+1}(x_0) \le \widehat{J}(x_0)$ and

$$\begin{aligned}
&\left(\|\widetilde{Z}(x_0, g_n(T(x_0)))\|_{\mathcal{Z}}^2 - \delta^2 \|g_n(T(x_0))\|_{\mathcal{V}}^2\right) \\
&\quad \le \widehat{J}_n(x_0) = \sup_{\mathbf{v} \in \mathcal{S}_n} \|\widetilde{Z}(x_0, \mathbf{v})\|_{\mathcal{Z}}^2 - \delta^2 \|\mathbf{v}\|_{\mathcal{V}}^2 \\
&\quad \le \sup_{\mathbf{v} \in \mathcal{V}} \left(\|\widetilde{Z}(x_0, \mathbf{v})\|_{\mathcal{Z}}^2 - \delta^2 \|\mathbf{v}\|_{\mathcal{V}}^2\right) = \widehat{J}(x_0) \\
&\quad = \left(\|\widetilde{Z}(x_0, (T(x_0)))\|_{\mathcal{Z}}^2 - \delta^2 \|T(x_0)\|_{\mathcal{V}}^2\right).
\end{aligned}$$

Since $g_n(T(x_0)) \to T(x_0)$ as $n \to \infty$, we have, by continuity of the norm and of $\widetilde{Z}(x_0, .)$, the desired result. □

Define the sequence $P_n \in B[H]$, $n \ge 0$, and $K_{un} \in B[H, H'']$, $K_{vn} \in B[H, H']$, $n \ge 1$, as follows:

$$\begin{aligned}
&P_0 = L \\
&P_{n+1} = M + (\mathcal{F}^{\#}_{BK_{un+1}})_{\frac{1}{\delta}}(P_n) + K_{un+1}^* K_{un+1} - K_{vn+1}^* K_{vn+1}
\end{aligned} \tag{22}$$

for $n \ge 0$, where

$$K_{un+1} = (I + B^* P_n B)^{-1} B^* P_n F_{\frac{1}{\delta} D(-K_{vn+1})} \in B[H, H''] \tag{23}$$

$$K_{vn+1} = (I - \frac{D^* P_n D}{\delta^2})^{-1} \frac{1}{\delta} D^* P_n F_{BK_{un+1}} \in B[H, H']. \tag{24}$$

That is,

$$\begin{aligned}
K_{un+1} &= \left(I + B^* P_n B + \frac{B^* P_n D}{\delta^2}\left(I - \frac{D^* P_n D}{\delta^2}\right)^{-1} D^* P_n B\right)^{-1} \\
&\quad \left(B^* \left(I + \frac{P_n D}{\delta^2}\left(I - \frac{D^* P_n D}{\delta^2}\right)^{-1} D^*\right) P_n F\right), \\
K_{vn+1} &= \left(I - \frac{D^* P_n D}{\delta^2} + \frac{D^* P_n B}{\delta^2}\left(I + B^* P_n B\right)^{-1} B^* P_n D\right)^{-1} \\
&\quad \left(\frac{D^*}{\delta}\left(I - P_n B (I + B^* P_n B)^{-1} B^*\right) P_n F\right),
\end{aligned} \tag{25}$$

where the inverse of the above operators will be established in the next proposition. For each $n \ge 0$ define the operators $\widehat{X}_n \in B[\mathcal{X}_0, \mathcal{X}]$, $\widehat{Z}_n \in$

$B[\mathcal{X}_0, \mathcal{Z}]$ and $T_n \in B[\mathcal{X}_0, \mathcal{S}_n]$ as follows (see Remark 3 below): for every $x_0 \in \mathcal{X}_0$,

$$\widehat{x}_{ni+1} = \left(A_0 + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle \right) \widehat{x}_{ni} + D\widehat{v}_{ni} + B\widehat{u}_{ni}, \quad \widehat{x}_{n0} = x_0, \quad i \geq 0$$

$$\widehat{u}_{ni} = \begin{cases} -K_{un-i}\widehat{x}_{ni} & \text{if } n-i-1 \geq 0 \\ -K_L \widehat{x}_{ni} & \text{otherwise} \end{cases} \qquad i \geq 0$$

$$\widehat{v}_{ni} = \begin{cases} \frac{1}{\delta} K_{vn-i}\widehat{x}_{ni} & \text{if } n-i-1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad i \geq 0$$

$$\widehat{z}_{ni} = \begin{bmatrix} M^{1/2}\widehat{x}_{ni} \\ \widehat{u}_{ni} \end{bmatrix} \qquad i \geq 0$$

and $T_n(x_0) = \widehat{\mathbf{v}}_n = (\widehat{v}_{n0}, \widehat{v}_{n1}, ...)$, $\widehat{X}_n(x_0) = (\widehat{x}_{n0}, \widehat{x}_{n1}, ...)$, $\widehat{Z}_n(x_0) = (\widehat{z}_{n0}, \widehat{z}_{n1}, ...)$ (note that for $i \geq n$, $\widehat{v}_{ni} = 0$ and $\widehat{u}_{ni} = -K_L \widehat{x}_{ni}$ and since $r((\mathcal{F}^{\#}_{BK_L})) < 1$, it follows that the linear operators T_n, $\widehat{X}_n$ and $\widehat{Z}_n$ are indeed bounded).

Proposition 13: *Consider the hypothesis of Lemma* 2*. For each* $n \geq 0$*,*

(a) $P_n \in B^+[H]$,

(b) $I - \frac{D^* P_n D}{\delta^2} \geq \frac{\alpha^2}{\delta^2} I$ (α defined in (19)),

(c) $\widehat{X}_n(x_0) = \widetilde{X}(x_0, \widehat{\mathbf{v}}_n) = X_{BK_L}(x_0, \widehat{v}_n, G\widehat{v}_n)$,
$\widehat{Z}_n(x_0) = \tilde{Z}_n(x_0, \widehat{\mathbf{v}}_n) = Z_{BK_L}(x_0, \widehat{v}_n, G\widehat{v}_n)$,

(d) $\widehat{J}_n(x_0) = \|P_n^{1/2} x_0\|^2_{\mathcal{H}} = \|\widehat{Z}_n(x_0)\|^2_{\mathcal{Z}} - \delta^2 \|T_n(x_0)\|^2_{\mathcal{V}}$
$= \|\widetilde{Z}(x_0, T_n(x_0))\|^2_{\mathcal{Z}} - \delta^2 \|T_n(x_0)\|^2_{\mathcal{V}}$.

Remark 3: Statement (d) says that $T_n(x_0)$ provides the maximizing disturbance of the optimization problem defined by $\widehat{J}_n(x_0)$ with $\widehat{J}_n(x_0) = \|P_n^{1/2} x_0\|^2_{\mathcal{H}}$, while statement (c) says that $\widehat{X}_n(x_0)$ and $\widehat{Z}_n(x_0)$ give the state sequence and output sequence when the maximizing disturbance $T_n(x_0)$ for $\mathbf{v}$ and the minimizing control $G(T_n(x_0))$ for $\mathbf{q}$ are used.

Proof: The proof goes by induction on n.

For $n = 0$ we get $P_0 = L \in B^+[H]$, $T_0(x_0) = 0$, $\widehat{X}_0(x_0) = \widetilde{X}(x_0, \mathbf{0}) = X_{BK_L}(x_0, \mathbf{0}, \mathbf{0})$, $\widehat{Z}_0(x_0) = \widetilde{Z}(x_0, \mathbf{0}) = Z_{BK_L}(x_0, \mathbf{0}, \mathbf{0})$ and, from (17), it follows that for every $x_0 \in \mathcal{X}_0$,

$$\widehat{J}_0(x_0) = \|\widetilde{Z}(x_0, \mathbf{0})\|^2_{\mathcal{Z}} = \|Z_{BK_L}(x_0, \mathbf{0}, \mathbf{0})\|^2_{\mathcal{Z}} = \|L^{1/2}x_0\|^2_{\mathcal{H}}$$

showing (a), (c) and (d). For any $v_0 \in \mathcal{V}_0$ consider the sequence $\mathbf{v} = (v_0, 0, 0, ...) \in \mathcal{V}$ and note that

$$\begin{aligned}\|Z^0_{BK_L}(\mathbf{v})\|^2_{\mathcal{Z}} &= \|Z_{BK_L}(Dv_0, \mathbf{0}, \mathbf{0})\|^2_{\mathcal{Z}} = \|\widetilde{Z}(Dv_0, \mathbf{0})\|^2_{\mathcal{Z}} \\ &= \|L^{1/2}Dv_0\|^2_{\mathcal{H}} = \langle D^*LDv_0; v_0\rangle_{\mathcal{H}'},\end{aligned}$$

$$\|Z^0_{BK}(\mathbf{v})\|^2_{\mathcal{Z}} = \|Z_{BK}(Dv_0, \mathbf{0}, \mathbf{0})\|^2_{\mathcal{Z}} = \|\bar{Z}(Dv_0, \mathbf{0})\|^2_{\mathcal{Z}},$$

so that, from Proposition 10, we obtain

$$\|\widetilde{Z}(Dv_0, \mathbf{0})\|^2_{\mathcal{Z}} \le \|\bar{Z}(Dv_0, \mathbf{0})\|^2_{\mathcal{Z}}$$

and from (19) it follows that

$$\begin{aligned}&\langle D^*LDv_0; v_0\rangle_{\mathcal{H}'} - \delta^2\|v_0\|^2_{\mathcal{H}'} = \\ &\quad = \|\widetilde{Z}(Dv_0, 0)\|^2_{\mathcal{Z}} - \delta^2\|v_0\|^2_{\mathcal{H}'} \le \|\tilde{Z}(Dv_0, 0)\|^2_{\mathcal{Z}} - \delta^2\|v_0\|^2_{\mathcal{H}'} \\ &\quad = \|Z^0_{BK}(\mathbf{v})\|^2_{\mathcal{Z}} - \delta^2\|\mathbf{v}\|^2_{\mathcal{V}} \le -\alpha^2\|\mathbf{v}\|^2_{\mathcal{V}} = -\alpha^2\|v_0\|^2_{\mathcal{H}'}\end{aligned}$$

which shows that, for every $v_0 \in H' \subset \mathcal{V}_0$,

$$\left\langle \left(I - \frac{D^*LD}{\delta^2}\right) v_0; v_0 \right\rangle \ \ge\ \frac{\alpha^2}{\delta^2}\|v_0\|^2.$$

Now we shall verify that, if the assertion holds for n, then it holds for $n+1$. Suppose the results holds for n. Then, for any $x_0 \in \mathcal{X}_0$ (see [1], Theorems 2.4 and 3.1),

$$\widehat{J}_{n+1}(x_0) = \sup_{v_0 \in \mathcal{V}_0} \inf_{q_0 \in \mathcal{U}_0} \left(\|z_0\|^2_{\mathcal{H}'''} - \delta^2\|v_0\|^2_{\mathcal{H}'} + \|P_n^{1/2}x_1\|^2_{\mathcal{H}}\right), \tag{26}$$

where $u_0 = -K_Lx_0 + q_0$ and x_1 is as in (1). Set

$$\begin{aligned}W_n &= (I + B^*P_nB) \in G^+[H''] \\ N_{n+1} &= (I + B^*P_nB)^{-1}B^*P_nF = W_n^{-1}B^*P_nF \\ P'_{n+1} &= \mathcal{F}^{\#}(P_n) + M - N^*_{n+1}(I + B^*P_nB)N_{n+1}\end{aligned}$$

$$\begin{aligned}\widehat{u}_0(x_0,v_0) &= W_n^{-1}(B^*P_nFx_0 + B^*P_nDv_0)\\ &= N_{n+1}x_0 + W_n^{-1}B^*P_nDv_0, \qquad x_0\in\mathcal{X}_0,\ \ v_0\in\mathcal{V}_0 \end{aligned} \tag{27}$$

and therefore

$$W_n\widehat{u}_0(x_0,v_0) = B^*P_nFx_0 + B^*P_nDv_0.$$

For simplicity we set $\widehat{u}_0 = \widehat{u}_0(x_0,v_0)$. A trivial but somewhat lengthy algebraic manipulation, similar to the one in the proof of Proposition 5, leads to

$$\begin{aligned}&\|P_n^{1/2}x_1\|^2_{\mathcal{H}} - \langle P'_{n+1}x_0;x_0\rangle_{\mathcal{H}}\\ &= -\left(\|M^{1/2}x_0\|^2_{\mathcal{H}} + \|u_0\|^2_{\mathcal{H}''}\right) + \langle D^*P_nFx_0;v_0\rangle_{\mathcal{H}'} + \langle v_0;D^*P_nFx_0\rangle_{\mathcal{H}'}\\ &+ \langle D^*P_nDv_0;v_0\rangle_{\mathcal{H}'} + \langle W_n(\widehat{u}_0+u_0);(\widehat{u}_0+u_0)\rangle_{\mathcal{H}''}\\ &+ \langle (D^*P_nBW_n^{-1}B^*P_nD)v_0;v_0)\rangle_{\mathcal{H}'}\\ &- \langle (B^*P_nFx_0 + B^*P_nDv_0);W_n^{-1}B^*P_nDv_0\rangle_{\mathcal{H}''}\\ &- \langle W_n^{-1}B^*P_nDv_0;(B^*P_nFx_0+B^*P_nDv_0)\rangle_{\mathcal{H}''}.\end{aligned}$$

Defining

$$U_n = D^*\left(I - P_nB(I+B^*P_nB)^{-1}B^*\right)P_nF \tag{28}$$

and

$$E_n = \delta^2\left(I - \frac{D^*P_nD}{\delta^2} + \frac{D^*P_nB}{\delta^2}\left(I+B^*P_nB\right)^{-1}B^*P_nD\right) \in G^+[H'] \tag{29}$$

we get

$$\begin{aligned}&\|P_n^{1/2}x_1\|^2_{\mathcal{H}} + \left(\|M^{1/2}x_0\|^2_{\mathcal{H}} + \|u_0\|^2_{\mathcal{H}''} - \delta^2\|v_0\|^2_{\mathcal{H}'}\right)\\ &= \langle P'_{n+1}x_0;x_0\rangle_{\mathcal{H}} - \|E_n^{1/2}v_0\|^2_{\mathcal{H}'} + \langle E_n(E_n^{-1}U_nx_0);v_0\rangle_{\mathcal{H}'}\\ &+ \langle v_0;E_n(E_n^{-1}U_nx_0)\rangle_{\mathcal{H}'} + \langle E_n(E_n^{-1}U_nx_0);(E_n^{-1}U_nx_0)\rangle_{\mathcal{H}'}\\ &- \langle E_n(E_n^{-1}U_nx_0);(E_n^{-1}U_nx_0)\rangle_{\mathcal{H}'} + \langle W_n(\widehat{u}_0+u_0);(\widehat{u}_0+u_0)\rangle_{\mathcal{H}''}\\ &= \langle\left(P'_{n+1} + U_n^*E_n^{-1}U_n\right)x_0;x_0\rangle_{\mathcal{H}} + \|W_n^{1/2}(\widehat{u}_0+u_0)\|^2_{\mathcal{H}''}\\ &- \|E_n^{1/2}\left(v_0 - E_n^{-1}U_nx_0\right)\|^2_{\mathcal{H}'}.\end{aligned}$$

From the above expressions it follows that the minimax problem defined in (26) is solved with:

$$q_0 = -\widehat{u}_0 + K_Lx_0 = -(I+B^*P_nB)^{-1}B^*P_n(Fx_0+Dv_0) + K_Lx_0$$

$$v_0 = E_n^{-1} U_n x_0 = \frac{1}{\delta} K_{vn+1} x_0;$$

and thus,

$$q_0 = K_L x_0 - \left((I + B^* P_n B)^{-1} B^* P_n F_{\frac{1}{\delta}\ D(-K_{vn+1})} \right) x_0 = (K_L - K_{un+1}) x_0.$$

Moreover, for any $x_0 \in \mathcal{X}_0$,

$$\widehat{J}_{n+1}(x_0) = \langle \left(P'_{n+1} + U_n^* E_n^{-1} U_n \right) x_0; x_0 \rangle_{\mathcal{H}} = \langle P_{n+1} x_0; x_0 \rangle_{\mathcal{H}} \geq 0$$

where, according to (25), (27), (28), (29), $P_{n+1} \geq 0$ is given by

$$\begin{aligned} P_{n+1} &= P'_{n+1} + U_n^* E_n^{-1} U_n \\ &= \mathcal{F}^{\#}(P_n) + M - N_{n+1}^* \left(I + B^* P_n B \right) N_{n+1} \\ &+ K_{vn+1}^* \left(I - \frac{D^* P_n D}{\delta^2} \right) K_{vn+1} \\ &+ K_{vn+1}^* \frac{D^* P_n B}{\delta} \left(I + B^* P_n B \right)^{-1} \frac{B^* P_n D}{\delta} K_{vn+1}. \end{aligned}$$

Thus,

$$\begin{aligned} &P_n - \mathcal{F}^{\#}(P_n) + N_{n+1}^* \left(I + B^* P_n B \right) N_{n+1} \\ &\quad = M + P_n - P_{n+1} + K_{vn+1}^* \left(I - \frac{D^* P_n D}{\delta^2} \right) K_{vn+1} \\ &\quad + K_{vn+1}^* \frac{D^* P_n B}{\delta} \left(I + B^* P_n B \right)^{-1} \frac{B^* P_n D}{\delta} K_{vn+1} \end{aligned}$$

and (23) and Proposition 3, (c)$\Rightarrow (b)$, lead to

$$\begin{aligned} &P_n - \mathcal{F}^{\#}_{BK_{un+1}}(P_n) + \left(N_{n+1} - K_{un+1} \right)^* \left(I + B^* P_n B \right) \left(N_{n+1} - K_{un+1} \right) \\ &= P_n - \mathcal{F}^{\#}_{BK_{un+1}}(P_n) + K_{vn+1}^* \frac{D^* P_n B}{\delta} \left(I + B^* P_n B \right)^{-1} \frac{B^* P_n D}{\delta} K_{vn+1} \\ &= M + P_n - P_{n+1} + K_{un+1}^* K_{un+1} + K_{vn+1}^* \left(I - \frac{D^* P_n D}{\delta^2} \right) K_{vn+1} \\ &+ K_{vn+1}^* \frac{D^* P_n B}{\delta} \left(I + B^* P_n B \right)^{-1} \frac{B^* P_n D}{\delta} K_{vn+1}; \end{aligned}$$

that is,

$$\begin{aligned} P_n - \mathcal{F}^{\#}_{BK_{un+1}}(P_n) + K_{vn+1}^* \left(\frac{D^* P_n D}{\delta^2} - I \right) K_{vn+1} \\ = M + P_n - P_{n+1} + K_{un+1}^* K_{un+1} \end{aligned}$$

and from (24) and Proposition 3, (c)$\Rightarrow$ (a), it follows that

$$P_{n+1} - (\mathcal{F}^{\#}_{BK_{un+1}})_{\frac{1}{\delta}\ D(-K_{vn+1})}(P_n) = M + K^*_{un+1}K_{un+1} - K^*_{vn+1}K_{vn+1}$$

which shows that P_{n+1} satisfies (22), proving (a), (c), (d) for $n+1$. Consider now (16) with $x_0 = 0$ and $\mathbf{v} = (v_0, v_1, ...) \in \mathcal{V}$, $\mathbf{q} = (q_0, q_1, ...) \in \mathcal{U}$ given by: $q_0 = 0, v_0 \in \mathcal{V}_0$,

$$q_i = \begin{cases} (K_L - K_{un+2-i})x_i & \text{if } \ n+1-i \geq 0, \\ 0 & \text{otherwise} \end{cases} \qquad i \geq 1$$

$$v_i = \begin{cases} \frac{1}{\delta}K_{vn+2-i}x_i & \text{if } \ n+1-i \geq 0, \\ 0 & \text{otherwise} \end{cases} \qquad i \geq 1$$

In other words, at time $i = 0$, we take $x_0 = 0$ and consider any disturbance $v_0 \in \mathcal{V}_0$, and for $i \geq 1$, we consider the maximizing disturbance $T_{n+1}(Dv_0))$ for the optimization problem $\widehat{J}_{n+1}(Dv_0)$ and the minimizing control $G(T_{n+1}(Dv_0))$ (note that this idea had already been applied for $n = 0$ above). We get

$$\begin{aligned} &\|Z_{BK_L}(0, \mathbf{v}, \mathbf{q})\|^2_{\mathcal{Z}} - \delta^2\|\mathbf{v}\|^2_{\mathcal{V}} \\ &\quad = \|\widehat{Z}_{n+1}(Dv_0)\|^2_{\mathcal{Z}} - \delta^2\|T_{n+1}(Dv_0)\|^2_{\mathcal{V}} - \delta^2\|v_0\|^2_{\mathcal{H}'} \\ &\quad = \langle P_{n+1}(Dv_0); (Dv_0)\rangle_{\mathcal{H}} - \delta^2\|v_0\|^2_{\mathcal{H}'} \end{aligned}$$

and from Proposition 10,

$$\begin{aligned} &\|\widehat{Z}_{n+1}(Dv_0)\|^2_{\mathcal{Z}} - \delta^2\|T_{n+1}(Dv_0)\|^2_{\mathcal{V}} \\ &\quad = \|\widetilde{Z}(Dv_0, T_n(Dv_0))\|^2_{\mathcal{Z}} - \delta^2\|T_n(Dv_0)\|^2_{\mathcal{V}} \\ &\quad \leq \|\bar{Z}(Dv_0, T_n(Dv_0))\|^2_{\mathcal{Z}} - \delta^2\|T_n(Dv_0)\|^2_{\mathcal{V}}. \end{aligned}$$

Equation (19) leads to

$$\begin{aligned} &\|\bar{Z}(Dv_0, T_n(Dv_0))\|^2_{\mathcal{Z}} - \delta^2\|T_n(Dv_0)\|^2_{\mathcal{V}} - \delta^2\|v_0\|^2_{\mathcal{H}'} \\ &\quad = \|\bar{Z}(0, \mathbf{v})\|^2_{\mathcal{Z}} - \delta^2\|\mathbf{v}\|^2_{\mathcal{V}} \leq -\alpha^2\|\mathbf{v}\|^2_{\mathcal{V}} \leq -\alpha^2\|v_0\|^2_{\mathcal{H}'} \end{aligned}$$

and thus, for every $v_0 \in H'$,

$$\langle (D^*P_{n+1}D)v_0; v_0\rangle - \delta^2\|v_0\|^2 \leq -\alpha^2\|v_0\|^2,$$

which implies that

$$\left\langle \left(I - \frac{1}{\delta^2}D^*P_{n+1}D\right)v_0; v_0 \right\rangle \ \geq \ \frac{\alpha^2}{\delta^2}\|v_0\|^2$$

concluding the induction and the proof of the Proposition. □

We can now proceed to the proof of Lemma 2.

Proof of Lemma 2: Consider P_n and P as defined in (22) and (21) respectively. We shall show that:

(I) $P_n \uparrow P$ strongly as $n \to \infty$ and P satisfies conditions (i) and (ii) of Theorem,

(II) By setting $\widehat{X}(x_0) = \widehat{\mathbf{x}} = (\widehat{x}_0, \widehat{x}_1, \widehat{x}_2, ...)$, $\widehat{x}_0 = x_0$, we shall prove that $Tx_0 = \widehat{\mathbf{v}} = (\widehat{v}_0, \widehat{v}_1, \widehat{v}_2, ...) = (\frac{K_v}{\delta}\widehat{x}_0, \frac{K_v}{\delta}\widehat{x}_1, \frac{K_v}{\delta}\widehat{x}_2, ...) = \frac{K_v}{\delta}\widehat{\mathbf{x}}$, and $G\widehat{\mathbf{v}} = \mathbf{q} = (\widehat{q}_0, \widehat{q}_1, \widehat{q}_2, ...) = ((K_L - K_u)\widehat{x}_0, (K_L - K_u)\widehat{x}_1, (K_L - K_u)\widehat{x}_2, ...) = (K_L - K_u)\widehat{\mathbf{x}}$, and

(III) P satisfies condition (iii) of Theorem.

Lemma 2 will follow from (I) and (III). Let us prove (I) first. From Propositions 12 and 13 we have, for each $x \in H$,

$$0 \leq \widehat{J}_n(x) = \langle P_n x; x\rangle \leq \widehat{J}_{n+1}(x) = \langle P_{n+1} x; x\rangle \leq \widehat{J}(x) = \langle Px; x\rangle$$

which shows that $0 \leq P_n \leq P_{n+1} \leq P$. Thus $\{P_n;\ n \geq 0\}$ is a bounded monotonic nondecreasing sequence of nonnegative operators, so that it converges strongly to an operator $P_\infty \in B^+[H]$ (see e.g. [21, p.79]). From Proposition 12 it follows that, for any $x \in H$,

$$\langle P_\infty x; x\rangle = \lim_{n\to\infty} \langle P_n x; x\rangle = \langle Px; x\rangle$$

and thus (cf. [16, p.374]) $P = P_\infty$. Since for every $v_0 \in H'$

$$\left\langle \left(I - \frac{1}{\delta^2} D^* P_n D\right) v_0; v_0 \right\rangle \ \geq\ \frac{\alpha^2}{\delta^2}\|v_0\|^2$$

we get

$$\left\langle \left(I - \frac{1}{\delta^2} D^* P D\right) v_0; v_0 \right\rangle \ \geq\ \frac{\alpha^2}{\delta^2}\|v_0\|^2$$

which shows that $\left(I - \frac{1}{\delta^2} D^* P D\right) \geq \frac{\alpha^2}{\delta^2}\ I$. By repeating the same arguments as in the proof of Lemma in [3] it follows that, as $n \to \infty$,

$$K_{un} \xrightarrow{s} K_u, \qquad K_{un}^* \xrightarrow{s} K_u^*, \qquad K_{un}^* K_{un} \xrightarrow{s} K_u^* K_u$$

$$K_{vn} \xrightarrow{s} K_v, \qquad K_{vn}^* \xrightarrow{s} K_v^*, \qquad K_{vn}^* K_{vn} \xrightarrow{s} K_v^* K_v$$

$$(\mathcal{F}^{\#}_{BK_{un+1}})_{\frac{1}{\delta}\ D(-K_{vn+1})}(P_n) \xrightarrow{s} (\mathcal{F}^{\#}_{BK_u})_{\frac{1}{\delta}\ D(-K_v)}(P).$$

Thus

$$P = M + (\mathcal{F}^{\#}_{BK_u})_{\frac{1}{\delta}\,D(-K_v)}(P) + K_u^* K_u - K_v^* K_v$$

showing (I). Since $\widehat{\mathbf{v}}_n = T_n(x_0) \in \mathcal{S}_n \subset \mathcal{V}$ is a maximizing sequence for $\sup_{\mathbf{v}\in\mathcal{V}} \widetilde{J}(x_0, \mathbf{v}) = \widehat{J}(x_0) = \widetilde{J}(x_0, \widehat{\mathbf{v}})$ and $\widehat{\mathbf{v}}$ is unique (Proposition 11) we conclude, from the same arguments as in the proof of Proposition 3 in [19], that $\widehat{\mathbf{v}}_n \to \widehat{\mathbf{v}}$ as $n \to \infty$. From continuity of the operator $\widetilde{X}$ it follows that, as $n \to \infty$,

$$\widehat{X}_n(x_0) = (\widehat{x}_{n0}, \widehat{x}_{n1}, ...) = \widetilde{X}(x_0, \widehat{\mathbf{v}}_n) \to \widetilde{X}(x_0, \widehat{\mathbf{v}}) = \widehat{X}(x_0) = (\widehat{x}_0, \widehat{x}_1, ...)$$

which implies, in particular, that $\widehat{x}_{ni} \to \widehat{x}_i$ as $n \to \infty$ for each $i \geq 0$. From strong convergence of K_{vn-i} to K_v in $B[\mathcal{H}, \mathcal{H}']$ (implied from strong convergence in $B[H, H']$, see [3]) as $n \to \infty$, we get for each $i \geq 0$,

$$K_{vn-i}\widehat{x}_{ni} \to K_v \widehat{x}_i \qquad \text{as} \qquad n \to \infty,$$

(reason: if $T_n \xrightarrow{s} T$ and $x_n \to x$, then $0 \leq \|T_n x_n - Tx\| \leq \|T_n x_n - T_n x\| + \|T_n x - Tx\| \leq \sup_{m\geq 0} \|T_m\| \, \|x_n - x\| + \|(T_n - T)x\| \to 0$ as $n \to \infty$) and thus,

$$\widehat{\mathbf{v}} = Tx_0 = \left(\frac{K_v}{\delta}\widehat{x}_0, \frac{K_v}{\delta}\widehat{x}_1, ... \right) = \frac{K_v}{\delta}\widehat{\mathbf{x}}.$$

From continuity of G we get that $G\widehat{\mathbf{v}}_n \to G\widehat{v} = \mathbf{q}$ and, by repeating the same arguments as above we conclude that, for each $i \geq 0$,

$$(K_L - K_{un-i})\widehat{x}_{ni} \to (K_L - K_u)\widehat{x}_i \qquad \text{as} \qquad n \to \infty,$$

which shows that

$$\widehat{\mathbf{q}} = ((K_L - K_u)\widehat{x}_0, (K_L - K_u)\widehat{x}_1, ...) = (K_L - K_u)\widehat{\mathbf{x}}$$

proving (II). Finally note that, for any $x_0 \in \mathcal{X}_0$, $\widehat{X}(x_0) = (\widehat{x}_0, \widehat{x}_1, ...) \in \mathcal{X}$ where

$$\begin{aligned}
\widehat{x}_{i+1} &= \left(A_0 + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle \right) \widehat{x}_i + D\widehat{v}_i + B(\widehat{q}_i - K_L \widehat{x}_i) \\
&= \left(A_0 + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle \right) \widehat{x}_i + D\frac{K_v}{\delta}\widehat{x}_i - BK_u \widehat{x}_i \\
&= \left(A_0 - BK_u + D\frac{K_v}{\delta} + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle \right) \widehat{x}_i, \qquad \widehat{x}_0 = x_0
\end{aligned}$$

and condition (iii) of Theorem follows from Proposition 1, showing (III).

□

Appendix

In this appendix we present the construction of the probability space (Ω, Σ, μ), $\mathcal{X}$, $\mathcal{X}_n$, $\mathcal{V}$, $\mathcal{V}_n$, $\mathcal{U}$, $\mathcal{U}_n$, and $\mathcal{Z}$, $\mathcal{Z}_n$ which lead to the independence conditions required in section 3. Let $(\bar{\Omega}, \bar{\Sigma}, \bar{\mu})$ be a probability space. For any family of subsets $\mathcal{G}$ of $\bar{\Omega}$ we denote by $\sigma(\mathcal{G})$ the σ- field generated by $\mathcal{G}$. We denote by $\prod_{i=0}^{n} \bar{\Omega}$ the product space formed by $\bar{\Omega}$ (n possibly ∞) and

$$\prod_{i=0}^{n} \bar{\Sigma} = \sigma\{E_0 \times ... \times E_n; \qquad E_i \in \bar{\Sigma}, i = 0, ..., n\} \quad \text{if} \quad n < \infty$$

$$\prod_{i=0}^{\infty} \bar{\Sigma} = \sigma \left\{ E_0 \times ... \times E_k \times \prod_{j=k+1}^{\infty} \bar{\Omega}; \quad E_i \in \bar{\Sigma}, i = 0, ..., k, k = 0, 1, ... \right\}$$

if $n = \infty$. Then (cf. [20]) there exists a unique probability measure μ defined on $\prod_{i=0}^{\infty} \bar{\Sigma}$ such that

$$\mu(E_0 \times ... \times E_k \times \prod_{j=k+1}^{\infty} \bar{\Omega}) = \mu_0(E_0)...\mu_k(E_k)$$

for $E_i \in \bar{\Sigma}$, $i = 0, ..., k$, $k = 0, 1, ...$ Set $\Omega = \prod_{i=0}^{\infty} \bar{\Omega}$, $\Sigma = \prod_{i=0}^{\infty} \Sigma$ and, for every $n \geq 0$,

$$\begin{aligned} \Sigma_n &= \sigma \left\{ E_0 \times ... \times E_n \times \prod_{j=n+1}^{\infty} \bar{\Omega}; \qquad E_i \in \bar{\Sigma}, i = 0, ..., n \right\} \\ &= \sigma \left\{ A \times \prod_{j=n+1}^{\infty} \bar{\Omega}; \qquad A \in \prod_{i=0}^{n} \bar{\Sigma} \right\} \subset \Sigma \end{aligned}$$

with μ the unique probability measure defined on Σ with the property seen above. This defines the probability space (Ω, Σ, μ) of section 3. Consider $w \in \bar{\mathcal{H}} = L_2(\bar{\Omega}, \bar{\Sigma}, \ \bar{\mu}; H)$ and define the stationary sequence of random variables $\{w_i \in \mathcal{H}; \quad i \geq 0\}$ (see notation in section 3) as follows: for $\omega = (\bar{\omega}_0, \bar{\omega}_1, ...) \in \Omega$, set $w_i(\omega) = w(\bar{\omega}_{i+1})$, $i \geq 0$. In this way, it is readily verifiable that $\{w_i; \ i \geq 0\}$ is an independent stationary random sequence. Define $\mathcal{X} \subset l_2(\mathcal{H})$ in the following way: $\mathbf{x} = (x_0, x_1, ...)$ belongs to $\mathcal{X}$ if $\mathbf{x} \in l_2(\mathcal{H})$ (i.e, $x_i \in \mathcal{H}$, $i \geq 0$, and $\|\mathbf{x}\|^2_{l_2(\mathcal{H})} = \sum_{i=0}^{\infty} \mathcal{E}(\|x_i\|^2) < \infty$) and, for each $i \geq 0$, $x_i \in L_2(\Omega, \Sigma_i, \mu; H)$. We have that $\mathcal{X}$ is a closed linear subspace of $l_2(\mathcal{H})$ and therefore a Hilbert space. In a similar way we define the Hilbert spaces $\mathcal{V} \subset l_2(\mathcal{H}')$,$\mathcal{U} \subset l_2(\mathcal{H}'')$ and $\mathcal{Z} \subset l_2(\mathcal{H}''')$ by replacing H and $\mathcal{H}$ in the definition of $\mathcal{X}$ by H' and $\mathcal{H}'$, H'' and $\mathcal{H}''$, and H''' and $\mathcal{H}'''$ respectively. From these definitions it is easy to verify that the independence properties of section 3 are satisfied. Finally we say that $\mathbf{x}_n = (x_0, x_1, ..., x_n) \in \mathcal{X}_n$ if $x_i \in L_2(\Omega, \Sigma_i, \mu; H)$ for each $i = 0, 1, ..., n$. The definitions of $\mathcal{V}_n$, $\mathcal{U}_n$ and $\mathcal{Z}_n$ are made in a similar way. Again it is easy to verify from the above construction the independence properties of section 3.

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