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CONVERGENCE OF POWER SEQUENCES OF OPERATORS VIA THEIR STABILITY

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Abstract. This paper is concerned with the convergence of power sequences and stability of Hilbert space operators, where "convergence" and "stability" are considered with respect to weak, strong and norm topologies. It is proved that an operator has a convergent power sequence if and only if it is a (not necessarily orthogonal) direct sum of an identity operator and a stable operator. This reduces the issue of convergence of the power sequence of an operator T to the study of stability of T. The question of when the limit of the power sequence is an orthogonal projection is investigated. Among operators sharing this property are hyponormal and contractive ones. In particular, a hyponormal or a contractive operator with no identity part is stable if and only if its power sequence is convergent. In turn, a unitary operator has a weakly convergent power sequence if and only if its singular-continuous part is weakly stable and its singular-discrete part is the identity. Characterizations of the convergence of power sequences and stability of subnormal operators are given in terms of semispectral measures.

1. Introduction

The notion of operator stability is linked to a discrete, time-invariant, free, linear dynamical system modelled by the autonomous homogeneous difference equation:

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots, \tag{1.1}$$

with the initial condition $x_0 = x \in \mathcal{X}$, where $\mathcal{X}$ is a normed space, x is a vector in $\mathcal{X}$ and T is a bounded linear operator on $\mathcal{X}$, whose solution is given by the formula

$$x_n = T^n x, \quad n = 0, 1, 2, \dots .$$

The above discrete system is asymptotically stable if $\{T^n x\}_{n=1}^{\infty}$ converges to zero for every initial condition x. Of course, the meaning of the term "convergence" depends on the topology with which $\mathcal{X}$ is equipped. For a given operator T, the operator-valued power sequence $\{T^n\}_{n=1}^{\infty}$ of T may converge to zero, in the sense that the $\mathcal{X}$-valued sequence $\{T^n x\}_{n=1}^{\infty}$ converges to zero for every x, either in the weak or norm topology of $\mathcal{X}$, or in the uniform norm topology of the corresponding operator space, giving rise to the notions of weak, strong, and uniform stability for the operator T. (These notions will be clarified in Section 2.) Thus, weak stability refers to the weakest way in which the linear discrete system (1.1) approaches asymptotically zero for all initial conditions. In infinite dimensions (which is our

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case), this concept has a continuous counterpart linked to mild solutions of certain partial differential equations. We refer the reader to [36] for a beautiful presentation of the limit properties of discrete-time distributed parameter systems and to [10] for the general theory of discrete time-invariant linear systems.

The purpose of this paper is to study the convergence of power sequences and stability of operators acting on complex Hilbert spaces. Convergence and stability refer to weak, strong and norm topologies. One of the main results of the paper, Theorem 3.2, states, inter alia, the following.

- 0° *An operator has a weakly* (*strongly, norm*) *convergent power sequence if and only if it is a* (*not necessarily orthogonal*) *direct sum of two operators, the first of which is the identity on a precisely described space, and the second is weakly* (*strongly, uniformly*) *stable.*

In other words, the issue of weak (strong, norm) convergence of power sequences reduces to the study of weak (strong, uniform) stability. As an immediate consequence, we obtain that an operator is weakly (strongly, uniformly) stable if and only if its power sequence converges weakly (strongly, in norm) and 1 is not its eigenvalue (see [21, Theorem 1]). In fact, operators whose power sequences are weakly (strongly, norm) convergent are completely characterized by the similarity to operators whose power sequences converge weakly (strongly, in norm) to orthogonal projections (see Corollary 3.4). For this reason, a significant part of the paper is devoted to the situation when the weak (strong, norm) limit of the power sequence of an operator is an orthogonal projection (see Section 4). Below are some important excerpts from this article.

- 1° *A hyponormal or a contractive operator with no identity part is weakly* (*strongly, uniformly*) *stable if and only if its power sequence converges weakly* (*strongly, in norm*) (see Corollary 4.5).
- 2° *If the power sequence of an operator T with $\liminf_{n\to\infty} \|T^n\| \leqslant 1$ is weakly convergent, then its limit is an orthogonal projection which commutes with T* (see Theorem 4.12).
- 3° *A unitary operator has a weakly convergent power sequence if and only if its singular-continuous part is weakly stable and its singular-discrete part is the identity operator; each part may be absent* (see Theorem 5.5).
- 4° *Characterizations of the weak* (*strong, uniform*) *convergence of power sequences and the corresponding stability of subnormal operators are stated in terms of their semispectral measures* (see Section 6).

The present paper was partially inspired by [4] and [16]. All concepts used above will be defined in the subsequent sections. The paper is organized as follows. Basic notation and terminology are summarized in Section 2. The weak (strong, norm) convergence of power sequences and the corresponding stability of general operators are studied in Section 3. In Section 4, we investigate the question of when the weak limit of the power sequence of an operator is an orthogonal projection. In particular, regarding 2°, we give an example of a weakly stable operator T such that $\liminf_{n\to\infty} \|T^n\| = 1$ and $\limsup_{n\to\infty} \|T^n\|$ takes a predetermined numerical value from the open interval $(1, \infty)$ (see Example 4.14). The special case of unitary operators is treated in Section 5. The weak (strong, uniform) convergence of power sequences and the related stability of subnormal operators is investigated in Section 6

2. Notation and Terminology

We write $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. By an operator T on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \text{-} \rangle$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. Let $\mathcal{B}(\mathcal{H})$ stand for the C^*-algebra of all operators on $\mathcal{H}$. We write I for the identity operator on $\mathcal{H}$. We use the same symbol $\|\cdot\|$ for the norm on $\mathcal{H}$ and for the induced operator norm on $\mathcal{B}(\mathcal{H})$. Let $T \in \mathcal{B}(\mathcal{H})$. The kernel and the range of T are denoted by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively. We write $\sigma(T)$, $r(T)$ and $w(T)$ for the spectrum, the spectral radius and the numerical radius of T, respectively. The operator T is a *normaloid* if $r(T) = \|T\|$, a *contraction* (or a *contractive operator*) if $\|T\| \leqslant 1$, a *strict contraction* if $\|T\| < 1$, and *power bounded* if $\sup_{n\geqslant 1} \|T^n\| < \infty$.

The power sequence $\{T^n\}_{n=1}^{\infty}$ of $T \in \mathcal{B}(\mathcal{H})$ converges *weakly* if the $\mathcal{H}$-valued sequence $\{T^n x\}_{n=1}^{\infty}$ converges weakly for every $x \in \mathcal{H}$, that is, the complex valued sequence $\{\langle T^n x, y\rangle\}_{n=1}^{\infty}$ converges for all $x, y \in \mathcal{H}$. Using the uniform boundedness principle and the Riesz representation theorem one can verify that $\{T^n\}_{n=1}^{\infty}$ converges weakly if and only if there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\{T^n x\}_{n=1}^{\infty}$ converges weakly to Ax for every $x \in \mathcal{H}$ (notation: $T^n x \xrightarrow{w} Ax$), or equivalently that $\langle T^n x, y\rangle \to \langle Ax, y\rangle$ as $n \to \infty$ for all $x, y \in \mathcal{H}$. With the above discussion in mind, we are ready to define the key concepts of this paper. Let $T, A \in \mathcal{B}(\mathcal{H})$. We say that the power sequence $\{T^n\}_{n=1}^{\infty}$ of T

- *converges weakly* to A (notation: $T^n \xrightarrow{w} A$) if $\langle T^n x, y\rangle \to \langle Ax, y\rangle$ as $n \to \infty$ for all $x, y \in \mathcal{H}$,
- *converges strongly* to A if $\{T^n x\}_{n=1}^{\infty}$ converges to Ax in norm as $n \to \infty$ for every $x \in \mathcal{H}$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be

- *weakly* (*strongly, uniformly*) *stable* if the power sequence $\{T^n\}_{n=1}^{\infty}$ of T converges weakly (strongly, in norm) to the zero operator.

The following fact is well known (cf. [18, p. 10]).

An operator $T \in \boldsymbol{B}(\mathcal{H})$ is uniformly stable if and only if $r(T) < 1$. Moreover, if T is a normaloid, then T is uniformly stable if and only if $\|T\| < 1$. (2.1)

By the uniform boundedness principle, the weak convergence of $\{T^n\}_{n=1}^{\infty}$ implies the power boundedness of T. Hence, using the spectral radius formula, we obtain.

If $T \in \boldsymbol{B}(\mathcal{H})$ is normaloid and $\{T^n\}_{n=1}^{\infty}$ is weakly convergent, then T is a contraction. (2.2)

We refer the reader to [8] and [18] for more information on this subject.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $TT^* = T^*T$, *hyponormal* if $TT^* \leqslant T^*T$, an *isometry* if $T^*T = I$, and *unitary* if $TT^* = T^*T = I$. The operator T is *subnormal* if T is the restriction of a normal operator to its invariant subspace (i.e., a closed vector subspace). If the only subspace of $\mathcal{K}$ reducing N and containing $\mathcal{H}$ is $\mathcal{K}$ itself, then N is called a *minimal normal extension* of T. These classes of operators are related to each other as follows (see [6, Propositions II.4.2 and II.4.6]).

Subnormal operators are hyponormal and hyponormal operators are normaloid. (2.3)

More details on the above-mentioned classes of operators can be found in [6].

If $\mathcal{X}$ is a subset of $\mathcal{H}$, then we write $\mathcal{X}^\perp = \{x \in \mathcal{H} : x \perp \mathcal{X}\}$. Let $T \in \boldsymbol{B}(\mathcal{H})$. A subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is reducing for T (or $\mathcal{M}$ reduces T) if $\mathcal{M}$ is invariant for both T and T^* or, equivalently, if $\mathcal{M}$ and $\mathcal{M}^\perp$ are both invariant for T. A *part* of T is understood as the restriction of T to any of its reducing subspaces. (Sometimes the term "part" is defined as the restriction of an operator to its invariant subspace, but this is not our case.) Weak convergence behaves well with respect to orthogonal parts and sums. Indeed, if $\mathcal{K} \oplus \mathcal{L}$ is the orthogonal sum of Hilbert spaces $\mathcal{K}$ and $\mathcal{L}$, $S \oplus T$ and $A \oplus B$ are the orthogonal sums of operators $S, A \in \boldsymbol{B}(\mathcal{K})$ and $T, B \in \boldsymbol{B}(\mathcal{L})$, then $(S \oplus T)^n = S^n \oplus T^n$ for every integer $n \geqslant 0$ and

$$(S \oplus T)^n \xrightarrow{w} A \oplus B \quad \textit{if and only if} \quad S^n \xrightarrow{w} A \textit{ and } T^n \xrightarrow{w} B. \tag{2.4}$$

In particular, $S \oplus T$ is weakly stable if and only if S and T are weakly stable. The same properties are shared by strong and operator norm topologies. Similar assertions can be stated for direct sums of operators, as discussed in Section 3.

3. Convergence of Power Sequences

In this section, we extend a result from [4] on the weak convergence of power sequences of unitary operators, showing that it is in fact valid for arbitrary operators, however, at the cost of losing the orthogonality of the limit projection (see Theorem 3.2). For the convenience of the reader, we state the original result explicitly.

Lemma 3.1 ([4, Lemma 3.6])**.** *Let U be a unitary operator on a Hilbert space.*

(i) *If $U^n \xrightarrow{w} P$, then P is an orthogonal projection such that $\mathcal{R}(P) = \mathcal{N}(I - U)$.*

(ii) *U is weakly stable if and only if $U^n \xrightarrow{w} P$ and $\mathcal{N}(I - U) = \{0\}$.*

It is worth pointing out that part (ii) of Lemma 3.1 holds for arbitrary operators (see [21, Theorem 1]).

Before we formulate the main result of this section, we will summarize the basic facts about (not necessarily orthogonal) projections in Hilbert spaces, where by a *projection* we mean an operator $P \in \boldsymbol{B}(\mathcal{H})$ which is an idempotent, that is, $P^2 = P$.

- If $P \in \boldsymbol{B}(\mathcal{H})$ is a projection, then $\mathcal{R}(P)$ and $\mathcal{R}(I - P)$ are closed and $\mathcal{H} = \mathcal{R}(P) \dotplus \mathcal{R}(I - P)$, where $\dotplus$ means direct (algebraic) sum. Moreover, $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(I - P)$ if and only if P is selfadjoint.
- Conversely, by the closed graph theorem, if $\mathcal{H} = \mathcal{H}_1 \dotplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are subspaces of $\mathcal{H}$, then there exists a unique projection $P \in \boldsymbol{B}(\mathcal{H})$ such that $\mathcal{R}(P) = \mathcal{H}_1$ and $\mathcal{R}(I - P) = \mathcal{H}_2$.
- Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \dotplus \mathcal{H}_2$, $T_1 \in \boldsymbol{B}(\mathcal{H}_1)$ and $T_2 \in \boldsymbol{B}(\mathcal{H}_2)$. We write $T = T_1 \dotplus T_2$ for the operator $T \in \boldsymbol{B}(\mathcal{H})$ given by $T(x_1 + x_2) = T_1x_1 + T_2x_2$ for $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$. Then $TP = PT$, where $P \in \boldsymbol{B}(\mathcal{H})$ is a unique projection with $\mathcal{R}(P) = \mathcal{H}_1$ and $\mathcal{R}(I - P) = \mathcal{H}_2$.
- Conversely, if $TP = PT$ for some projection $P \in \boldsymbol{B}(\mathcal{H})$, then T decomposes as $T = T_1 \dotplus T_2$, where $T_1 \in \boldsymbol{B}(\mathcal{R}(P))$ and $T_2 \in \boldsymbol{B}(\mathcal{R}(I - P))$.

Theorem 3.2. *Let $T \in \boldsymbol{B}(\mathcal{H})$. Then the following statements hold:*

(i) *if $\{T^n\}_{n=1}^\infty$ converges weakly (strongly, in norm) to $P \in \boldsymbol{B}(\mathcal{H})$, then P is a projection with $\mathcal{R}(P) = \mathcal{N}(I - T)$ and T decomposes as $T = I \dotplus L$ with respect to the direct decomposition $\mathcal{H} = \mathcal{R}(P) \dotplus \mathcal{R}(I - P)$, where L is weakly (strongly, uniformly) stable,*

(ii) *if* $P \in \boldsymbol{B}(\mathcal{H})$ *is a projection and* T *decomposes as* $T = I \dotplus L$ *with respect to the direct decomposition* $\mathcal{H} = \mathcal{R}(P) \dotplus \mathcal{R}(I-P)$, *where* L *is weakly* (*strongly, uniformly*) *stable, then* $\{T^n\}_{n=1}^\infty$ *converges weakly* (*strongly, in norm*) *to* P,

(iii) T *is weakly* (*strongly, uniformly*) *stable if and only if* $\mathcal{N}(I-T) = \{0\}$ *and* $\{T^n\}_{n=1}^\infty$ *is weakly* (*strongly, norm*) *convergent.*

Proof. First, we deal with the case of weak topology.

(i) Suppose that $\{T^n\}_{n=1}^\infty$ converges weakly to $P \in \boldsymbol{B}(\mathcal{H})$. It follows from [21, Theorem 1] that P is a projection which commutes with T. For the reader's convenience, we sketch a slightly different proof of this fact. Since $T^n \xrightarrow{w} P$ as $n \to \infty$, we see that for $k \geqslant 1$, $T^{n+k} \xrightarrow{w} P$ as $n \to \infty$. Using the fact that multiplication in $\boldsymbol{B}(\mathcal{H})$ is separately weakly continuous, we deduce that for $k \geqslant 1$, $T^{n+k} \xrightarrow{w} PT^k$ as $n \to \infty$. Thus, $P = PT^k$ for $k \geqslant 1$. Passing to the limit as $k \to \infty$, we conclude that $P = P^2$. Arguing as above, we get

$$PT = (\text{weak}) \lim_{n\to\infty} T^n T = P = (\text{weak}) \lim_{n\to\infty} TT^n = TP. \tag{3.1}$$

Hence, P commutes with T.

Now, we show that $\mathcal{R}(P) = \mathcal{N}(I-T)$. Take $x \in \mathcal{R}(P)$. Since $P = P^2$, we see that $Px = x$. Hence, by (3.1), $(I-T)x = (I-T)P = 0$, so $x \in \mathcal{N}(I-T)$. Conversely, if $x \in \mathcal{N}(I-T)$, then $T^n x = x$ for all $n \geqslant 1$. Passing to the limit as $n \to \infty$ yields $Px = x$, so $x \in \mathcal{R}(P)$. This shows that $\mathcal{R}(P) = \mathcal{N}(I-T)$. By (3.1), T decomposes as $T = I \dotplus L$ with respect to the direct decomposition $\mathcal{H} = \mathcal{R}(P) \dotplus \mathcal{R}(I-P)$.

It remains to show that L is weakly stable. For, since $\{T^n\}_{n=1}^\infty$ is weakly convergent, so is $\{L^n\}_{n=1}^\infty$. Applying what was proved above to L instead of T, we see that the weak limit of $\{L^n\}_{n=1}^\infty$ is a projection acting on $\mathcal{R}(I-P)$ with range $\mathcal{N}(I-L)$. Noting that $\mathcal{N}(I-L) = \{0\}$, we conclude that L is weakly stable.

(ii) It is easy to verify that under the assumptions of (ii), $\{T^n\}_{n=1}^\infty$ converges weakly to $Q \in \boldsymbol{B}(\mathcal{H})$, where Q decomposes as $Q = I \dotplus 0$ with respect to $\mathcal{H} = \mathcal{R}(P) \dotplus \mathcal{R}(I-P)$. This implies that $Q = P$.

(iii) This statement follows easily from (i).

Our next goal is to cover the cases of strong and norm topologies. Due to the similarity of the proof, we will only consider the case of strong topology. Thus, if $\{T^n\}_{n=1}^\infty$ is strongly convergent to $P \in \boldsymbol{B}(\mathcal{H})$, then it is weakly convergent to P and so, by (i), P is a projection with $\mathcal{R}(P) = \mathcal{N}(I-T)$ and $T = I \dotplus L$ with respect to $\mathcal{H} = \mathcal{R}(P) \dotplus \mathcal{R}(I-P)$, where L is weakly stable. However, the strong convergence of $\{T^n\}_{n=1}^\infty$ implies the strong convergence of $\{L^n\}_{n=1}^\infty$, which ultimately, due to the uniqueness of limit, gives the strong stability of L. This proves (i) for strong topology. Similar arguments can be used to prove statements (ii) and (iii) in the case of strong topology. This completes the proof. □

Remark 3.3. It is a matter of routine to verify that statements (i) and (ii) of Theorem 3.2 are equivalent to (i′) and (ii′), respectively, where

(i′) if $\{T^n\}_{n=1}^\infty$ converges weakly (strongly, in norm) to $P \in \boldsymbol{B}(\mathcal{H})$, then P is a projection with $\mathcal{R}(P) = \mathcal{N}(I-T)$ such that $TP = PT$ and $T|_{\mathcal{R}(I-P)}$ is weakly (strongly, uniformly) stable, (ii′) if $P \in \boldsymbol{B}(\mathcal{H})$ is a projection with $\mathcal{R}(P) = \mathcal{N}(I-T)$ such that $TP = PT$ and $T|_{\mathcal{R}(I-P)}$ is weakly (strongly, uniformly) stable, then $\{T^n\}_{n=1}^{\infty}$ converges weakly (strongly, in norm) to P. ♢

Corollary 3.4. *Let $T \in \boldsymbol{B}(\mathcal{H})$. Then $\{T^n\}_{n=1}^{\infty}$ converges weakly (strongly, in norm) to $P \in \boldsymbol{B}(\mathcal{H})$ if and only if T is similar to an operator $R \in \boldsymbol{B}(\mathcal{K})$ with the property that $\{R^n\}_{n=1}^{\infty}$ converges weakly (strongly, in norm) to an orthogonal projection $Q \in \boldsymbol{B}(\mathcal{K})$. If this is the case, then P and Q are similar via the same similarity as T and R.*

Proof. Suppose that $\{T^n\}_{n=1}^{\infty}$ converges weakly (strongly, in norm) to $P \in \boldsymbol{B}(\mathcal{H})$. Then by Theorem 3.2(i), P is a projection. Set $\mathcal{K} = \mathcal{R}(P) \oplus \mathcal{R}(I-P)$ (the exterior orthogonal sum) and define the operator $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ by $S(x,y) = x + y$ for $x \in \mathcal{R}(P)$ and $y \in \mathcal{R}(I-P)$. Since P is a projection, we have $\mathcal{H} = \mathcal{R}(P) \dotplus \mathcal{R}(I-P)$, so by the inverse mapping theorem, S is invertible. It is easy to verify that the operator $Q = S^{-1}PS$ is an orthogonal projection. Set $R = S^{-1}TS$. Since the map

$$\boldsymbol{B}(\mathcal{H}) \ni X \longmapsto S^{-1}XS \in \boldsymbol{B}(\mathcal{K})$$

is a unital algebra isomorphism which is a weak (strong, norm) homeomorphism, we see that $\{R^n\}_{n=1}^{\infty}$ converges weakly (strongly, in norm) to $Q \in \boldsymbol{B}(\mathcal{H})$. Reversing the above reasoning completes the proof. □

Remark 3.5. Regarding the recently published [4, Theorem 2.1], it is worth noting that the limit operators P and A appearing there are, respectively, the orthogonal projection of $\mathcal{H}_{1\mathrm{u}}$ onto $\mathcal{N}(I-U)$ and the idempotent with $\mathcal{R}(A) = \mathcal{N}(I-X)$. The first fact is a direct consequence of Lemma 3.1. The second follows immediately from Theorem 3.2. ♢

4. Orthogonality of limit projection

In this section, we generalize part (i) of Lemma 3.1 to cover the cases of hyponormal and contractive operators (see Corollaries 4.4 and 4.5, see also Remark 4.6), as well as operators T with $\liminf_{n\to\infty} \|T^n\| \leqslant 1$ (see Theorem 4.12; see also Remark 4.13).

In view of Theorem 3.2(i), if the power sequence $\{T^n\}_{n=1}^{\infty}$ of an operator T converges weakly to P, then P is a projection. On the other hand, by Corollary 3.4, such a T is similar to an operator whose power sequence is weakly convergent to an orthogonal projection. It is therefore of interest to answer the question of when the weak limit of power sequence is an orthogonal projection. We will begin with the following general observation that will shed more light on the answer to the above question.

Proposition 4.1. *Let $T \in \boldsymbol{B}(\mathcal{H})$. Then the following conditions are equivalent:*

(i) *$\mathcal{N}(I-T)$ reduces T,*
(ii) *$T^*|_{\mathcal{N}(I-T)}$ is the identity operator,*
(iii) *$T^*|_{\mathcal{N}(I-T)}$ is an isometry,*
(iv) *$\mathcal{N}(I-T) \subseteq \mathcal{N}(I-T^*)$.*

Proof. First, observe that

$$I|_{\mathcal{N}(I-T)} = (T|_{\mathcal{N}(I-T)})^* = QT^*|_{\mathcal{N}(I-T)}, \tag{4.1}$$

where $Q \in \boldsymbol{B}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I-T)$.

(i)⇒(ii) This is a direct consequence of (4.1).

(ii)⇒(iii) Trivial.

(iii)⇒(i) Note that

$$\|x\| \overset{(4.1)}{=} \|QT^*x\| \leqslant \|T^*x\| \overset{\text{(iii)}}{=} \|x\|, \quad x \in \mathcal{N}(I-T).$$

Hence $\|QT^*x\| = \|T^*x\|$ for every $x \in \mathcal{N}(I-T)$. This implies that $\mathcal{N}(I-T)$ is invariant for T^*, and thus $\mathcal{N}(I-T)$ reduces T.

The equivalence (ii)⇔(iv) is obvious. □

The general answer to our question is as follows.

Theorem 4.2. *Suppose that $T, P \in \boldsymbol{B}(\mathcal{H})$ and $T^n \xrightarrow{w} P$ as $n \to \infty$. Then the following statements are equivalent:*

(i) *P is an orthogonal projection,*

(ii) *$\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$,*

(iii) *$\mathcal{N}(I-T)$ reduces T.*

If (i) *holds, then P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I-T)$.*

Proof. (i)⇒(ii) Suppose that P is an orthogonal projection. Using the fact that the adjoint operation on $\boldsymbol{B}(\mathcal{H})$ is weakly continuous, we see that $T^{*n} \xrightarrow{w} P^*$ as $n \to \infty$. Since $P = P^*$, we deduce from Theorem 3.2(i) applied to T and T^* that

$$\mathcal{N}(I-T) = \mathcal{R}(P) = \mathcal{R}(P^*) = \mathcal{N}(I-T^*).$$

(ii)⇒(iii) This implication is obvious.

(iii)⇒(i) By (iii), T decomposes as

$$T = I \oplus L \tag{4.2}$$

with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{N}(I-T) \oplus \mathcal{N}(I-T)^{\perp}$. Since $\{T^n\}_{n=1}^{\infty}$ is weakly convergent, we infer from (2.4) and (4.2) that $\{L^n\}_{n=1}^{\infty}$ is also weakly convergent. Noting that $\mathcal{N}(I-L) = \{0\}$, we deduce from Theorem 3.2(iii) that L is weakly stable. Using (4.2) again, we conclude that $\{T^n\}_{n=1}^{\infty}$ converges weakly to $I \oplus 0$, which is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I-T)$. □

It is worth mentioning that implication (ii)⇒(iii) of Theorem 4.2 is valid for any Hilbert space operator T regardless of whether the sequence $\{T^n\}_{n=1}^{\infty}$ is weakly convergent or not (see also Proposition 4.1). The proofs of the remaining implications (i)⇒(ii) and (iii)⇒(i) appeal to Theorem 3.2.

With $\subsetneq$ denoting proper inclusion, we obtain the following result, which is a direct consequence of Proposition 4.1 and Theorem 4.2.

Corollary 4.3. *If $T \in \boldsymbol{B}(\mathcal{H})$ is such that $\mathcal{N}(I-T) \subsetneq \mathcal{N}(I-T^*)$, then $\mathcal{N}(I-T)$ reduces T and the power sequence $\{T^n\}_{n=1}^{\infty}$ is not weakly convergent to an orthogonal projection, and thus T is not weakly stable.*

An explicit example of an operator $T \in \boldsymbol{B}(\mathcal{H})$ for which $\mathcal{N}(I-T)$ reduces T, but $\mathcal{N}(I-T) \subsetneq \mathcal{N}(I-T^*)$, is given in Example 4.7.

According to Theorem 4.2, if the power sequence $\{T^n\}_{n=1}^{\infty}$ of an operator T converges weakly to P, then the requirement that $\mathcal{N}(I-T)$ reduces T is necessary and sufficient for P to be an orthogonal projection. Under this assumption, the weak convergence of the power sequence $\{T^n\}_{n=1}^{\infty}$ is completely determined by the weak stability of L, where L is as in (4.2).

Corollary 4.4. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be an operator such that $\mathcal{N}(I-T)$ reduces T and let*

$$T = I \oplus L$$

be the orthogonal decomposition of T with respect to the decomposition

$$\mathcal{H} = \mathcal{N}(I-T) \oplus \mathcal{N}(I-T)^{\perp}.$$

Then the power sequence $\{T^n\}_{n=1}^{\infty}$ is weakly (strongly, norm) convergent if and only if L is weakly (strongly, uniformly) stable, and if this is the case, then the weak (strong, norm) limit of $\{T^n\}_{n=1}^{\infty}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I-T)$ and $\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$.

Proof. As in the proof of implication (iii)⇒(i) of Theorem 4.2, we see that if the power sequence $\{T^n\}_{n=1}^{\infty}$ is weakly (strongly, norm) convergent, then L is weakly (strongly, uniformly) stable. The converse implication is obvious. It follows from Theorem 4.2 that the weak (strong, norm) limit of $\{T^n\}_{n=1}^{\infty}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I-T)$. □

Corollary 4.5. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a hyponormal (resp., contractive) operator. Then*

(i) *$\mathcal{N}(I-T)$ reduces T and T decomposes as $T = I \oplus L$ with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{N}(I-T) \oplus \mathcal{N}(I-T)^{\perp}$, where L is a hyponormal (resp., contractive) operator on $\mathcal{N}(I-T)^{\perp}$,*

(ii) *$\{T^n\}_{n=1}^{\infty}$ is weakly (strongly, norm) convergent if and only if L is weakly (strongly, uniformly) stable, and if this is the case, then the weak (strong, norm) limit of $\{T^n\}_{n=1}^{\infty}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I-T)$ and $\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$.*

Proof. If T is hyponormal, then the restriction of T to its invariant subspace is hyponormal and $\mathcal{N}(I-T) \subseteq \mathcal{N}(I-T^*)$ (see [6, Proposition II.4.4]), so $\mathcal{N}(I-T)$ reduces T. In turn, if T is contractive, then by [35, Proposition I.3.1], $\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$, so $\mathcal{N}(I-T)$ reduces T. Now we can apply Corollary 4.4. □

For an operator $T \in \boldsymbol{B}(\mathcal{H})$ reduced by $\mathcal{N}(I-T)$, the restriction of T to $\mathcal{N}(I-T)$ is called the *identity part* of T. By Corollary 4.5, we get the following.

> *A hyponormal or a contractive operator with no identity part is weakly (strongly, uniformly) stable if and only if its power sequence converges weakly (strongly, in norm).*

Remark 4.6. There are more classes of Hilbert space operators T for which the eigenspace $\mathcal{N}(I-T)$ reduces T. For instance, this is the case for operators having the property that $\mathcal{N}(\alpha I - T) \subseteq \mathcal{N}(\overline{\alpha} I - T^*)$ for every $\alpha \in \mathbb{C}$. This property, in turn, is possessed by so-called *dominant operators*, i.e., operators T such that $\mathcal{R}(\alpha I - T) \subseteq \mathcal{R}(\overline{\alpha} I - T^*)$ for every $\alpha \in \mathbb{C}$ (see [33]). Let us mention that the class of dominant operators on an infinite dimensional Hilbert space is essentially larger than the class of hyponormal operators. For the discussion of the case of numerical contractions, i.e., operators T with $w(T) \leqslant 1$ (less restrictive than $\|T\| \leqslant 1$), we refer the reader to Remark 4.13. ♢

Example 4.7. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Take a bounded sequence $\{\lambda_n\}_{n=0}^{\infty}$ of positive real numbers. Then there exists a unique operator $T \in \boldsymbol{B}(\mathcal{H})$, called a weighted shift (with weights $\{\lambda_n\}_{n=0}^{\infty}$), such that $Te_n = \lambda_n e_{n+1}$ for all integers $n \geqslant 0$. Assume that the sequence $\{\lambda_n\}_{n=0}^{\infty}$ is monotonically increasing to $\lambda_\infty \in (1,\infty)$. Then the weighted shift T is hyponormal (see [30, p. 83, Lemma]). It is well known that the point spectrum of T is empty (see [30, Theorem 8(i)]), so $\mathcal{N}(I-T) = \{0\}$ reduces T. It follows from [32, Theorem 1] that the point spectrum of T^* is equal to $\{z \in \mathbb{C} : |z| < \lambda_\infty\}$. Since $\lambda_\infty > 1$, we see that 1 is in the point spectrum of T^*. Hence, $\mathcal{N}(I-T) \subsetneq \mathcal{N}(I-T^*)$, thus by Corollary 4.3 the power sequence $\{T^n\}_{n=1}^{\infty}$ is not weakly convergent to an orthogonal projection and so T is not weakly stable. Using [32, Theorem 5] and the fact that subnormal operators are hyponormal, we can modify the weights $\{\lambda_n\}_{n=0}^{\infty}$ of T so that T is not only hyponormal, but also subnormal (see also [30, Proposition 25]). Finally, by considering the operator $I \oplus T$ with the above T, we obtain an operator S with the property that $\mathcal{N}(I-S)$ reduces S, $\mathcal{N}(I-S) \neq \{0\}$ and $\mathcal{N}(I-S) \subsetneq \mathcal{N}(I-S^*)$. ♢

In Theorem 4.12 below, we will continue the discussion of the question of when the weak limit of the power sequence of an operator T is an orthogonal projection. We will give a sufficient condition for this, assuming that $\liminf_{n\to\infty} \|T^n\| \leqslant 1$ (see also Remark 4.13). Before stating the result, we show that the new assumption is closely related to the spectral radius of T and to the uniform stability of T. For the reader's convenience, we sketch the proof of the equivalence of conditions (i)-(iv) below (cf. the proof of [22, Theorem 3]).

Lemma 4.8. *Suppose that $T \in \boldsymbol{B}(\mathcal{H})$. Then the following conditions are equivalent:*

(i) *T is uniformly stable,*

(ii) $\liminf_{n\to\infty} \|T^n\| < 1$*,*

(iii) $\inf_{n\geqslant k} \|T^n\| < 1$ *for some integer* $k \geqslant 1$*,*

(iv) $r(T) < 1$*.*

Moreover, if $\liminf_{n\to\infty} \|T^n\| = 1$*, then* $r(T) = 1$*.*

Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are trivial.

(iii)⇒(iv) Since $\inf_{n\geqslant k} \|T^n\| < 1$, there exists an integer $\ell \geqslant k$ such that $\|T^\ell\| < 1$. Then, by the spectral radius formula (see [29, Theorem 10.13]), we get

$$r(T) = \inf_{n\geqslant 1} \|T^n\|^{1/n} \leqslant \|T^\ell\|^{1/\ell} < 1.$$

(i)⇔(iv) This equivalence is well known (see [18, p. 10]).

Assume that $\liminf_{n\to\infty} \|T^n\| = 1$. Then there exists a subsequence $\{\|T^{k_n}\|\}_{n=1}^{\infty}$ tending to 1. Since $\lim_{n\to\infty} \alpha_n = 1$ implies that $\lim_{n\to\infty} \alpha_n^{1/\ell_n} = 1$ for any sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers and any sequence $\{\ell_n\}_{n=1}^{\infty}$ of positive integers tending to ∞, we deduce that $\lim_{n\to\infty} \|T^{k_n}\|^{1/\ell_n} = 1$. Applying this to $\ell_n = k_n$ and using the spectral radius formula, we see that

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{n\to\infty} \|T^{k_n}\|^{1/k_n} = 1.$$

This completes the proof. □

Corollary 4.9. *If $T \in \boldsymbol{B}(\mathcal{H})$, then the following conditions are equivalent:*

(i) $\liminf_{n\to\infty} \|T^n\| \leqslant 1$*,*

(ii) *either T is uniformly stable or* $\liminf_{n\to\infty} \|T^n\| = 1$*.*

As shown in an example below, the converse of the implication

$$\liminf_{n\to\infty} \|T^n\| = 1 \implies r(T) = 1$$

appearing in the "moreover" part of Lemma 4.8 does not hold in general.

Example 4.10. Let $T \in \boldsymbol{B}(\mathcal{H})$ be a 2-*isometry* on a Hilbert space $\mathcal{H}$, that is, $I - 2T^*T + T^{*2}T^2 = 0$. It follows from [14, Proposition 4.5] that there exists a positive operator $C \in \boldsymbol{B}(\mathcal{H})$ such that

$$T^{*n}T^n = I + nC \text{ for all integers } n \geqslant 0. \tag{4.3}$$

Using the spectral radius formula, we deduce that $r(T) = 1$ (in fact, this is a consequence of a more general fact, see [1, Lemma 1.21]). Assume that T is not an isometry, that is, $C \neq 0$. Then, by (4.3), $\lim_{n\to\infty} \|T^n\| = \infty$. To have an example of a non-isometric 2-isometry, consider the weighted shift T with weights $\{\sigma_n(\lambda)\}_{n=0}^\infty$ defined by

$$\sigma_n(\lambda) = \sqrt{\frac{1 + (n+1)(\lambda^2 - 1)}{1 + n(\lambda^2 - 1)}}, \quad n \geqslant 0,$$

where $\lambda \in (1, \infty)$ (see, e.g., [15, Lemma 6.1]; see also [31, Example 2.7]). ♢

Next, we need the following characterization of orthogonal projections.

Lemma 4.11 ([11, Lemma])**.** *Let $\mathcal{H}$ be a Hilbert space and $P \in \boldsymbol{B}(\mathcal{H})$ be an idempotent. Then the following conditions are equivalent:*

(i) *P is an orthogonal projection,*

(ii) *$|\langle Px, x\rangle| \leqslant \|x\|^2$ for all $x \in \mathcal{H}$, or, equivalently, $w(P) \leqslant 1$.*

Now we provide yet another criterion for the limit of a weakly convergent power sequence of an operator to be an orthogonal projection.

Theorem 4.12. *Suppose that $T, P \in \boldsymbol{B}(\mathcal{H})$ are such that $T^n \xrightarrow{w} P$ as $n \to \infty$ and*

$$\liminf_{n\to\infty} \|T^n\| \leqslant 1. \tag{4.4}$$

Then P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I-T)$, $\mathcal{N}(I-T)$ reduces T, and $\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$.

Proof. By Theorem 3.2(i), $P = P^2$ and $\mathcal{R}(P) = \mathcal{N}(I-T)$. Since

$$|\langle Px, x\rangle| = \lim_{n\to\infty} |\langle T^n x, x\rangle| \leqslant \liminf_{n\to\infty} \|T^n\|\|x\|^2 \leqslant \|x\|^2, \quad x \in \mathcal{H},$$

we deduce from Lemma 4.11 that P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I-T)$. The remaining part of the conclusion is a direct consequence of Theorem 4.2. □

Remark 4.13. Arguing as in the above proof, we see that Theorem 4.12 remains valid if the assumption (4.4) is replaced by

$$\liminf_{n\to\infty} \|T^n x\| \leqslant \|x\|, \quad x \in \mathcal{H}. \tag{4.5}$$

Repeating the above reasoning again, we deduce that Theorem 4.12 is also true if the assumption (4.4) is replaced by

$$\liminf_{n\to\infty} w(T^n) \leqslant 1. \tag{4.6}$$

In turn, using the Berger inequality $w(T^n) \leqslant w(T)^n$, which holds for all nonnegative integers n (see [13, Problem 221]), we conclude that the inequality

$$w(T) \leqslant 1 \tag{4.7}$$

implies (4.6), which gives yet another version of Theorem 4.12. It is obvious that (4.4) implies both (4.5) and (4.6). Note also that any contraction T satisfies conditions (4.4)-(4.7), and that for any T with $w(T) \leqslant 1$, $\mathcal{N}(I-T)$ reduces T and $\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$ (the second fact is well known, although not easy to find in the literature; see Appendix for the proof). Therefore, it is an open question which of the conditions (4.4)-(4.6) implies that the space $\mathcal{N}(I-T)$ reduces T or/and $\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$. $\diamondsuit$

We conclude this section by showing that for every $\vartheta \in (1, \infty]$, there exists an operator $T \in \boldsymbol{B}(\mathcal{H})$ such that

$$\liminf_{n\to\infty} \|T^n\| = 1 \text{ (so } T \text{ is not uniformly stable)}, \tag{4.8}$$

$$\limsup_{n\to\infty} \|T^n\| = \vartheta \text{ (so } T \text{ is not a contraction)}, \tag{4.9}$$

$$T \text{ is weakly stable if and only if } \vartheta < \infty. \tag{4.10}$$

Certainly, by (4.8) and the moreover part of Lemma 4.8, $r(T) = 1$ regardless of whether ϑ is finite or not.

Example 4.14. Let $T \in \boldsymbol{B}(\mathcal{H})$ be a weighted shift with positive real weights $\{\lambda_n\}_{n=0}^\infty$ relative to the orthonormal basis $\{e_n\}_{n=0}^\infty$ of $\mathcal{H}$ (see Example 4.7). Then

$$T^k e_n = \lambda_n \cdots \lambda_{n+k-1} e_{n+k}, \quad k \geqslant 1,\, n \geqslant 0, \tag{4.11}$$

$$\|T^k\| = \sup_{n\geqslant 0} \lambda_n \cdots \lambda_{n+k-1}, \quad k \geqslant 1. \tag{4.12}$$

Let $\{q_n\}_{n=1}^\infty$ be a strictly decreasing sequence of positive real numbers such that $\lim_{n\to\infty} q_n = 1$. To determine the weights that meet our requirements, we will use the inflation method, which amounts to constructing the sequences $\{x_j\}_{j=1}^\infty$, $\{t_j\}_{j=1}^\infty$ and $\{l_j\}_{j=1}^\infty$ of integers greater than 2 such that

$$t_j = s_j x_j,\ l_j > m_j = t_j + \sum_{i=1}^{j-1}(t_i + l_i) \textit{ and } x_{j+1} > m_j + l_j \textit{ for all } j \geqslant 1, \tag{4.13}$$

with convention $\sum_{i=1}^{0} \xi_i = 0$, where $\{s_j\}_{j=1}^\infty$ is an arbitrary sequence of integers greater than 2 (later in the proof, a condition will be imposed on the $\{s_j\}_{j=1}^\infty$). Namely, using induction, we can construct a sequence of finite segments $\mathcal{P}_j$ and $\mathcal{Q}_j$, $j \geqslant 1$, of the form

$$\underbrace{\overbrace{\underbrace{q_j, 1, \ldots, 1}_{x_j}, \underbrace{q_j, 1, \ldots, 1}_{x_j}, \ldots, \underbrace{q_j, 1, \ldots, 1}_{x_j}, \underbrace{q_j, 1, \ldots, 1, q_j^{-s_j}}_{x_j}}^{\text{segment } \mathcal{P}_j}}_{t_j = s_j x_j}, \overbrace{\underbrace{1, \ldots\ldots\ldots\ldots, 1}_{l_j}}^{\text{segment } \mathcal{Q}_j}, \tag{4.14}$$

arranged in the order $\mathcal{P}_1, \mathcal{Q}_1, \mathcal{P}_2, \mathcal{Q}_2, \ldots, \mathcal{P}_n, \mathcal{Q}_n, \ldots$, where the length of the j-th segment $\mathcal{P}_j$ is t_j and the length the j-th segment $\mathcal{Q}_j$ is l_j; the j-th segment $\mathcal{P}_j$ itself is partitioned into $s_j - 1$ segments of the form $q_j, 1, \ldots, 1$, each of length x_j, plus a single segment of the form $q_j, 1, \ldots, 1, q_j^{-s_j}$ of the same length x_j; finally, the segment $\mathcal{Q}_j$ consists of l_j units 1. Having done this, we define the weights $\{\lambda_n\}_{n=0}^\infty$ of the weighted shift T via concatenation of the (finite) sequences $\mathcal{P}_1, \mathcal{Q}_1, \mathcal{P}_2, \mathcal{Q}_2, \ldots$ as follows

$$\lambda_0, \lambda_1, \lambda_2, \ldots = \mathcal{P}_1, \mathcal{Q}_1, \mathcal{P}_2, \mathcal{Q}_2, \ldots, \mathcal{P}_n, \mathcal{Q}_n, \ldots. \tag{4.15}$$

By (4.13), we have

$$x_{j+1} > m_j \geqslant t_j = s_j x_j > x_j, \quad j \geqslant 1. \tag{4.16}$$

The number of occurrences of q_j in the j-th segment $\mathcal{P}_j$ is equal to s_j (the expression $q_j^{-s_j}$ is not counted). First, we prove that $\liminf_{n\to\infty} \|T^n\| = 1$. Since $\{q_n\}_{n=1}^\infty$ is decreasing, we deduce from (4.12)-(4.16) that

$$\|T^n\| = \begin{cases} q_{j+1} & \text{if } m_j \leqslant n \leqslant m_j + l_j \text{ with } j \geqslant 1, \\ q_j^{s_j} & \text{if } m_j - x_j + 1 \leqslant n < m_j \text{ with } j \geqslant 1, \end{cases} \tag{4.17}$$

and

$$q_j \leqslant \|T^n\| \leqslant q_j^{s_j} \quad \text{if } m_j - t_j + 1 \leqslant n < m_j - x_j + 1 \text{ with } j \geqslant 1. \tag{4.18}$$

By (4.17), (4.18) and the fact that $q_j \searrow 1$ as $j \to \infty$, we have

$$\liminf_{n\to\infty} \|T^n\| = \lim_{j\to\infty} q_j = 1 \quad \text{and} \quad \limsup_{n\to\infty} \|T^n\| = \limsup_{j\to\infty} q_j^{s_j}. \tag{4.19}$$

In particular, this proves (4.8). Write q_j as $q_j = \mathrm{e}^{\varepsilon_j}$ with $\varepsilon_j = \log q_j$. Then $\{\varepsilon_n\}_{n=1}^\infty$ is a strictly decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. Assume that $\{s_j\}_{j=1}^\infty$ is strictly increasing. Then $\lim_{j\to\infty} s_j = \infty$. We show that for every $\vartheta \in (1, \infty]$, there exists $\{\varepsilon_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} s_j\varepsilon_j = \log\vartheta$. For, consider two cases. If $\vartheta = \infty$, then we set $\varepsilon_j = \frac{1}{\sqrt{s_j}}$ for $j \geqslant 1$. In turn, if $\vartheta \in (1, \infty)$, then we set $\varepsilon_j = \frac{\log\vartheta}{s_j}$ for $j \geqslant 1$. Since $q_j^{s_j} = \mathrm{e}^{s_j\varepsilon_j}$, we deduce from (4.19) that (4.9) holds.

As weak stability always implies uniform boundedness, which by (4.9) is equivalent to $\vartheta < \infty$, it remains to prove that T is weakly stable provided $\vartheta < \infty$. Using (4.11), we see that $\lim_{n\to\infty}\langle T^n e_k, e_l\rangle = 0$ for all integers $k, l \geqslant 0$. Hence, $\lim_{n\to\infty}\langle T^n x, y\rangle = 0$ for all $x, y \in \mathcal{X}$, where $\mathcal{X}$ stands for the linear span of $\{e_n\}_{n=0}^\infty$. Assume that $\vartheta < \infty$. Since $\mathcal{X}$ is dense in $\mathcal{H}$ and, by (4.9), $\sup_{n\geqslant 1} \|T^n\| < \infty$, we conclude that T is weakly stable (see e.g., [21, Lemma 1]), that is (4.10) holds. ♢

Theorem 4.12 and Corollary 4.5 naturally hold for unitary operators. Outgrowths of such a particularisation to unitary operators are the subject of the next section.

5. Weak Convergence of Power Sequences of Unitary Operators

Take a unitary operator $U \in \boldsymbol{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. Since U is hyponormal, $\mathcal{N}(I-U)$ reduces U, so U decomposes relative to $\mathcal{H} = \mathcal{N}(I-U) \oplus \mathcal{N}(I-U)^\perp$ as $U = I \oplus W$, where W is unitary on $\mathcal{N}(I-U)^\perp$ (any part of the decomposition of U may be absent). Thus, by Corollary 4.5 we obtain the following.

Proposition 5.1 ([4, Corollary 3.7]). *Let $U \in \boldsymbol{B}(\mathcal{H})$ be a unitary operator on a Hilbert space $\mathcal{H}$. Then $\mathcal{N}(I-U)$ reduces U and U decomposes as the orthogonal sum*

$$U = I \oplus W$$

relative to the orthogonal decomposition $\mathcal{H} = \mathcal{N}(I-U) \oplus \mathcal{N}(I-U)^\perp$, where W is a unitary operator on $\mathcal{N}(I-U)^\perp$. Moreover, $\{U^n\}_{n=1}^\infty$ is weakly convergent if and *only if W is weakly stable, and if this is the case, then the weak limit of $\{U^n\}_{n=1}^{\infty}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I-U)$.*

Note that the power sequences $\{U^n\}_{n=1}^{\infty}$ and $\{W^n\}_{n=1}^{\infty}$ may not coverage at all; for example, this is the case if U is a symmetry (i.e., a unitary involution) such as

$$U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1 \oplus (-1).$$

According to Proposition 5.1, the following holds.

The power sequence of a unitary operator with no identity part is weakly convergent if and only if it is weakly stable.

The weak stability of a unitary operator U can be completely characterized by the requirement that the spectral measure of U is Rajchman (see Corollary 6.2; see also (5.3)). Let us also note that, since unitary operators are clearly not strongly stable (and therefore not uniformly stable), it follows from Corollary 4.5 that the only unitary operator whose power sequence is strongly or norm convergent is the identity operator.

Let λ and μ be σ-finite measures on the σ-algebra $\mathcal{A}_{\mathbb{T}}$ of Borel subsets of the unit circle $\mathbb{T}$ centered at the origin of the complex plane $\mathbb{C}$. The Lebesgue decomposition theorem implies that the measure μ has a unique decomposition $\mu = \mu_a + \mu_s$ relative to λ, where μ_a and μ_s are measures on $\mathcal{A}_{\mathbb{T}}$ that are absolutely continuous and singular relative to λ, respectively. In turn, the measure μ_s has a unique decomposition $\mu_s = \mu_{sc} + \mu_{sd}$, where μ_{sc} and μ_{sd} are measures on $\mathcal{A}_{\mathbb{T}}$ that are continuous and discrete (i.e., pure point), respectively (cf. [28, Theorem I.13]). Call the measures μ_{sc} and μ_{sd} *singular-continuous* and *singular-discrete* relative to λ, respectively (cf. [19, Proposition 7.13]).

A unitary operator is *absolutely continuous*, *singular-continuous*, or *singular-discrete* if its spectral measure is absolutely continuous, singular-continuous, or singular-discrete relative to the normalized Lebesgue measure on $\mathcal{A}_{\mathbb{T}}$, respectively. By the spectral theorem, every unitary operator $U \in \boldsymbol{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ decomposes as the orthogonal sum

$$U = U_a \oplus U_{sc} \oplus U_{sd} \tag{5.1}$$

of unitary operators relative to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{sd}$ of Hilbert spaces, where $U_a \in \boldsymbol{B}(\mathcal{H}_a)$ is absolutely continuous, $U_{sc} \in \boldsymbol{B}(\mathcal{H}_{sc})$ is singular-continuous and $U_{sd} \in \boldsymbol{B}(\mathcal{H}_{sd})$ is singular-discrete. Note that any part in the decomposition (5.1) may be absent.

The weak stability of unitary operators can be characterised in terms of the decomposition (5.1) as follows.

Remark 5.2. Consider the decomposition (5.1) of a unitary operator $U \in \boldsymbol{B}(\mathcal{H})$ relative to the decomposition $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{sd}$. The following facts are well known (see, e.g., [20, p. 48]).

(i) *An absolutely continuous unitary is always weakly stable, i.e.,* $U_a^n \xrightarrow{w} 0$.

(ii) *A singular-discrete unitary is never weakly stable, i.e.,* $U_{sd}^n \overset{w}{\not\longrightarrow} 0$.

This in turn is equivalent to the following statement.

A unitary operator is weakly stable if and only if its singular-continuous part is weakly stable and its singular-discrete part is absent. (5.2)

Recall also the following two observations from [20, Propositions 3.2 and 3.3].

(iii) *There exist weakly stable singular-continuous unitary operators.*

(iv) *There exist weakly unstable singular-continuous unitary operators.*

Now, we provide more details.

(i′) Absolutely continuous unitaries as parts of bilateral shifts. A unitary operator is absolutely continuous if and only if it is a part of a bilateral shift (see [9, p. 56, Exercise 8]).

(ii′) Singular-discrete unitaries via eigenvalues. A complex number α is an eigenvalue of a unitary operator U with the spectral measure E if and only if $E(\{\alpha\}) \neq 0$ (see [29, Theorem 12.29]). Hence, U has no singular-discrete (equivalently, discrete) part if and only if it has an eigenvalue.

(ii″) Weakly unstable singular-discrete unitaries with full spectrum. If $\{\alpha_k\}_{k=1}^{\infty}$ is an enumeration of the rationals in $[0,1)$, then the diagonal operator U with diagonal $\{e^{2\pi i\alpha_k}\}_{k=1}^{\infty}$ on ℓ^2_+ is a singular-discrete unitary whose spectrum is the whole unit circle $\mathbb{T}$, and which is not weakly stable according to (ii) (see also [7, Example 13.5]).

(iii′) Weakly stable singular-continuous unitaries. Let μ be a finite measure on $\mathcal{A}_{\mathbb{T}}$ and let $L^2(\mathbb{T},\mu)$ be the Hilbert space of square integrable Borel complex functions on $\mathbb{T}$ with respect to μ. Consider the unitary multiplication operator $U_{\varphi,\mu}$ on $L^2(\mathbb{T},\mu)$, which is defined by

$$U_{\varphi,\mu}\psi = \varphi\cdot\psi \text{ a.e. } [\mu], \quad \psi \in L^2(\mathbb{T},\mu),$$

where $\varphi\colon \mathbb{T}\to\mathbb{T}$ is the identity map. Note that the measure μ can be regarded as the scalar spectral measure of $U_{\varphi,\mu}$. Recall that a finite measure ν on $\mathcal{A}_{\mathbb{T}}$ is a *Rajchman measure* if $\int_{\mathbb{T}} z^k\, d\nu(z) \to 0$ as $|k|\to\infty$ (equivalently as $k\to\infty$). For the operator $U_{\varphi,\mu}$, we have

$$\mu \text{ is a Rajchman measure if and only if } U^n_{\varphi,\mu} \xrightarrow{w} 0. \tag{5.3}$$

(Indeed, $U^n_{\varphi,\mu} \xrightarrow{w} 0 \implies \langle U^n_{\varphi,\mu}1,1\rangle \to 0 \iff \int_{\mathbb{T}} z^n\, d\mu(z) \to 0 \implies U^n_{\varphi,\mu} \xrightarrow{w} 0$; the proof of the last implication is straightforward (cf. [2, pp. 1383/1384]). A Rajchman measure i always continuous (see [27], see also [24, p. 364]); however there are singular Rajchman measures [24, Theorem 3.4]. Thus a singular Rajchman measure is singular-continuous, so, by (5.3), we have

if μ is a singular Rajchman measure, then the multiplication operator $U_{\varphi,\mu}$ on $L^2(\mathbb{T},\mu)$ is a weakly stable singular-continuous unitary.

(iv′) Weakly unstable singular-continuous unitaries. Let μ be the Borel-Stieltjes measure on $\mathcal{A}_{\mathbb{T}}$ generated by the Cantor function associated with the Cantor set Γ over the unit circle $\mathbb{T}$ (cf. [28, p. 20, Example 3] and [19, p. 128, Problem 7.15(c)]). Then μ is singular-continuous, and thus the multiplication operator $U_{\varphi,\mu}$ on $L^2(\mathbb{T},\mu)$ is a singular-continuous unitary whose spectrum is Γ, but μ is not a Rajchman measure (see [24, p. 364]), so by (5.3), the operator $U_{\varphi,\mu}$ is not weakly stable. ♢

Now, we apply Proposition 5.1 to the decomposition (5.1) of a unitary operator U and conclude that the power sequence $\{U^n\}_{n=1}^{\infty}$ is weakly convergent if and only if its singular-continuous part U_{sc} is weakly stable and $U_{sd} = I$ (see Theorem 5.5).

Lemma 5.3. *The power sequence of a singular-continuous unitary operator U converges weakly if and only if U is weakly stable.*

Proof. Only the necessity part requires proof. Decompose U as in Proposition 5.1, that is $U = I \oplus W$. Since U is singular-continuous, U has no eigenvalue (see Remark 5.2(ii′)), and thus $U = W$. Using the "moreover" part of Proposition 5.1 completes the proof. □

Lemma 5.4. *The power sequence of a singular-discrete unitary operator U converges weakly if and only if U is the identity operator.*

Proof. As above, we only discuss the necessity part. Decompose U as in Proposition 5.1. Since U is singular-discrete, so is W. Therefore, by the "moreover" part of Proposition 5.1 and Remark 5.2(ii), $U = I$. □

Theorem 5.5. *Let U be a unitary operator on a Hilbert space $\mathcal{H}$ and let $U = U_a \oplus U_{sc} \oplus U_{sd}$ be the decomposition as in* (5.1)*. Then $\{U^n\}_{n=1}^\infty$ is weakly convergent if and only if U_{sc} is weakly stable and U_{sd} is the identity operator* (*any of U_a, U_{sc}, and U_{sd} may be absent*)*. Moreover, if this is the case, then the weak limit of $\{U^n\}_{n=1}^\infty$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I - U) = \mathcal{H}_{sd}$.*

Proof. By (2.4), $\{U^n\}_{n=1}^\infty$ converges weakly if and only if every sequence $\{U_a^n\}_{n=1}^\infty$, $\{U_{sc}^n\}_{n=1}^\infty$, and $\{U_{sd}^n\}_{n=1}^\infty$ converges weakly. It follows from Remark 5.2(i) that $U_a^n \xrightarrow{w} 0$ as $n \to \infty$. Now, using Lemmas 5.3 and 5.4, we can prove the "if and only if" part. The "moreover" part follows from the "if and only if" part and Proposition 5.1. □

Observe that Theorem 5.5 extends the weak stability criterion (5.2).

6. The case of subnormal operators

In this section, we characterize the weak, strong and uniform stability of subnormal operators in terms of their semispectral measures. As a consequence, we obtain results on the convergence of the power sequence of a subnormal operator with respect to the weak, strong and norm topologies.

We begin by reviewing the concepts of the semispectral integral and the semispectral measure of a subnormal operator. Suppose that $F\colon \mathcal{A}_\Omega \to \boldsymbol{B}(\mathcal{H})$ is a semispectral measure on a σ-algebra $\mathcal{A}_\Omega$ of subsets of a set Ω, i.e., $\mu_x = \langle F(\cdot)x, x\rangle$ is a positive measure for every $x \in \mathcal{H}$, and $F(\Omega) = I$. Also recall that if $\varphi \in \bigcap_{x\in\mathcal{H}} L^1(\Omega, \mu_x)$, then there exists a unique operator in $\boldsymbol{B}(\mathcal{H})$, denoted by $\int_\Omega \varphi(\omega)\, F(\mathrm{d}\omega)$, such that

$$\Big\langle \int_\Omega \varphi(\omega)\, F(\mathrm{d}\omega)x, x\Big\rangle = \int_\Omega \varphi(\omega)\langle F(\mathrm{d}\omega)x, x\rangle, \quad x \in \mathcal{H}. \tag{6.1}$$

The same applies to positive operator valued measures.

If $T \in \boldsymbol{B}(\mathcal{H})$ is a subnormal operator and E is the spectral measure of a minimal normal extension $N \in \boldsymbol{B}(\mathcal{K})$ of T, then the Borel semispectral measure F on $\mathbb{C}$ with values in $\boldsymbol{B}(\mathcal{H})$ defined by

$$F(\varDelta) = P_{\mathcal{H}} E(\varDelta)|_{\mathcal{H}}, \quad \varDelta\text{-Borel subset of } \mathbb{C}, \tag{6.2}$$

where $P_{\mathcal{H}} \in \boldsymbol{B}(\mathcal{K})$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, is called the *semispectral measure* of T. The definition of F is independent of the choice of a minimal normal extension of T and F is a unique representing semispectral measure of the operator valued complex moment sequence $\{T^{*n}T^m\}_{m,n=0}^\infty$, i.e.,

$$T^{*n}T^m = \int_{\mathbb{C}} z^m \bar{z}^n F(\mathrm{d}z), \quad m, n \geqslant 0. \tag{6.3}$$

The closed support of F is equal to $\sigma(N)$. See [34, Appendix] and [17, Section 3] (see also [3, 6]) for more information.

If F is a $\boldsymbol{B}(\mathcal{H})$-valued Borel semispectral measure on $\mathbb{C}$, then $F_{\mathbb{T}}$ denotes the positive operator-valued measure defined as $F_{\mathbb{T}}(\varDelta) = F(\varDelta \cap \mathbb{T})$ for any Borel subset of $\mathbb{C}$.

We begin by discussing the weak stability of subnormal operators.

Proposition 6.1. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a subnormal operator with the semispectral measure F. Then T is weakly stable if and only if $\|T\| \leqslant 1$ and $\langle F_{\mathbb{T}}(\cdot)x, x\rangle$ is a Rajchman measure for every $x \in \mathcal{H}$, that is,*

$$\lim_{n\to\infty} \int_{\mathbb{T}} z^n \langle F(\mathrm{d}z)x, x\rangle = 0, \quad x \in \mathcal{H}. \tag{6.4}$$

Proof. If T is weakly stable, then by (2.2) and (2.3), $\|T\| \leqslant 1$. Hence, $\sigma(T) \subseteq \mathbb{D} \sqcup \mathbb{T}$ and thus, by (6.1) and (6.3), the following identity holds

$$\langle T^n x, x\rangle = \int_{\mathbb{D}} z^n \langle F(\mathrm{d}z)x, x\rangle + \int_{\mathbb{T}} z^n \langle F(\mathrm{d}z)x, x\rangle, \quad x \in \mathcal{H},\ n \geqslant 0. \tag{6.5}$$

By the Lebesgue dominated convergence theorem, the first summand in (6.5) tends to zero as $n \to \infty$. Therefore, T is weakly stable if and only if (6.4) holds. □

The following result is a direct consequence of Proposition 6.1 and the obvious observation that semispectral measures of normal (and therefore unitary) operators are spectral. It can also be derived from (5.3) and two facts, the first of which says that any unitary operator is an orthogonal sum (of arbitrary cardinality) of unitary multiplication operators, as in Remark 5.2(iii′), and the second states that a finite Borel measure on $\mathbb{T}$, which is the sum of a series of any (not necessarily countable) family of Rajchman measures, is a Rajchman measure.

Corollary 6.2. *A unitary operator $U \in \boldsymbol{B}(\mathcal{H})$ with the spectral measure E is weakly stable if and only if $\langle E(\cdot)x, x\rangle$ is a Rajchman measure for every $x \in \mathcal{H}$.*

Since absolutely continuous measures with respect to the normalized Lebesgue measure on $\mathcal{A}_{\mathbb{T}}$ are Rajchman measures (see [24, p. 364]), we get

Corollary 6.3. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a subnormal contraction with the semispectral measureF. If $F_{\mathbb{T}}$ is absolutely continuous with respect to the normalized Lebesgue measure on $\mathcal{A}_{\mathbb{T}}$, then T is weakly stable. In particular, this is the case if $F_{\mathbb{T}} = 0$.*

Using Corollary 4.5 and Proposition 6.1 (see also (2.4)), we can characterize the weak convergence of the power sequence of a subnormal operator as follows.

Proposition 6.4. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a subnormal operator. Then $\{T^n\}_{n=1}^\infty$ is weakly convergent if and only if T decomposes as $T = I \oplus L$ with respect to a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where L is a subnormal contraction with the semispectral measure G such that $\langle G_{\mathbb{T}}(\cdot)x, x\rangle$ is a Rajchman measure for every $x \in \mathcal{H}_2$. If this is the case, then the weak limit of $\{T^n\}_{n=1}^\infty$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I - T)$. Moreover, the orthogonal decomposition of T is unique and $\mathcal{H}_1 = \mathcal{N}(I - T)$.*

As for the strong stability of subnormal operators, we have the following result.

Proposition 6.5 ([16, Proposition 4.2(ii)]). *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a subnormal operator with the semispectral measure F. Then T is strongly stable if and only if $\|T\| \leqslant 1$ and $F(\mathbb{T}) = 0$, or equivalently, if and only if T is power bounded and $F(\mathbb{T}) = 0$.*

It turns out that weakly stable normal operators are simply orthogonal sums of weakly stable unitary operators and strongly stable normal operators. In fact, as the proof of Corollary 6.6 shows, any normal contraction T has the form $T = U \oplus S$, where U is unitary and S is a strongly stable subnormal contraction. This kind of orthogonal decompositions for a class of contractions, covering the case of normal operators, can be found in [23].

Corollary 6.6. *A normal operator $T \in \boldsymbol{B}(\mathcal{H})$ is weakly stable if and only if $T = U \oplus S$, where U is a weakly stable unitary operator and S is a strongly stable normal operator.*

Proof. First, observe that the semispectral measure E of T is the spectral measure of T. Suppose that T is weakly stable. According to Proposition 6.1, $\|T\| \leqslant 1$ and

$$\langle E_{\mathbb{T}}(\cdot)x, x\rangle \text{ is a Rajchman measure for every } x \in \mathcal{H}. \tag{6.6}$$

By [3, Theorem 6.6.3], the spaces $\mathcal{M} = \mathcal{R}(E(\mathbb{T}))$ and $\mathcal{N} = \mathcal{R}(E(\mathbb{D}))$ reduce T, $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ and $T = U \oplus S$, where $U = T|_{\mathcal{M}}$ and $S = T|_{\mathcal{N}}$. Clearly, U is unitary (see [3, Theorem 6.1.2]) and S is a normal contraction. It is easy to see that $E_{\mathbb{T}}$ is the spectral measure of U and the measure $E_{\mathbb{D}}$ defined by $E_{\mathbb{D}}(\varDelta) = E(\varDelta \cap \mathbb{D})$ for any Borel subset $\varDelta$ of $\mathbb{C}$ is the spectral measure of S. Hence, by (6.6) and Corollary 6.2, U is weakly stable. Since $E_{\mathbb{D}}(\mathbb{T}) = 0$, we deduce from Proposition 6.5 that S is strongly stable. In view of (2.4), the converse implication is obvious. □

Before we deal with the strong convergence of power sequences of subnormal operators, we prove the following

Lemma 6.7. *Let F be a $\boldsymbol{B}(\mathcal{H})$-valued Borel semispectral measure on $\mathbb{C}$. Then*

$$\Big\{\int_{\mathbb{D}} z^n F(\mathrm{d}z)\Big\}_{n=1}^{\infty} \text{ converges strongly to } 0. \tag{6.7}$$

Proof. By Naimark's dilation theorem (see [26] and [25, Theorem 6.4])), there exist a Hilbert space $\mathcal{K}$ and a spectral measure $E\colon \mathscr{A}_{\mathbb{C}} \to \boldsymbol{B}(\mathcal{K})$ such that $\mathcal{H} \subseteq \mathcal{K}$ and

$$\langle F(\varDelta)x, y\rangle = \langle E(\varDelta)x, y\rangle, \quad x, y \in \mathcal{H},\ \varDelta \in \mathscr{A}_{\mathbb{C}},$$

where $\mathscr{A}_{\mathbb{C}}$ stands for the σ-algebra of all Borel subsets of $\mathbb{C}$. This, together with (6.1), implies that

$$\begin{aligned} \Big|\Big\langle \int_{\mathbb{D}} z^n F(\mathrm{d}z)x, y\Big\rangle\Big| &= \Big|\int_{\mathbb{D}} z^n \langle E(\mathrm{d}z)x, y\rangle\Big| \\ &= \Big|\Big\langle \int_{\mathbb{D}} z^n E(\mathrm{d}z)x, y\Big\rangle\Big| \\ &\leqslant \Big\|\int_{\mathbb{D}} z^n E(\mathrm{d}z)x\Big\|\|y\| \\ &= \Big(\int_{\mathbb{D}} |z|^{2n} \langle F(\mathrm{d}z)x, x\rangle\Big)^{1/2}\|y\|, \quad x, y \in \mathcal{H},\ n \geqslant 0. \end{aligned}$$

As a consequence, we have

$$\Big\| \int_{\mathbb{D}} z^n F(\mathrm{d}z)x \Big\| \leqslant \Big(\int_{\mathbb{D}} |z|^{2n} \langle F(\mathrm{d}z)x, x\rangle \Big)^{1/2}, \quad x \in \mathcal{H},\ n \geqslant 0.$$

By the Lebesgue dominated convergence theorem, $\int_{\mathbb{D}} |z|^{2n}\langle F(\mathrm{d}z)x, x\rangle \to 0$ as $n \to \infty$, so (6.7) holds. □

Now we can characterize the strong convergence of power sequences of subnormal operators.

Theorem 6.8. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a subnormal operator with the semispectral measure F. Then the following conditions are equivalent:*

- (i) *$\{T^n\}_{n=1}^{\infty}$ is strongly convergent,*
- (ii) *T decomposes as $T = I \oplus L$ with respect to a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where L is a subnormal contraction whose semispectral measure vanishes on $\mathbb{T}$,*
- (iii) *T is a contraction and*

$$F(\varDelta) = \delta_1(\varDelta)F(\mathbb{T}) \text{ for every Borel subset } \varDelta \text{ of } \mathbb{T}, \tag{6.8}$$

where δ_1 is the Dirac measure at 1.

Moreover, if (i) *holds, then $F(\mathbb{T})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I - T)$ and the strong limit of $\{T^n\}_{n=1}^{\infty}$ is equal to $F(\mathbb{T})$. Furthermore, the orthogonal decomposition of T in* (ii) *is unique and $\mathcal{H}_1 = \mathcal{N}(I - T)$.*

Proof. There is no loss of generality in assuming that $\|T\| \leqslant 1$ (see the proof of Proposition 6.1).

(i)⇒(ii) It follows from Corollary 4.5 that T decomposes as $T = I \oplus L$ with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{N}(I - T) \oplus \mathcal{N}(I - T)^{\perp}$, where L is a subnormal contraction. Hence, if $\{T^n\}_{n=1}^{\infty}$ is strongly convergent, then by Corollary 4.5, L is strongly stable. Now we can apply Proposition 6.5 to get (ii).

(ii)⇒(iii) Denote by G the semispectral measure of L. Then (see [4, Lemma 3.1])

$$F(\varDelta) = \delta_1(\varDelta)I \oplus G(\varDelta), \quad \varDelta\text{-Borel subset of} \mathbb{C}. \tag{6.9}$$

Substituting $\varDelta = \mathbb{T}$ and using $G(\mathbb{T}) = 0$, we get $F(\mathbb{T}) = I \oplus 0$. Since $G(\mathbb{T}) = 0$ implies that $G(\varDelta) = 0$ for every Borel subset $\varDelta$ of $\mathbb{T}$, we see that (6.9) implies (6.8), so (iii) holds. According to Proposition 6.5, the subnormal contraction L is strongly stable, so $\{T^n\}_{n=1}^{\infty}$ converges strongly to $I \oplus 0$. In turn, by Corollary 4.5, $I \oplus 0 = F(\mathbb{T})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I - T)$. This implies that $\mathcal{H}_1 = \mathcal{N}(I - T)$, and consequently shows that the orthogonal decomposition of T in (ii) is unique. In summary, we have proven both the "moreover" and "furthermore" parts.

(iii)⇒(i) It follows from the inclusion $\sigma(T) \subseteq \mathbb{D} \sqcup \mathbb{T}$ that

$$\begin{aligned} T^n &\overset{(6.3)}{=} \int_{\mathbb{D}} z^n F(\mathrm{d}z) + \int_{\mathbb{T}} z^n F(\mathrm{d}z) \\ &\overset{(6.8)}{=} \int_{\mathbb{D}} z^n F(\mathrm{d}z) + F(\mathbb{T}), \quad n \geqslant 0. \end{aligned} \tag{6.10}$$

Therefore, by Lemma 6.7, $\{T^n\}_{n=1}^{\infty}$ converges strongly to $F(\mathbb{T})$. □

Remark 6.9. It is possible to prove the implication (iii)⇒(ii) of Theorem 6.8 directly without using Lemma 6.7. We are still assuming that $\|T\| \leqslant 1$. Let $T = I \oplus L$ be the orthogonal decomposition of T as in Corollary 4.5. According to (6.10), we have

$$T^n = \int_{\mathbb{D}} z^n F(\mathrm{d}z) + F(\mathbb{T}), \quad n \geqslant 0. \tag{6.11}$$

It follows from (6.1) and the Lebesgue dominated convergence theorem that the sequence $\big\{\int_{\mathbb{D}} z^n F(\mathrm{d}z)\big\}_{n=1}^{\infty}$ converges weakly to 0 as $n \to \infty$ (this is less than what is postulated in (6.7)). Hence, by (6.11), the power sequence $\{T^n\}_{n=1}^{\infty}$ of T converges weakly to $F(\mathbb{T})$. In view of Corollary 4.5, $F(\mathbb{T})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(I - T)$. Let G be the semispectral measure of L. Then

$$0 \overset{(6.8)}{=} F(\mathbb{T} \setminus \{1\}) \overset{(6.9)}{=} 0 \oplus G(\mathbb{T} \setminus \{1\}),$$

so $G_{\mathbb{T}}(\varDelta) = \delta_1(\varDelta) G(\{1\})$ for all Borel subsets $\varDelta$ of $\mathbb{T}$. However, by Proposition 6.4, $\langle G_{\mathbb{T}}(\cdot)x, x\rangle$ is a Rajchman measure for every $x \in \mathcal{H}$, so $\langle G(\mathbb{T})x, x\rangle = 0$ for all $x \in \mathcal{H}$ or, equivalently, $G(\mathbb{T}) = 0$. This yields (ii). ♢

In the last part of the paper, we will focus on the uniform stability and the associated norm convergence of power sequences of subnormal operators. It follows from (2.1) and (2.3) that

$$\text{a subnormal operator is uniformly stable if and only if it is a strict contraction.} \tag{6.12}$$

First, we show that the uniform stability of subnormal operators can be characterized in terms of strong stability as follows.

Theorem 6.10. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a subnormal operator with the semispectral measure F. Then the following statements are equivalent:*

(i) *T is uniformly stable,*

(ii) *T is strongly stable and $\int_{\mathbb{D}} \frac{1}{1-|z|^2} \langle F(\mathrm{d}z)x, x\rangle < \infty$ for every $x \in \mathcal{H}$,*

(iii) *$r(T) \leqslant 1$, $F(\mathbb{T}) = 0$ and $\int_{\mathbb{D}} \frac{1}{1-|z|^2} \langle F(\mathrm{d}z)x, x\rangle < \infty$ for every $x \in \mathcal{H}$.*

Moreover, if T is uniformly stable, $N \in \boldsymbol{B}(\mathcal{K})$ is a minimal normal extension of T and $P_{\mathcal{H}}$ in $\boldsymbol{B}(\mathcal{K})$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, then $\|N\| < 1$ and

$$\sum_{j=0}^{\infty} T^{*j} T^j = \int_{\mathbb{D}} \frac{1}{1-|z|^2} F(\mathrm{d}z) = P_{\mathcal{H}} (I - N^*N)^{-1}|_{\mathcal{H}}, \tag{6.13}$$

where the series is norm convergent.

Proof. Let E be the spectral measure of N. Using (2.3) and [6, Corollary II.2.17], we deduce that

$$\theta := \|N\| = \|T\| = r(T) \text{ and both measures } E \text{ and } F \text{ are supported in } \{z \in \mathbb{C} \colon |z| \leqslant \theta\}. \tag{6.14}$$

(i)⇒(iii) This is a direct consequence of (6.12).

(iii)⇒(i) By (6.14), $T^*T \leqslant I$. Since $\int_{\mathbb{D}} \frac{1}{1-|z|^2} \langle F(\mathrm{d}z)x, x\rangle < \infty$ for every $x \in \mathcal{H}$, there exists the semispectral integral $R = \int_{\mathbb{D}} \frac{1}{1-|z|^2} F(\mathrm{d}z) \in \boldsymbol{B}(\mathcal{H})$ (see [34, Theorem A.1]). It follows from (iii) that

$$F(\mathbb{C} \setminus \mathbb{D}) = 0. \tag{6.15}$$

Hence, we have

$$\langle Rx, x\rangle \overset{(6.1)}{=} \int_{\mathbb{D}} \frac{1}{1-|z|^2}\langle F(\mathrm{d}z)x, x\rangle \geqslant \|x\|^2, \quad x \in \mathcal{H},$$

and thus $Rneq0$. Using the Cauchy-Schwarz inequality, (6.1) and (6.3), we obtain

$$\begin{aligned}
\|x\|^2 &\overset{(6.15)}{=} \int_{\mathbb{D}} 1\,\langle F(\mathrm{d}z)x, x\rangle \\
&\leqslant \bigg(\int_{\mathbb{D}} (1-|z|^2)\langle F(\mathrm{d}z)x, x\rangle\bigg)^{1/2} \bigg(\int_{\mathbb{D}} \frac{1}{1-|z|^2}\langle F(\mathrm{d}z)x, x\rangle\bigg)^{1/2} \\
&= \langle (I-T^*T)x, x\rangle^{1/2}\langle Rx, x\rangle^{1/2} \\
&\leqslant \|(I-T^*T)^{1/2}x\|\|R\|^{1/2}\,\|x\|, \quad x \in \mathcal{H}.
\end{aligned}$$

Therefore, we have

$$\|(I-T^*T)^{1/2}x\| \geqslant \frac{1}{\|R\|^{1/2}}\|x\|, \quad x \in \mathcal{H}.$$

This implies that $(I-T^*T)^{1/2}$ is invertible. Thus, $0 \notin \sigma(I-T^*T)$, or equivalently $1 \notin \sigma(T^*T)$. Since $T^*T \leqslant I$, we conclude that $\|T\| < 1$. Hence, in view of (6.12), statement (i) holds.

(ii)⇔(iii) This equivalence is a direct consequence of Proposition 6.5 and (2.3).

It remains to prove the "moreover" part. Assume that T is uniformly stable. By (6.12), $\|T\| < 1$, so the series $\sum_{j=0}^\infty T^{*j}T^j$ is norm convergent. Using (6.3), we get

$$\begin{aligned}
\bigg\langle\bigg(\sum_{j=0}^\infty T^{*j}T^j\bigg)x, x\bigg\rangle &= \sum_{j=0}^\infty \int_{\mathbb{D}} |z|^{2j}\langle F(\mathrm{d}z)x, x\rangle \\
&= \int_{\mathbb{D}} \bigg(\sum_{j=0}^\infty |z|^{2j}\bigg)\langle F(\mathrm{d}z)x, x\rangle \\
&\overset{(6.1)}{=} \bigg\langle \int_{\mathbb{D}} \frac{1}{1-|z|^2} F(\mathrm{d}z)x, x\bigg\rangle, quad x \in \mathcal{H}.
\end{aligned}$$

This proves the first equality in (6.13). By (6.14), the operator $I-N^*N$ is invertible. Hence, by (6.1), (6.2), (6.14) and the Stone-von Neumann calculus, we have

$$\begin{aligned}
\bigg\langle \int_{\mathbb{D}} \frac{1}{1-|z|^2} F(\mathrm{d}z)x, x\bigg\rangle &= \bigg\langle \int_{\mathbb{D}} \frac{1}{1-|z|^2} E(\mathrm{d}z)x, x\bigg\rangle \\
&= \langle (I-N^*N)^{-1}x, x\rangle \\
&= \langle P_{\mathcal{H}}(I-N^*N)^{-1}|_{\mathcal{H}}x, x\rangle, \quad x \in \mathcal{H}.
\end{aligned}$$

This proves the second equality in (6.13) and completes the proof. □

Now, we can describe the norm convergence of power sequences of subnormal operators. The result below is a direct consequence of (6.12) and Corollary 4.5.

Proposition 6.11. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a subnormal operator. Then $\{T^n\}_{n=1}^\infty$ is norm convergent if and only if T decomposes as $T = I \oplus L$, where L is a strict contraction.*

We conclude the paper by relating the stability of a subnormal operator to the stability of its minimal normal extension.

Theorem 6.12. *Let $T \in \boldsymbol{B}(\mathcal{H}))$ be a subnormal operator and $N \in \boldsymbol{B}(\mathcal{K})$ be a minimal normal extension of T. Then the following assertions hold:*

(i) *T is weakly (strongly, uniformly) stable if and only if N is weakly (strongly, uniformly) stable,*

(ii) *$\{T^n\}_{n=1}^{\infty}$ converges weakly (strongly, in norm) if and only if $\{N^n\}_{n=1}^{\infty}$ converges weakly (strongly, in norm),*

(iii) *if $\{T^n\}_{n=1}^{\infty}$ converges weakly (strongly, in norm), then $\mathcal{N}(I-T) = \mathcal{N}(I-N)$, $T = I \oplus L$ relative to $\mathcal{H} = \mathcal{N}(I-T) \oplus \mathcal{N}(I-T)^{\perp}$ and $N = I \oplus M$ relative to $\mathcal{K} = \mathcal{N}(I-N) \oplus \mathcal{N}(I-N)^{\perp}$, where L and M are weakly (strongly, uniformly) stable and M is a minimal normal extension of L.*

Proof. (i) First, we will address the issue of weak stability. Suppose that T is weakly stable. Then $\|N\| = \|T\| \leqslant 1$ (see (6.14) and Proposition 6.1). It follows from [6, Proposition II.2.4] that

$$\mathcal{K} = \bigvee_{j=0}^{\infty} N^{*j}\mathcal{H}. \tag{6.16}$$

Then for all $j, k \geqslant 0$ and $n \geqslant 1$,

$$\langle N^n(N^{*j}x), N^{*k}y\rangle = \langle N^{n+k}x, N^j y\rangle = \langle T^n(T^k x), T^j y\rangle, \quad x, y \in \mathcal{H}.$$

By the weak stability of T, this implies that $\langle N^n f, g\rangle \to 0$ as $n \to \infty$ for all $f, g \in \mathcal{X}$, where $\mathcal{X}$ is the linear span of $\bigcup_{j=0}^{\infty} N^{*j}\mathcal{H}$. Since by (6.16), $\mathcal{X}$ is dense in $\mathcal{K}$, and $\sup_{n\geqslant 1} \|N^n\| < \infty$, we conclude that N is weakly stable (see e.g., [21, Lemma 1]). The converse implication is obvious.

The case of strong stability can be considered similarly (see also [16, Proposition 4.2(iii)]). In turn, the case of uniform stability follows from the fact that $\|T^n\| = \|N^n\|$ for all $n \geqslant 1$ (see [6, Corollary II.2.17]).

(ii) & (iii) Assume that $\{T^n\}_{n=1}^{\infty}$ converges weakly to $P \in \boldsymbol{B}(\mathcal{H})$. Then, by Corollary 4.5, P is an orthogonal projection with $\mathcal{R}(P) = \mathcal{N}(I-T)$ and T decomposes as $T = I \oplus L$ with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(I-P)$, where L is a weakly stable subnormal operator. It follows from [5, Lemma 3.4] that $N = I \oplus M$ relative to $\mathcal{K} = \mathcal{R}(P) \oplus \mathcal{R}(P)^{\perp}$, where M is a minimal normal extension of L. By (i), M is weakly stable, so $\{N^n\}_{n=1}^{\infty}$ converges weakly to $Q \in \boldsymbol{B}(\mathcal{K}))$, where Q is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{R}(P)$ (clearly, Q extends P). Applying Corollary 4.5 to N in place of T, we conclude that

$$\mathcal{N}(I-N) = \mathcal{R}(Q) = \mathcal{R}(P) = \mathcal{N}(I-T).$$

The "if" part of (ii) for the weak topology is again obvious. This proves both (ii) and (iii) in the case of weak topology. A similar argument can be used in the case of strong and norm topologies. □

Appendix A. Numerical contractions

In this appendix we prove the fact announced in Remark 4.13.

Proposition A.1. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is such $w(T) \leqslant 1$. Then the eigenspace $\mathcal{N}(I-T)$ reduces T and $\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$.*

Proof provided by M. Choi. Since $w(T^*) = w(T)$, it suffices to show that $\mathcal{N}(I-T) \subseteq \mathcal{N}(I-T^*)$. Take $x \in \mathcal{N}(I-T)$ such that $\|x\| = 1$. Set $\mathcal{M} = \operatorname{span}\{x, T^*x\}$. If $\dim \mathcal{M} = 1$, then $T^*x = \lambda x$ for some $\lambda \in \mathbb{C}$, so $1 = \langle T^*x, x\rangle = \lambda$, which implies $x \in \mathcal{N}(I-T^*)$. Thus the remaining possibility is that $\dim \mathcal{M} = 2$. Then there exists $z \in \mathcal{M}$ such that $\|z\| = 1$, $z \perp x$ and $\mathcal{M} = \operatorname{span}\{x, z\}$. This implies that $T^*x = \xi x + \eta z$ for some $\xi, \eta \in \mathbb{C}$. Hence $1 = \langle Tx, x\rangle = \langle T^*x, x\rangle = \xi$, so

$$T^*x = x + \eta z. \tag{A.1}$$

Let $P \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. Set $B = PT|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M})$. Then $Bx = x$ and $Bz = PTz = \alpha x + \beta z$ for some $\alpha, \beta \in \mathbb{C}$. Thus B has the matrix representation $B = \left[\begin{smallmatrix} 1 & \alpha \\ 0 & \beta \end{smallmatrix}\right]$ relative to the orthonormal basis $\{x, z\}$ of $\mathcal{M}$. Since $w(B) \leqslant w(T) \leqslant 1$ and $\langle Bx, x\rangle = 1$, we have $w(B)1$. We show that $\alpha = 0$. Suppose, to the contrary, that $\alpha \neq 0$. If $\beta \neq 1$, then it follows from the proof of [12, Lemma 1.1-1] that the numerical range $W(B)$ of B is the ellipse with foci 1 and β and minor axis $|\alpha| > 0$, so $w(B) > 1$, a contradiction. This means that $\beta = 1$. Again, by [12], $W(B) = 1 + \left\{z \in \mathbb{C}\colon |z| \leqslant \frac{|\alpha|}{2}\right\}$, so $1 = w(B) \geqslant 1 + \frac{|\alpha|}{2}$, which is a contradiction. Summarizing, $\alpha = 0$. Then $Bz = \beta z$, which implies that $P(\beta z - Tz) = 0$, so $(\beta z - Tz) \perp x$. Therefore we have

$$0 = \beta\langle z, x\rangle = \langle Tz, x\rangle = \langle z, T^*x\rangle \overset{\text{(A.1)}}{=} \bar{\eta},$$

and consequently by (A.1), $T^*x = x$, or equivalently $x \in \mathcal{N}(I-T^*)$. □

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References

1. J. Agler, M. Stankus, m-isometric transformations of Hilbert spaces, I, II, III, *Integr. Equ. Oper. Theory* **21, 23, 24** (1995, 1995, 1996), 383-429, 1–48, 379–421.
2. C. Badea, V. Müller, On weak orbits of operators, *Topology Appl.* **156** (2009), 1381–1385.
3. M.Sh. Birman, M.Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, D. Reidel Publishing Co., Dordrecht, 1987.
4. S. Chavan, Z.J. Jabłoński, I.B. Jung, J. Stochel, Convergence of power sequences of B-operators with applications to stability, *Proc. Amer. Math. Soc.* **152** (2024), 2035–2050.
5. S. Chavan, Z.J. Jabłoński, I.B. Jung, J. Stochel, Lifting B-subnormal operators, submitted 2023.
6. J.B. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monographs, **36**, Amer. Math. Soc. Providence, RI 1991.
7. H.R. Dowson, *Spectral Theory of Linear Operators*, Academic Press, London, 1978.
8. T. Eisner, *Stability of operators and operator semigroups*, Operator Theory: Advances and Applications, 209, Birkhäuser Verlag, Basel, 2010.
9. P.A. Fillmore, *Notes on Operator Theory*, Van Nostrand, New York, 1970.
10. P.A. Fuhrmann, *Linear systems and operators in Hilbert space*, McGraw-Hill, New York, 1981.
11. T. Furuta, R. Nakamoto, Certain numerical radius contraction operators, *Proc. Amer. Math. Soc.* **29** (1971), 521–524.
12. K. E. Gustafson, D. K. M. Rao, *Numerical range. The field of values of linear operators and matrices*, Universitext Springer-Verlag, New York, 1997.
13. P.R. Halmos, *A Hilbert space problem book*, Springer-Verlag, New York. 1982.

14. Z.J. Jabłoński, Complete hyperexpansivity, subnormality and inverted boundedness conditions, *Integr. Equ. Oper. Theory* **44** (2002), 316–336.
15. Z.J. Jabłoński, J. Stochel, Unbounded 2-hyperexpansive operators, *Proc. Edinburgh Math. Soc.* **44** (2001), 613–629.
16. Z.J. Jabłoński, I.B. Jung, J. Stochel, Criteria for algebraic operators to be unitary, *Kyungpook Math. J.* **63** (2023), 1–10.
17. I.B. Jung, J. Stochel, Subnormal operators whose adjoints have rich point spectrum, *J. Funct. Anal.* **255** (2008), 1797–1816.
18. C.S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory*, Birkhäuser, Boston, 1997.
19. C.S. Kubrusly, *Essentials of Measure Theory*, Springer, Cham, 2015.
20. C.S. Kubrusly, Singular-continuous unitaries and weak dynamics, *Math. Proc. R. Ir. Acad.* **116A** (2016), 45–56.
21. C.S. Kubrusly, P.C.M. Vieira, Weak asymptotic stability for discrete linear distributed systems, Proceedings of the 5th IFAC Symposium on Control of Distributed Parameter Systems, Perpignan, pp. 69–73, June 1989 (Pergamon Press, Oxford).
22. C.S. Kubrusly, P.C.M. Vieira, Convergence and decomposition for tensor products of Hilbert space operators, *Oper. Matrices* **2** (2008), 407–416.
23. C.S. Kubrusly, P.C.M. Vieira, D.O. Pinto, A decomposition for a class of contractions, *Adv. Math. Sci. Appl.* **6** (1996), 523–530.
24. R. Lyons, Seventy years of Rajchman measures, Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993) *J. Fourier Anal. Appl.*, Special Issue (1995), 363–377.
25. W. Mlak, *Dilations of Hilbert space operators* (*general theory*), Dissertationes Math. **153** (1978), 61 pp.
26. M.A. Naimark, On a Representation of Additive Operator Set Functions, *C.R. Acad. Sci. URSS* **41** (1943), 359–361.
27. L. Neder, Über die Fourierkoeffizienten der Funktionen von beschränkter Schwankung, *Math. Z.* **6** (1920), 270–273.
28. M. Reed, B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York-London, 1972.
29. W. Rudin, *Functional Analysis*, 2nd edn. McGraw-Hill, New York, 1991.
30. A. L. Shields, Weighted shift operators and analytic function theory, *Topics in operator theory*, pp. 49–128. Math. Surveys, No. 13, Amer. Math. Soc., Providence, 1974.
31. V.M. Sholapurkar, A. Athavale, Completely and alternatingly hyperexpansive operators, *J. Operator Theory* **43** (2000), 43–68.
32. J. G. Stampfli, Which weighted shifts are subnormal, *Pacific J. Math.* **17** (1966), 367–379.
33. J.G. Stampfli, B.L. Wadhwa, On dominant operators, *Monatsh. Math.* **84** (1977), 143–153.
34. J. Stochel, Decomposition and disintegration of positive definite kernels on convex $*$-semigroups, *Ann. Polon. Math.* **56** (1992), 243–294.
35. B. Sz.-Nagy, C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York; Akadémiai Kiadó, Budapest, 1970.
36. J. Zabczyk, Remarks on the control of discrete-time distributed parameter systems, *SIAM J. Control* **12** (1974), 721–735.

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