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WEAK SUPERCYCLICITY — AN EXPOSITORY SURVEY

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Dedicated to the memory of my PhD supervisor Ruth F. Curtain (1941–2018)

Abstract. This is an exposition on supercyclicity and weak supercyclicity, especially designed to advance further developments in weakly supercyclicity, which is a recent research field showing significant momentum during the past two decades. For operators on a normed space, the present paper explores the relationship among supercyclicity and three versions of weak supercyclicity, namely, weak l-sequential, weak sequential, and weak supercyclicities, and their connection with strong stability, weak stability, and weak quasistability. A survey of the literature on weak supercyclicity is followed by an analysis of the interplay between supercyclicity and strong stability, as well as between weak supercyclicity and weak stability. A description of the spectrum of weakly l-sequentially supercyclic operators is also given.

1. Introduction

This is an expository-survey paper focusing on weak forms of supercyclicity and their connection with weak stability of bounded linear operators on a normed space. It has been devised to offer an organised, summarised, and unified collection of results in such a recent research area, as well as to exhibit some motivating open questions on it. The target is weak supercyclicity, in particular, weak l-sequential supercyclicity, whose relatively recent mathematical developments began around 2005, and have become a particularly active research field since then.

It has been known for some time that supercyclicity in the norm topology, also referred to as strong supercyclicity, implies strong stability. This means that if the projective orbit of a power bounded linear operator T on a normed space $\mathcal{X}$ is dense in $\mathcal{X}$ for some vector y in $\mathcal{X}$, then $T^n x \to 0$ for every $x \in \mathcal{X}$, with both density and convergence in the norm topology. Symbolically: $\mathcal{O}_T([y])^- = \mathcal{X} \Longrightarrow T^n \xrightarrow{s} O$. This is an important result from [2, Theorem 2.2], which becomes a major motivation behind the quest for a possible relationship between some form of weak supercyclicity and weak stability. Does such an implication survive the transition from norm topology to weak topology? In fact, it was asked in [34] whether a stronger version of weak supercyclicity, viz., weak l-sequential supercyclicity, implies weak stability. As discussed in Section 3, weak l-sequential supercyclicity is weaker than supercyclicity in the norm topology but stronger than supercyclicity in the weak sequential topology, and so stronger than supercyclicity in the weak topology.

The present paper brings an exposition on weak supercyclicity in general, including a survey of the recent literature on weak supercyclicity and, in particular, a discussion on the above mentioned question.

All terms used above will be defined here in due course. The paper is organised into six more sections. Since it deals with many aspects related to linear dynamics, involving a large amount of specific jargon and concepts, a section on notation and

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terminology will be split into two parts in order to make it easier to read. Basic notation and terminology are posed in Section 2, while notions involving cyclicity, hypercyclicity, supercyclicity, and three forms of weak supercyclicity are considered in Section 3. That being said, the literature specifically on forms of *weak supercyclicity* is surveyed in Section 4. Such literature is classified according to the main target of each paper, roughly in chronological order, into eight subgroups, namely,

(1) pre-weak supercyclicity,
(2) the beginning of weak supercyclicity,
(3) further forms of weak supercyclicity,
(4) clarifying three forms of weak supercyclicity,
(5) weak supercyclicity for operators close to normal,
(6) weak supercyclicity for compact operators,
(7) weak supercyclicity for composition operators,
(8) weak supercyclicity and weak stability.

The fundamental arguments ensuring that strong supercyclicity implies strong stability are considered in detail in Section 5. The special themes of weak supercyclicity (in particular, weak l-sequential supercyclicity) and weak stability are reviewed in Chapter 6. We close the paper by proving a characterisation of the spectrum of weakly l-sequentially supercyclic operators in Section 7.

2. Notation and Terminology, Part I: Basic Notions

Throughout the paper, linear spaces are over a scalar field $\mathbb{F}$, which here is always $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B}[\mathcal{X}]$ denote the normed algebra of all bounded linear transformations of a normed space $\mathcal{X}$ onto itself. Elements of $\mathcal{B}[\mathcal{X}]$ will be referred to as operators. Both norms on a normed space $\mathcal{X}$ and the induced uniform norm on $\mathcal{B}[\mathcal{X}]$ will be denoted by the same symbol $\|\cdot\|$. Let $\mathcal{X}^*$ be the dual of a normed space $\mathcal{X}$, and let $T^*\in\mathcal{B}[\mathcal{X}^*]$ stand for the normed-space adjoint of $T\in\mathcal{B}[\mathcal{X}]$. We use the same notation $T^*\in\mathcal{B}[\mathcal{H}]$ for the Hilbert-space adjoint if $T\in\mathcal{B}[\mathcal{H}]$ acts on a Hilbert space $\mathcal{H}$. The null operator and the identity operator will be denoted by O and I, respectively.

We begin with a standard definition (see, e.g., [42, Definition 1.13.2]).

Definition. Let $\mathcal{X}$ be a normed space. An $\mathcal{X}$-valued sequence $\{x_n\}$ converges weakly, or $\{x_n\}$ is weakly convergent, if the scalar-valued sequence $\{f(x_n)\}$ converges for every $f\in\mathcal{X}^*$ in the sense that there exists $z\in\mathcal{X}$ for which the scalar-valued sequence $\{f(x_n)\}$ converges to $f(z)$ for every $f\in\mathcal{X}^*$.

Therefore, an $\mathcal{X}$-valued sequence $\{x_n\}$ converges weakly if $\lim_n f(x_n)\to f(z)$ for some $z\in\mathcal{X}$, for every $f\in\mathcal{X}^*$. Usual notation: $x_n\xrightarrow{w} z$ or $z=w\text{-}\lim_n x_n$. An $\mathcal{X}$-valued sequence $\{x_n\}$ is strongly convergent, or converges in the norm topology, if $\lim_n\|x_n-z\|\to 0$ for some $z\in\mathcal{X}$. Usual notation: $x_n\to z$.

For any $T\in\mathcal{B}[\mathcal{X}]$, take its $\mathcal{B}[\mathcal{X}]$-valued power sequence $\{T^n\}$ (with n running over the nonnegative integers) and, for each $x\in\mathcal{X}$, consider the $\mathcal{X}$-valued sequence $\{T^nx\}$. An operator T is power bounded if $\sup_n\|T^n\|<\infty$. If $\mathcal{X}$ is a Banach space, then this is equivalent to $\sup_n\|T^nx\|<\infty$ for every $x\in\mathcal{X}$ by the Banach Steinhaus Theorem. Moreover, an operator T is of class $C_{1\cdot}$ if

$$T^nx\not\to 0\ \text{ for every nonzero } x\in\mathcal{X}$$

(i.e., $\|T^n x\| \not\to 0$ for every $0 \neq x \in \mathcal{X}$). The opposite notion is that of strongly stable operators. An operator T is strongly stable, or T is of class $C_{0}\cdot$, if

$$T^n x \to 0 \;\text{ for every } x \in \mathcal{X}$$

(i.e., $\|T^n x\| \to 0$ for every $x \in \mathcal{X}$). On the other hand, T is weakly stable if

$$T^n x \xrightarrow{\;w\;} 0 \;\text{ for every } x \in \mathcal{X}.$$

Equivalently, an operator T is weakly stable if

$$\lim_n |f(T^n x)| = 0 \;\text{ for every } x \in \mathcal{X}, \;\text{ for every } f \in \mathcal{X}^*.$$

Usual notation for strong and weak stability: $T^n \xrightarrow{\;s\;} O$ and $T^n \xrightarrow{\;w\;} O$, respectively.

In general, strong convergence implies weak convergence, and so, in particular, strong stability implies weak stability. If $\mathcal{X}$ is a Banach space, then weak stability implies power boundedness by the Banach–Steinhaus Theorem. If $\mathcal{X}$ is a complex Banach space, then let $\sigma(T)$ denote the spectrum of T and let $r(T) = \sup_{\lambda\in\sigma(T)} |\lambda|$ stand for the spectral radius of T. Recall that $0 \leq r(T) \leq |T\|$, and T is quasinilpotent or normaloid if $r(T) = 0$ or $r(T) = \|T\|$, respectively. The spectrum of a power bounded operator is included in the closed unit disk. Summing up:

$$T^n \xrightarrow{\;s\;} O \quad\Longrightarrow\quad T^n \xrightarrow{\;w\;} O \quad\Longrightarrow\quad \sup_n \|T^n\| < \infty \quad\Longrightarrow\quad r(T) \leq 1.$$

A final note on basic notation: we write ℓ^p for $\ell^p(\mathbb{Z})$ and ℓ^p_+ for $\ell^p(\mathbb{N})$ or $\ell^p(\mathbb{N}_0)$, for any $p \geq 1$, where $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ stand for the sets of all integers, all positive integers, and all nonnegative integers, respectively.

3. Notation and Terminology, Part II: Notions of Cyclicity

This section pulls together the specific notation and terminology required to describe all kinds of cyclicity. Before introducing the standard notions of hypercyclicity, supercyclicity, and cyclicity, and also the variants of weak supercyclicity, we need the notions of weak l-sequential denseness and projective orbit.

3.1. weak l-sequential denseness. Let A be a subset of a normed space $\mathcal{X}$. Its closure in the norm topology on $\mathcal{X}$ is denoted by A^-, and its weak closure in the weak topology on $\mathcal{X}$ is denoted by A^{-w}. Thus A is dense (in the norm topology) or weakly dense (in the weak topology) if $A^- = \mathcal{X}$ or $A^{-w} = \mathcal{X}$, respectively. Since the weak topology is not metrizable, the notions of weak closure and weak sequential closure do not necessarily coincide. A set A is weakly sequentially closed if every A-valued weakly convergent sequence has its limit in A. Let the weak sequential closure of A be denoted by A^{-ws}, which is the smallest weakly sequentially closed subset of $\mathcal{X}$ including A. That is, A^{-ws} is the intersection of all weakly sequentially closed subsets of $\mathcal{X}$ that include A. So a set A is weakly sequentially dense if $A^{-ws} = \mathcal{X}$. Differently, the *weak limit set* A^{-wl} of A is the set of all weak limits of weakly convergent A-valued sequences, that is,

$$A^{-wl} = \big\{x \in \mathcal{X} \colon x = w\text{-}\lim_n x_n \text{ with } x_n \in A\big\}.$$

We say that A is *weakly l-sequentially dense* if $A^{-wl} = \mathcal{X}$. It is known that (see, e.g., [33, Proposition 2.1])

$$A^- \subseteq A^{-wl} \subseteq A^{-ws} \subseteq A^{-w}.$$

These inclusions may be proper in general (see, e.g., [53, pp.38,39], [5, pp.259,260], [33, pp.10,11]). Consequently, if A is dense in the norm topology (i.e., $A^- = \mathcal{X}$), then it is dense with respect to all notions of denseness defined above. Recall that if a set A is convex, then $A^- = A^{-w}$ (see, e.g., [42, Theorem 2.5.16]). Thus, if A is convex, then the above inclusions become identities, and so for a convex set all the above notions of denseness coincide.

A vector $x \in \mathcal{X}$ is a *weak limit point* of a set $A \subseteq \mathcal{X}$ if $x \in A^{-wl}$, and it is a *weak accumulation point* of A if $x \in A^{-w}$. Thus the above inclusions show that weak limit points are weak accumulation points, but the converse, in general, fails.

3.2. PROJECTIVE ORBIT. Given an operator $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$, consider again its power sequence $\{T^n\}$ (with n running over the nonnegative integers). The *orbit* $\mathcal{O}_T(y)$ or $\mathrm{Orb}(T, y)$ of a vector $y \in \mathcal{X}$ under T is the set

$$\mathcal{O}_T(y) = \bigcup_n T^n\{y\} = \bigcup_n \{T^n y\} = \big\{T^n y \in \mathcal{X} \colon \text{for every nonnegative integer } n\big\}.$$

The orbit $\mathcal{O}_T(A)$ of a set $A \subseteq \mathcal{X}$ under T is likewise defined: $\mathcal{O}_T(A) = \bigcup_n T^n(A) = \bigcup_{z \in A} \mathcal{O}_T(z)$. Let $[x] = \mathrm{span}\{x\}$ stand for the subspace of $\mathcal{X}$ spanned by a singleton $\{x\}$ at a vector $x \in \mathcal{X}$, which is a one-dimensional subspace of $\mathcal{X}$ whenever x is nonzero. The *projective orbit* of a vector $y \in \mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the orbit of span of $\{y\}$; that is, the orbit $\mathcal{O}_T([y])$ of $[y]$:

$$\mathcal{O}_T([y]) = \mathcal{O}_T(\mathrm{span}\{y\}) = \bigcup_{z \in [y]} \mathcal{O}_T(z) = \big\{\alpha T^n y \in \mathcal{X} \colon \alpha \in \mathbb{F},\ n \in \mathbb{N}_0\big\}.$$

3.3. HYPERCYCLICITY, SUPERCYCLICITY, AND CYCLICITY. It is natural and usual to classify the concepts of cyclicity into three classes, namely, hypercyclicity, supercyclicity, and cyclicity. According to the type of density (cf. Subsection 3.1) upon which these notions are defined, a further split goes as follows.

(A) A vector y in $\mathcal{X}$ is a *hypercyclic vector* for T if its orbit $\mathcal{O}_T(y)$ is dense in $\mathcal{X}$ (i.e., dense in the norm topology):

$$\mathcal{O}_T(y)^- = \mathcal{X}.$$

As the norm topology is metrizable, a nonzero vector y in the normed space $\mathcal{X}$ is a hypercyclic vector for T if for every x in $\mathcal{X}$ there is a sequence with elements in $\mathcal{O}_T(y)$ converging to it. That is, $y \in \mathcal{X}$ is a hypercyclic vector for T if for every x in $\mathcal{X}$ there is a sequence $\{T^{n_k}\}_{k \geq 0}$ (which depends on x) with entries from $\{T^n\}_{n \geq 0}$ such that

$$T^{n_k} y \to x \qquad (\text{i.e., } \|T^{n_k} y - x\| \to 0).$$

An operator T is a *hypercyclic operator* if it has a hypercyclic vector. Clearly, a hypercyclic operator is not power bounded.

(B) A vector y in $\mathcal{X}$ is a *supercyclic vector* for T if its projective orbit $\mathcal{O}_T([y])$ is dense in $\mathcal{X}$. That is, if the orbit of the span of $\{y\}$ is dense in $\mathcal{X}$ (i.e., dense in the norm topology):

$$\mathcal{O}_T([y])^- = \mathcal{X} \qquad (\text{i.e., } \mathcal{O}_T(\mathrm{span}\{y\})^- = \mathcal{X}).$$

Again, as the norm topology is metrizable, a nonzero vector y in $\mathcal{X}$ is a supercyclic vector for an operator T if for every x in $\mathcal{X}$ there is an $\mathbb{F}$-valued sequence $\{\alpha_k\}_{k \geq 0}$ such that, for some sequence $\{T^{n_k}\}_{k \geq 0}$ with entries from $\{T^n\}_{n \geq 0}$, the sequence $\{\alpha_k T^{n_k} y\}_{k \geq 0}$ converges to x (in the norm topology):

$$\alpha_k T^{n_k} y \to x \qquad (\text{i.e., } \|\alpha_k T^{n_k} y - x\| \to 0).$$

The scalar sequence $\{\alpha_k\}_{k\geq 0}$ consists of nonzero numbers if $x\neq 0$. Both sequences $\{\alpha_k\}_{k\geq 0}$ and $\{T^{n_k}\}_{k\geq 0}$ depend on x for each y. An operator T is a *supercyclic operator* if it has a supercyclic vector.

The next two definitions are weak counterparts of the above convergence criteria.

(C) A nonzero vector y in $\mathcal{X}$ is a *weakly l-sequentially hypercyclic vector* for T if for every x in $\mathcal{X}$ there exists a sequence $\{T^{n_k}\}_{k\geq 0}$ with entries from $\{T^n\}_{n\geq 0}$ such that the $\mathcal{X}$-valued sequence $\{T^{n_k}y\}_{k\geq 0}$ converges weakly to x,

$$T^{n_k}y \xrightarrow{w} x \qquad (\text{i.e.,}\quad f(T^{n_k}y-x)\to 0 \quad \text{for every} \quad f\in\mathcal{X}^*).$$

The sequence $\{T^{n_k}\}_{k\geq 0}$ depends on x for each y but does not depend on f. This means the orbit $\mathcal{O}_T(y)$ of y under T is weakly l-sequentially dense in $\mathcal{X}$ in the sense that the weak limit set of $\mathcal{O}_T(y)$ coincides with $\mathcal{X}$:

$$\mathcal{O}_T(y)^{-wl}=\mathcal{X}.$$

An operator T is said to be a *weakly l-sequentially hypercyclic operator* if it has a weakly l-sequentially hypercyclic vector.

(D) A nonzero vector y in $\mathcal{X}$ is a *weakly l-sequentially supercyclic vector* for T if for every x in $\mathcal{X}$ there exists an $\mathbb{F}$-valued sequence $\{\alpha_k\}_{k\geq 0}$ such that, for some sequence $\{T^{n_k}\}_{k\geq 0}$ with entries from $\{T^n\}_{n\geq 0}$, the $\mathcal{X}$-valued sequence $\{\alpha_k T^{n_k}y\}_{k\geq 0}$ converges weakly to x. In other words, if

$$\alpha_k T^{n_k}y \xrightarrow{w} x \qquad (\text{i.e.,}\quad f(\alpha_k T^{n_k}y-x)\to 0 \quad \text{for every} \quad f\in\mathcal{X}^*).$$

The scalar sequence $\{\alpha_k\}_{k\geq 0}$ consists of nonzero numbers if $x\neq 0$, and both sequences $\{\alpha_k\}_{k\geq 0}$ and $\{T^{n_k}\}_{k\geq 0}$ depend on x for each y but do not depend on f. This means the projective orbit $\mathcal{O}_T([y])$ of y under T is weakly l-sequentially dense in $\mathcal{X}$ in the sense that the weak limit set of $\mathcal{O}_T([y])$ coincides with $\mathcal{X}$:

$$\mathcal{O}_T([y])^{-wl}=\mathcal{X}.$$

An operator T is a *weakly l-sequentially supercyclic operator* if it has a weakly l-sequentially supercyclic vector.

Before proceeding, it is convenient to comment on two important points. Remark 1 below shows that weak l-sequential supercyclicity and weak l-sequential hypercyclicity *are not topological notions*. In the subsequent Remark 2, it is recalled that any form of cyclicity makes nontrivial sense only when approaching points outside the projective orbit, or outside the orbit, respectively.

Remark 1. WEAK L-SEQUENTIAL SUPERCYCLICITY. It was observed in [33] that, for any subset A of $\mathcal{X}$, the map $A\mapsto A^{-wl}$ is not a topological closure operation. Thus weak l-sequential supercyclicity (and weak l-sequential hypercyclicity) are not topological notions. Indeed, as defined in (D) (and in (C)) above, weak l-sequential supercyclicity (and weak l-sequential hypercyclicity) come as weak versions of a norm convergence criterion, but not as denseness of the projective orbit (or of the orbit) in any topology. In particular, weak l-sequential supercyclicity (and weak l-sequential hypercyclicity) do not mean denseness of the projective orbit (or of the orbit) in the weak topology or in the weak sequential topology. Therefore, weak l-sequential supercyclicity (and weak l-sequential hypercyclicity) are not weak topology notions. Consequently, it is not the case of applying weak topology techniques when dealing with weak l-sequential supercyclicity (or with weak l-sequential hypercyclicity). For a discussion along this line, see also [53].

Remark 2. SEQUENCE WITH ENTRIES FROM A SEQUENCE. Take a nonzero $x \in \mathcal{X}$. If $x \in \mathcal{O}_T([y])$ (or $x \in \mathcal{O}_T(y)$) for some vector y, then the sequence $\{T^{n_k}\}_{k\geq 0}$ mentioned in (A) to (D) can be thought of as a constant infinite sequence or, equivalently, as a one-entry, thus finite, subsequence of $\{T^n\}_{n\geq 0}$. For instance, in the supercyclicity cases described in (B) and (D), if $x \in \mathcal{O}_T([y])$, then $x = \alpha_0 T^{n_0} y$ for some $\alpha_0 \neq 0$ and some $n_0 \geq 0$. In such cases, both notions of $\alpha_k T^{n_k} y \xrightarrow{w} x$ and $\alpha_k T^{n_k} y \to x$, associated with weak l-sequential supercyclicity and supercyclicity, coincide, where convergence of the constant or finite sequence $\{\alpha_k T^{n_k} y\}$ means that a limit is eventually reached. Thus, in general, we use the expression

$$\{T^{n_k}\}_{k\geq 0} \textit{ is a sequence with entries from } \{T^n\}_{n\geq 0}.$$

However, if $x \notin \mathcal{O}_T([y])$ (or $x \notin \mathcal{O}_T(y)$), this means

$$\{T^{n_k}\}_{k\geq 0} \textit{ is a subsequence of } \{T^n\}_{n\geq 0}.$$

Since the weak topology is not metrizable, the notions of weak closure and weak sequential closure do not necessarily coincide.

(E) A vector y in $\mathcal{X}$ is a *weakly sequentially hypercyclic vector* for T if its orbit $\mathcal{O}_T(y)$ is weakly sequentially dense in $\mathcal{X}$ (i.e., dense in the weak sequential topology):

$$\mathcal{O}_T(y)^{-ws} = \mathcal{X}.$$

An operator T is a *weakly sequentially hypercyclic operator* if it has a weakly sequentially hypercyclic vector.

(F) A vector y in $\mathcal{X}$ is a *weakly sequentially supercyclic vector* for T if its projective orbit $\mathcal{O}_T([y])$ is weakly sequentially dense in $\mathcal{X}$ (i.e., dense in the weak sequential topology):

$$\mathcal{O}_T([y])^{-ws} = \mathcal{X}.$$

An operator T is a *weakly sequentially supercyclic operator* if it has a weakly sequentially supercyclic vector.

(G) A vector y in $\mathcal{X}$ is a *weakly hypercyclic vector* for T if its orbit $\mathcal{O}_T(y)$ is weakly dense in $\mathcal{X}$ (i.e., dense in the weak topology):

$$\mathcal{O}_T(y)^{-w} = \mathcal{X}.$$

An operator T is a *weakly hypercyclic operator* if it has a weakly hypercyclic vector.

(H) A vector y in $\mathcal{X}$ is a *weakly supercyclic vector* for T if its projective orbit $\mathcal{O}_T([y])$ is weakly dense in $\mathcal{X}$ (i.e., dense in the weak topology):

$$\mathcal{O}_T([y])^{-w} = \mathcal{X}.$$

An operator T is a *weakly supercyclic operator* if it has a weakly supercyclic vector.

(I) A vector y in $\mathcal{X}$ is a *cyclic vector* for an operator T if the span of its orbit span $\mathcal{O}_T(y)$ is dense in $\mathcal{X}$ (i.e., dense in the norm topology):

$$\big(\operatorname{span} \mathcal{O}_T(y)\big)^{-} = \mathcal{X}.$$

An operator T is a *cyclic operator* if it has a cyclic vector.

3.4. The relation among hypercyclicity, supercyclicity, and cyclicity. Since span $\mathcal{O}_T(y)$ is a convex subset of a normed space, its (norm) closure coincides with its weak closure, $(\operatorname{span}\mathcal{O}_T(y))^- = (\operatorname{span}\mathcal{O}_T(y))^{-w}$, and hence weak cyclicity boils down to cyclicity, and so do the intermediate concepts:

$$\big(\operatorname{span}\mathcal{O}_T(y)\big)^{-} = \big(\operatorname{span}\mathcal{O}_T(y)\big)^{-wl} = \big(\operatorname{span}\mathcal{O}_T(y)\big)^{-ws} = \big(\operatorname{span}\mathcal{O}_T(y)\big)^{-w}.$$

The above notions are related as follows.

$$\begin{array}{ccccc}
\text{HYPERCYCLIC} & \Longrightarrow & \text{SUPERCYCLIC} & \Longrightarrow & \text{CYCLIC} \\
\Downarrow & & \Downarrow & & \\
\begin{array}{c}\text{WEAKLY L-SEQUENTIALLY}\\ \text{HYPERCYCLIC}\end{array} & \Longrightarrow & \begin{array}{c}\text{WEAKLY L-SEQUENTIALLY}\\ \text{SUPERCYCLIC}\end{array} & & \\
\Downarrow & & \Downarrow & & \Updownarrow \\
\begin{array}{c}\text{WEAKLY SEQUENTIALLY}\\ \text{HYPERCYCLIC}\end{array} & \Longrightarrow & \begin{array}{c}\text{WEAKLY SEQUENTIALLY}\\ \text{SUPERCYCLIC}\end{array} & & \\
\Downarrow & & \Downarrow & & \\
\text{WEAKLY HYPERCYCLIC} & \Longrightarrow & \text{WEAKLY SUPERCYCLIC} & \Longrightarrow & \text{WEAKLY CYCLIC.}
\end{array}$$

It is known that there is no hypercyclic operator on a finite-dimensional normed space (see, e.g.,[5, Proposition 1.1] and [21, Theorem 2.58]), although every nonzero vector is trivially supercyclic for every operator on a one-dimensional space. Recall that weak and norm topologies coincide in a finite-dimensional space, and also that cyclicity implies separability for any normed space, as it is spanned by a countable orbit, and therefore all forms of cyclicity in the above diagram imply separability. This paper focuses mainly on three forms of weak supercyclicity, and so all normed spaces here are supposed to be infinite-dimensional and separable.

For basic properties concerning these notions of cyclicity, see, e.g., [53, pp.38,39], [5, pp.259,260], [21, pp.159,232], [34, pp.50,51]. In particular, questions related to some converses in the above diagram asking, for instance, whether weak l-sequential supercyclicity differs from weak sequential supercyclicity,

is there a weakly sequentially supercyclic not weakly l-sequentially supercyclic?

as well as whether weak l-sequential hypercyclicity differs from weak sequential hypercyclicity, have been addressed in [53, pp.71,72]. However, by omitting the four entries in the above diagram containing the term "sequentially", the remaining converses have been shown to fail, which has been summarised in [34, Proposition 3.1]. We will return to some of these nonreversible implications in Section 6.

4. A Review of Forms of Weak Supercyclicity

There is no reason to investigate any form of weak cyclicity since, as we saw above, weak cyclicity coincides with (strong) cyclicity. Our focus in this section is narrowed exclusively to

forms of *weak supercyclicity* only,

as described in Section 3. Historically, weak supercyclicity followed the first works on weak hypercyclicity. So, before discussing forms of weak supercyclicity properly, we take as a starting point a few pioneering works on weak hypercyclicity that explicitly motivated weak supercyclicity. Since the purpose of this section is to present a survey on forms of weak supercyclicity, aiming to discuss the literature in a rough chronological order, all references cited in this section will be followed by their respective year of publication.

4.1. PRE-WEAK SUPERCYCLICITY: 2002–2004. Feldman [17] (2002) asked whether every weakly hypercyclic operator is hypercyclic (i.e., strong hypercyclic — in the norm topology). Chan and Sanders [13] (2004) answered that question by exhibiting a weakly hypercyclic operator that is not hypercyclic, and they have asked if the spectrum of a weakly hypercyclic operator meets the unit circle. As it happened in the norm topology case shown by Kitai [30] (1982)), Dilworth and Troitsky [14] (2003) have shown that every component of the spectrum of a weakly hypercyclic operator intersects the unit circle, thus answering the question posed in [13] (2004). Although the above mentioned works did not deal with weak supercyclicity, but with hypercyclicity, weak or strong, they motivated and influenced the subsequent works on the several forms of weak supercyclicity.

4.2. THE BEGINNING OF WEAK SUPERCYCLICITY: 2004–2005. It seems that the first results specifically on weak supercyclicity are due to Sanders in [50] (2004) and in [51] (2005). In [50] (2004) she constructed a weakly supercyclic counterpart of the above mentioned result from [13] (2004) by exhibiting a weakly supercyclic operator that is not weakly hypercyclic. Also, it was exhibited in [50] (2004) a weakly supercyclic operator that is not supercyclic, and it was also shown that the set of weakly supercyclic vectors is dense in the norm topology. Moreover, it was shown in [50] (2004) that hyponormal operators on a Hilbert space are not weakly hypercyclic, neither supercyclic as shown by Bourdon [12] (1997), and the question of whether there exists a weakly supercyclic hyponormal operator closed the paper [50] (2004) (which was affirmatively answered later, as we will see below). In contrast with a result by Ansari and Bourdon [2] (1997), which says that there is no supercyclic isometry on Banach spaces, it was exhibited in [51] (2005) a weakly supercyclic isometry on the Banach space $c_0(\mathbb{Z})$ of bilateral sequences converging to zero on both sides. In [48] (2005) Prăjitură gave a collection of spectral properties for weakly supercyclic operators on a Hilbert space $\mathcal{H}$. (For supercyclic operators, a spectral description had been given by Herrero in [24] (1991).) It was also shown in [48] (2005) that the collection of all operators that are not weakly supercyclic is dense in $\mathcal{B}[\mathcal{H}]$.

4.3. FURTHER FORMS OF WEAK SUPERCYCLICITY: 2005–2006. Although the term weak l-sequential supercyclicity had not been coined at that time, Bès, Chan, and Sanders considered the notion in [8] (2005) to show that a bilateral weighted shift, on ℓ^p for $1 \leq p < \infty$ or on $c_0(\mathbb{Z})$, is weakly l-sequentially supercyclic if and only if it is (strong) supercyclic. Forms of weak* sequential supercyclicity have also been investigated by Bès, Chan, and Sanders in another paper [9] (2006). The notion of weak l-sequential supercyclicity was also implicitly used in at least one part (proof of Lemma 3.3) of [4] (2006). Bayart and Matheron [4] (2006) proved that if a hyponormal operator is weakly supercyclic, then it is a multiple of a unitary They also proved the existence of a weakly supercyclic unitary operator, in fact, the existence of a weakly l-sequentially supercyclic unitary operator, thus answering the question posed in [50] (2004) — there are weakly supercyclic hyponormal operators — and also extending to Hilbert spaces the existence of weakly supercyclic isometries on $c_0(\mathbb{Z})$ proved in [51] (2005). The question of whether there exists a weakly hypercyclic operator that is not weakly sequentially supercyclic closed the paper [4] (2006) (which was affirmatively answered later, as we will see below).

4.4. CLARIFYING THREE FORMS OF WEAK SUPERCYCLICITY: 2007. The notions of weak l-sequential supercyclicity, weak sequential supercyclicity, and weak supercyclicity, as presented in Section 3, have been considered by Shkarin [53] (2007), where the notion of weak l-sequential supercyclicity, stronger than the usual weak sequential supercyclicity but weaker than (strong) supercyclicity, was discussed in detail, thus amounting to the following inclusions:

$$\text{WEAK L-SEQUENTIAL SUPERCYCLICITY} \subseteq \text{WEAK SEQUENTIAL SUPERCYCLICITY} \subset \text{WEAK SUPERCYCLICITY.}$$

Special attention was given to whether these were proper inclusions. It was shown in [53] (2007) the existence of a weakly supercyclic unitary operator that is not weakly sequentially supercyclic, and so not weakly l-sequentially supercyclic. It was also asked in [53] (2007) whether there is a weakly sequentially supercyclic operator that is not weakly l-sequentially supercyclic. (Is there?) The original terminology for weak l-sequential supercyclicity is different from ours: we use the letter “l” for “limit”, borrowed from [34] (2018), while the numeral “1” was used in [53] (2007) — and there are reasons for both terminologies.

4.5. WEAK SUPERCYCLICITY FOR OPERATORS CLOSE TO NORMAL: 2006–2020. Consider the well-known chain of classes of Hilbert-space operators:

$$\text{UNITARY} \subset \text{NORMAL} \subset \text{HYPONORMAL} \subset \text{QUASIHYPONORMAL} \subset \text{SEMI-QUASIHYPONORMAL} \subset \text{PARANORMAL} \subset \text{NORMALOID.}$$

For the above classes, set $|T| = (T^*T)^{1/2}$ and recall that an operator T on a Hilbert space $\mathcal{H}$ is, by definition, unitary if $|T^*|^2 = |T|^2 = I$, normal if $|T^*|^2 = |T|^2$, hyponormal if $|T^*|^2 \le |T|^2$, quasihyponormal if $|T|^4 \le |T^2|^2$, semi-quasihyponormal if $|T|^2 \le |T^2|$ (also known as class $\mathcal{A}$), paranormal if $\|Tx\|^2 \le \|T^2x\|\,\|x\|$ for every x in $\mathcal{H}$, and normaloid if $\|T^n\| = \|T\|^n$ for every $n \ge 0$, equivalently, if $r(T) = \|T\|$.

There is no supercyclic unitary operator. Indeed, as we have seen in Subsection 4.2, Ansari and Bourdon [2, Proof of Theorem 2.1] (1997) have shown that there is no supercyclic isometry on a Banach space — as a matter of fact, Hilden and Wallen [26, p.564] (1974) had already shown that the unilateral shift is not supercyclic. Actually, Bourdon [12, Theorem 3.1] (1997) had shown that there is no supercyclic hyponormal operator on a Hilbert space, as we have also seen in Subsection 4.2, extending Kitai's result [30, Corollary 4.5] (1982) where she had shown that there is no hypercyclic hyponormal operator; and also extending Hilden and Wallen's result [26, p.564] (1974) where they had shown that there is no supercyclic normal operator. In spite of this, there are weakly supercyclic unitary (thus normal, and hence hyponormal) operators on a Hilbert space. In fact, we saw in Subsection 4.3 that Bayart and Matheron [4, Example 3.6] (2006) have shown that there exist weakly l-sequentially supercyclic (thus weakly supercyclic) unitary operators, and also that if a hyponormal operator is weakly supercyclic, then it is a multiple of a unitary operator [4, Theorem 3.4] (2006). Shkarin [53, Corollary 1.3] (2007) has shown the existence of a weakly supercyclic unitary operator that is not weakly sequentially (thus not weakly l-sequentially) supercyclic. Results about reduction to multiples of unitary operators have been extended beyond the hyponormal operators. For instance, Jung, Kim, and Ko [27, Theorem 4.2] (2014) proved that if an injective semi-quasihyponormal (i.e., class $\mathcal{A}$) operator is weakly supercyclic, then it is a scalar multiple of a unitary. This was followed by Duggal, Kubrusly, and Kim

[16, Theorem 2.7] (2015), where the injectivity assumption was dismissed. Duggal [15, Corollary 3.2] (2016) has shown that if T is such that $\|T^*x\|^2 \leq \|T^2x\|\,\|x\|$ for every $x \in \mathcal{H}$ (such operators are called $*$-paranormal) and is weakly supercyclic, then it is a scalar multiple of a unitary (among other results). Still along this line, Shen and Ji [52] (2020) have shown that weakly supercyclic quasi-2-expansive operator (i.e., an operator T for which $T^{*3}T^3 - 2T^{*2}T^2 + T^*T \leq O$) is unitary. All the above classes include the unitary operators, as discussed in [4], [53], [27], [16], [15], [52]. Weak forms of supercyclicity for unitary operators will be considered in Section 6.

4.6. WEAK SUPERCYCLICITY FOR COMPACT OPERATORS: 2007–2018. It had already been shown by León-Saavedra and Piqueiras-Lerena [41] (2003) and Gallardo-Gutiérrez and Montes-Rodríques [20] (2004) that the Volterra operator on $L^p[0,1]$, $p \geq 1$ is not supercyclic. A nonelementary proof that the Volterra operator is not even weakly supercyclic was given by Montes-Rodrígues and Shkarin [45] (2007), and so, in particular, it is not weakly l-sequentially supercyclic. Shkarin [54] (2010) has shown that any operator on $L^p[0,1]$ commuting with the Volterra operator is not weakly supercyclic. Also, in a paper linked to Volterra operators, Hedayatian and Faghih-Ahmadi [23] have observed that there is no weakly supercyclic operator in the commutant of a cyclic convolution operator. It has been verified by Kubrusly and Duggal [35] (2018) that for a compact operator on a normed space, weak l-sequential supercyclicity coincides with supercyclic. Thus, since Volterra operators are compact and not supercyclic, as we saw above, the non-weak l-sequential supercyclicity comes at once. It was also shown in [35] (2018) that a weakly l-sequentially supercyclic compact operator on an infinite-dimensional Banach space is quasinilpotent (recall that Volterra operators are also quasinilpotent).

Remark 3. STILL ON COMPACT OPERATORS.

(i) As weak l-sequential supercyclicity coincides with supercyclicity for compact operators, weak supercyclicity coincides with supercyclicity for bilateral weighted shifts on ℓ^p for $1 \leq p < 2$ [45, Theorem 6.3] (2007) — for weak l-sequential supercyclicity and for $1 \leq p < \infty$, see [8, Theorem 1] (2005), as mentioned in 4.3 above.

(ii) A hypercyclic operator is not compact, and there is no supercyclic operator on a complex normed space with a finite dimension greater than 1 [25, Section 4] (1992). There are, however, compact supercyclic operators on a separable infinite-dimensional complex Banach space [25, Theorem 1 and Section 4]. Consequently, there are weakly supercyclic compact operators for any form of weak supercyclicity.

(iii) The Volterra operator is an example of a compact quasinilpotent that is not supercyclic, and so it is not weakly l-sequentially supercyclic, as we saw above. But, as we saw in Subsection 4.6, there exist quasinilpotent supercyclic operators. It was shown in [49, Corollary 5.3] (1999) that if the adjoint of a bilateral weighted shift on ℓ^2, or of a unilateral weighted shift on ℓ^2_+, has a weighting sequence possessing a subsequence that goes to zero, then there is an infinite-dimensional subspace whose all nonzero vectors are supercyclic for it. In particular, this holds for the adjoint of weighted shifts with weighting sequences converging to zero, and so this happens for the adjoint of compact quasinilpotent weighted shifts.

4.7. WEAK SUPERCYCLICITY FOR COMPOSITION OPERATORS: 2010–2021. There are a large number of works dealing with cyclicity and hypercyclicity (weak or not), as well as with supercyclicity, for composition operators. The number, however, is not so large when dealing with weak supercyclicity. Kamali, Hedayatian, and Khani Robati [29] (2010) investigated conditions for weak supercyclicity of weighted composition operators, in particular on the Hilbert space of holomorphic functions on the unit disk. Weak supercyclicity, and weak hypercyclicity, of composition operators on some Hilbert spaces of analytic functions (e.g., on certain weighted Hardy spaces) were investigated by Kamali [28] (2013). Bès [7] (2013) has given necessary and sufficient conditions for a composition operator to be weakly supercyclic. Conditions for weak supercyclicity of some classes of weighted composition operators were given by Moradi, Khani Robati, and Hedayatian [44] (2017), where specific spaces on which weak supercyclicity of composition operators never holds were also investigated. Beltrán-Meneu, Jordá, and Murillo-Arcila [6] (2020) also considered conditions ensuring that certain weighted composition operators are not weakly supercyclic. See also Mengestie [43] (2021) for non-weak supercyclicity of weighted composition operators on certain Fock spaces.

4.8. WEAK SUPERCYCLICITY AND WEAK STABILITY: 2016–2024. This is a central topic in the present exposition. A review of current literature dealing with weak supercyclicity and its relation to weak stability will be given in Section 6. Before discussing if (or when) weak supercyclicity implies weak stability, we need to consider the motivating fact that is behind such a problem, viz., supercyclicity implies strong stability. This fundamental result will be treated in detail in the next section.

Remark 4. FURTHER RESULTS RELATED TO WEAK SUPERCYCLICITY.

(i) The term *weak supercyclicity* has also been used to mean a notion different from those considered above. For instance, a subset S of a Hilbert space $\mathcal{H}$ is said to be *n-weakly dense* in $\mathcal{H}$ if for every n-dimensional subspace $\mathcal{M}$ of $\mathcal{H}$, the image of S under the orthogonal projection E onto $\mathcal{M}$ is dense in $\mathcal{M}$ (i.e., if $E(S)^- = \mathcal{M}$ for the orthogonal projection $E\colon \mathcal{H} \to \mathcal{H}$ such that $\operatorname{range}(\mathrm{E}) = \mathcal{M}$ if $\dim \mathcal{M} = n$). An operator T in $\mathcal{B}[\mathcal{H}]$ is *n-weakly supercyclic* if it has a projective orbit $\mathcal{O}([y])$ for some y in $\mathcal{H}$ that is n-weakly dense in $\mathcal{H}$ (see, e.g., [18, 19] (2011, 2012)).

(ii) If an operator is invertible and supercyclic, then so is its inverse [2, Section 4] (1997), [49, Corollary 2.4] (1999). There are, however, invertible weakly supercyclic operators on ℓ^p for $p \in [2, \infty)$ whose inverses are not weakly supercyclic [50, Corollary 2.5] (2004). Is the inverse (i.e., the adjoint) of a weakly supercyclic unitary operator weakly supercyclic? Recall that the adjoint of a supercyclic coisometry may not be supercyclic. Example: a backward unilateral shift S^* is a supercyclic coisometry [26, Theorem 3] (1974), while its adjoint, the unilateral shift S, being an isometry, is not supercyclic [26, p.564] (1974) — also see [2, Proof of Theorem 2.1] (1997) or [34, Lemma 4.1(b)] (2018). The same happens with weak supercyclicity: the adjoint of a weakly supercyclic coisometry may not be weakly supercyclic. Example: S is not weakly supercyclic — since a weakly supercyclic hyponormal operator is a multiple of a unitary operator [4, Theorem 3.4] (2006) — but S^* is weakly supercyclic, since it is supercyclic. However, if an Hilbert-space isometry is invertible (i.e., if it is unitary) and weakly l-sequentially supercyclic, then it has a weakly l-sequentially supercyclic adjoint [35, Theorem 3.3] (2018).

(iii) The notion of hypercyclicity, in particular, has been investigated in Fréchet spaces, or F-spaces, or locally convex spaces (see, e.g., [1] (1997), [11] (1998), [46] (2001), [10] (2005), [5] (2009), [21] (2011), among others). Our discussion in the present paper is restricted to normed (or Banach) spaces.

5. Strong Supercyclicity and Strong Stability

An important motivation for an investigation on weak supercyclicity and its relationship with weak stability is the fact that (strong) supercyclicity and strong stability behave nicely: *supercyclicity implies strong stability*. The verification of this fact is all but trivial, and the proof itself naturally impels the quest for a weak counterpart. For this reason, we begin by looking at the strong case first.

It was proved in [2, Theorem 2.1] that *if a power bounded operator T on a Banach space $\mathcal{X}$ is such that $\|T^n x\| \not\to 0$ for every nonzero $x \in \mathcal{X}$, then T has no supercyclic vector*. In other words, a power bounded operator of class $C_{1\cdot}$ is not supercyclic. Based on this result, more was proved in [2, Theorem 2.2], namely, *if a power bounded operator $T \in \mathcal{B}[\mathcal{X}]$ is such that there exists a vector $y \in \mathcal{X}$ for which there is a scalar sequence $\{\alpha_k\}_{k\geq 0}$ and a sequence $\{T^{n_k}\}_{k\geq 0}$ with entries from $\{T^n\}_{n\geq 0}$ such that $\alpha_k T^{n_k} y \to x$ for each $x \in \mathcal{X}$, then $T^n z \to 0$ for every $z \in \mathcal{X}$*. In other words, a power bounded supercyclic operator on a normed space is strongly stable.

These significant results by Ansari and Bourdon [2] (1997) are essential in linking supercyclicity to stability, establishing a direct implication from (strong) supercyclicity to strong stability. Such an implication will be crucial in any attempt to extend it, expecting that some form of weak supercyclicity will imply weak stability.

The above mentioned results are stated below, followed by detailed proofs. This is done here not only for sake of completeness but mainly because a close analysis of the structure of their proofs will be fundamental for the next section. In fact, the arguments in the next proofs are required to yield a possible counterpart of Theorem 5.2 below for weak l-sequential supercyclicity and weak stability.

Theorem 5.1. [2] *A power bounded operator of class $C_{1\cdot}$ acting on a normed space is not supercyclic.*

Proof. The well-known notion of Banach limit is a central argument in the proof. For a summary of all Banach limit properties required here, see, e.g., [36, Section 6]. Let the functional $\varphi\colon \ell^\infty_+ \to \mathbb{C}$, assigning a complex number $\varphi(\{\xi_n\})$ to each bounded complex-valued sequence $\{\xi_n\} = \{\xi_n\}_{n\geq 0} \in \ell^\infty_+$, be a Banach limit. This means that φ is linear, bounded, positive (i.e., $0 \leq \varphi(\{\xi_n\})$ whenever $0 \leq \xi_n$ for every integer $n \geq 0$), backward-shift-invariant (i.e., $\varphi(\{\xi_n\}) = \varphi(\{\xi_{n+1}\})$), with $\|\varphi\| = 1$, and assigns to each convergent sequence its own limit. Let $(\mathcal{X}, \|\cdot\|)$ be any normed space. Suppose $T \in \mathcal{B}[\mathcal{X}]$ is power bounded. For each $x \in \mathcal{X}$ consider the bounded sequence of nonnegative numbers $\{\|T^n x\|\}_{n\geq 0}$, and set

$$\|x\|_\varphi = \varphi(\{\|T^n x\|\})$$

for any Banach limit φ. Since a Banach limit φ is order-preserving for real-valued bounded sequences (i.e., $\varphi(\{\xi_n\}) \leq \varphi(\{\upsilon_n\})$ whenever ξ_n and υ_n are real numbers such that $\xi_n \leq \upsilon_n$), with $\varphi(\{1,1,1,\dots\}) = 1$, and since T is power bounded, this defines an $\mathbb{R}$-valued functional $\|\cdot\|_\varphi\colon \mathcal{X} \to \mathbb{R}$ with the property that, for every $x \in \mathcal{X}$,

$$\|x\|_\varphi \leq (\sup_n \|T^n\|)\, \|x\|.$$

But this defines a seminorm on $\mathcal{X}$. Indeed, since φ is a positive functional, we get $0 \leq \|x\|_\varphi$ for every $x \in \mathcal{X}$. Since, in addition, φ is linear (as well as T) and order-preserving, we get $|\alpha|\|x\|_\varphi = \|\alpha x\|_\varphi$ and $\|x+y\|_\varphi \leq \|x\|_\varphi + \|y\|_\varphi$ for every $x, y \in \mathcal{X}$

and every $\alpha \in \mathbb{C}$. However, as the power bounded operator T is of class $C_{1\cdot}$, it follows that $\|\cdot\|_\varphi$ is a norm, as we will see next.

Claim 0. If T is power bounded operator on a normed space, then

$$T^n x \not\to 0 \quad \text{implies} \quad 0 < \liminf_n \|T^n x\| \quad \text{which implies} \quad 0 < \inf_n \|T^n x\|.$$

Proof of Claim 0. (i) Suppose $\{\beta_n\}_{n\geq 0}$ is a real-valued sequence such that

$$0 \leq \beta_n \leq \beta\, \beta_m \quad \text{whenever} \quad m \leq n$$

for some positive constant β. If $\beta_n \not\to 0$, then $0 < \beta_n \leq \beta\, \beta_0$ for every n. So $\{\beta_n\}_{n\geq 0}$ is a bounded sequence of positive numbers for which there is a subsequence $\{\beta_{n_i}\}_{i\geq 0}$ such that $0 < \lim_i \beta_{n_i}$ and $\beta_{n_i} \leq \beta\, \beta_n$ if $n < n_i$. Hence $0 < \lim_i \beta_{n_i} = \liminf_i \beta_{n_i} \leq \beta \liminf_n \beta_n$, which ensures $0 < \inf_n \beta_n$ (since $0 < \beta_n$ for every n). Thus

$$\beta_n \not\to 0 \qquad \Longrightarrow \qquad 0 < \liminf_n \beta_n \qquad \Longrightarrow \qquad 0 < \inf_n \beta_n.$$

(ii) Since $0 \leq \|T^n x\| = \|T^{n-m}\, T^m x\| \leq (\sup_n \|T^n\|)\, \|T^m x\|$ for every x whenever $m \leq n$, the proof of Claim 0 is complete by setting $\beta_n = \|T^n x\|$ and $\beta = \sup_n \|T^n\|$. $\square$

So, according to Claim 0, if T is of class $C_{1\cdot}$ (i.e., if $T^n x \not\to 0$ for every $0 \neq x \in \mathcal{X}$), then $0 < \liminf_n \|T^n x\|$ for every $0 \neq x \in \mathcal{X}$. Moreover, as $\liminf_n \xi_n \leq \varphi(\{\xi_n\}) \leq \limsup_n \xi_n$ for every real-valued bounded sequence $\{\xi_n\}$, we get $0 < \varphi(\{\|T^n x\|\}) = \|x\|_\varphi$ if $x \neq 0$. Consequently,

$$T \text{ is of class } C_{1\cdot} \quad \Longrightarrow \quad \|\cdot\|_\varphi \text{ is a norm on } \mathcal{X}.$$

Thus assuming that T is of class $C_{1\cdot}$ is required to ensure that $\|\cdot\|_\varphi$ is a norm. Then consider the normed space $\mathcal{X}_\varphi = (\mathcal{X}, \|\cdot\|_\varphi)$ wherein the operator T acts as an isometry. In fact, since φ is backward-shift-invariant, we get for every $x \in \mathcal{X}$

$$\|Tx\|_\varphi = \varphi(\{\|(T^{n+1}x)\|\}) = \varphi(\{\|(T^n x)\|\}) = \|x\|_\varphi.$$

Now consider the completion $\widehat{\mathcal{X}} = (\widehat{\mathcal{X}}, \|\cdot\|_{\widehat{\mathcal{X}}})$ of the normed space $\mathcal{X}_\varphi = (\mathcal{X}, \|\cdot\|_\varphi)$, so tat $\mathcal{X}_\varphi$ is isometrically and densely embedded in $\widehat{\mathcal{X}}$, which means $\mathcal{X}_\varphi$ is isometrically isomorphic to a dense linear manifold $\widetilde{\mathcal{X}}$ of the Banach space $\widehat{\mathcal{X}}$. In other words, there is an isometric isomorphism $J\colon \mathcal{X}_\varphi \to J(\mathcal{X}_\varphi) = \widetilde{\mathcal{X}}$ where $\widetilde{\mathcal{X}}$ is dense in $\widehat{\mathcal{X}}$. Consider the extension over completion $\widehat{T} \in \mathcal{B}[\widehat{\mathcal{X}}]$ of $T \in \mathcal{B}[\mathcal{X}_\varphi]$, and let $\widetilde{T} = JTJ^{-1}$ be the restriction of $\widehat{T}$ to $\widetilde{\mathcal{X}}$. That is, $\widetilde{T} = \widehat{T}|_{\widetilde{\mathcal{X}}} = JTJ^{-1} \in \mathcal{B}[\widetilde{\mathcal{X}}]$. Thus $\widetilde{\mathcal{X}}$ is $\widehat{T}$-invariant and $T = J^{-1}\widetilde{T}J = J^{-1}\widehat{T}|_{\widetilde{\mathcal{X}}}J = J^{-1}\widehat{T}J \in \mathcal{B}[\mathcal{X}_\varphi]$, as in the following commutative diagram, where the symbol $\bar{\subseteq}$ means densely included:

$$\begin{array}{ccccccc}
\mathcal{X}_\varphi & \xrightarrow{\;J\;} & \widetilde{\mathcal{X}} & = & J(\mathcal{X}_\varphi) & \bar{\subseteq} & \widehat{\mathcal{X}} \\
{\scriptstyle T}\big\uparrow & & \big\uparrow{\scriptstyle \widetilde{T}=\widehat{T}|_{\widetilde{\mathcal{X}}}=JTJ^{-1}} & & & & \big\uparrow{\scriptstyle \widehat{T}} \\
\mathcal{X}_\varphi & \xrightarrow{\;J\;} & \widetilde{\mathcal{X}} & = & J(\mathcal{X}_\varphi) & \bar{\subseteq} & \widehat{\mathcal{X}}.
\end{array}$$

Next suppose T is supercyclic, and observe that supercyclicity easily survives the change of norms from $\|\cdot\|$ to $\|\cdot\|_\varphi$; that is,

(a) *if $y \in \mathcal{X}$ is a supercyclic vector for T when it acts on $\mathcal{X} = (\mathcal{X}, \|\cdot\|)$, then this same $y \in \mathcal{X}$ is a supercyclic vector for T when it acts on $\mathcal{X}_\varphi = (\mathcal{X}, \|\cdot\|_\varphi)$.*

Indeed, since $0 < \sup_n \|T^n\| < \infty$ and $\|z\|_\varphi \leq (\sup_n \|T^n\|)\, \|z\|$ for every $z \in \mathcal{X}$, it follows at once that $\|\alpha_i T^{n_i} y - x\|_\varphi \leq (\sup_n \|T^n\|) \|\alpha_i T^{n_i} y - x\|$, and so y is a supercyclic vector in $\mathcal{X}_\varphi$ whenever it is a supercyclic vector in $\mathcal{X}$.

This ensures that supercyclicity can be extended from T on $\mathcal{X}$ to $\widehat{T}$ on $\widehat{\mathcal{X}}$. In fact,

(b) *if $y \in \mathcal{X}$ is a supercyclic vector for T when it acts on $\mathcal{X}_\varphi$, then its image $\widetilde{y} = Jy \in \widetilde{\mathcal{X}}$ under the isometric isomorphism J of $\mathcal{X}_\varphi$ onto the dense linear manifold $\widetilde{\mathcal{X}}$ of the Banach space $\widehat{\mathcal{X}}$ is supercyclic for the extension $\widehat{T}$ on $\widehat{\mathcal{X}}$ of T.*

The above result was applied in the proof of Theorem 2.1 from [2] (on p.198, lines 11,12) but a proof was omitted. We give a proof of it in Claim I below.

Claim I. If there is a supercyclic vector $y \in \mathcal{X}_\varphi$ for $T \in \mathcal{B}[\mathcal{X}_\varphi]$, then there is a supercyclic vector $\widetilde{y} \in \widehat{\mathcal{X}}$ for $\widehat{T} \in \mathcal{B}[\widehat{\mathcal{X}}]$.

Proof of Claim I. Since $\widetilde{T} = \widehat{T}|_{\widetilde{\mathcal{X}}} = JTJ^{-1}$, where $\widetilde{\mathcal{X}}$ is $\widehat{T}$-invariant, we get $\widetilde{T}^n = (\widehat{T}|_{\widetilde{\mathcal{X}}})^n = \widehat{T}^n|_{\widetilde{\mathcal{X}}} = JT^nJ^{-1}$ for every integer $n \geq 0$. Now split the proof into two parts.

(1) If there is a supercyclic vector $y \in \mathcal{X}_\varphi$ for $T \in \mathcal{B}[\mathcal{X}_\varphi]$, then there is a supercyclic vector $\widetilde{y} \in \widetilde{\mathcal{X}}$ for $\widetilde{T} \in \mathcal{B}[\widetilde{\mathcal{X}}]$.

(2) If there is a supercyclic vector $\widetilde{y} \in \widetilde{\mathcal{X}}$ for $\widetilde{T} \in \mathcal{B}[\widetilde{\mathcal{X}}]$, then $\widetilde{y}$ is a supercyclic vector for $\widehat{T} \in \mathcal{B}[\widehat{\mathcal{X}}]$.

Proof of (1). If there is a supercyclic vector $y \in \mathcal{X}_\varphi$ for $T \in \mathcal{B}[\mathcal{X}_\varphi]$, then for every $0 \neq x \in \mathcal{X}_\varphi$ there exists a sequence of nonzero numbers $\{\alpha_j\}_{j\geq 0}$ that depends on x, and so we write $\{\alpha_j(x)\}_{j\geq 0}$, and a sequence $\{T^{n_j}\}_{j\geq 0}$ with entries from $\{T^n\}_{n\geq 0}$, which also depends on x, such that, in $\mathcal{X}_\varphi = (\mathcal{X}, \|\cdot\|_\varphi)$,

$$\alpha_j(x)T^{n_j}y \longrightarrow x.$$

Set $\widetilde{y} = Jy$ in $\widetilde{\mathcal{X}}$. As every $\widetilde{x}$ in $\widetilde{\mathcal{X}}$ is of the form $\widetilde{x} = Jx$ for some x in $\mathcal{X}_\varphi$, thus for every $\widetilde{x}$ in $\widetilde{\mathcal{X}}$ set $\widetilde{\alpha}_j(\widetilde{x}) = \widetilde{\alpha}_j(Jx) = \alpha_j(x)$ for some $x \in \mathcal{X}_\varphi$. Take an arbitrary $\widetilde{x} \in \widetilde{\mathcal{X}}$. Then there exists a sequence $\{\widetilde{\alpha}_j(\widetilde{x})\}_{j\geq 0}$ such that (with $\widetilde{T}^{n_j} = J\,T^{n_j}J^{-1}$)

$$\begin{aligned}\|\widetilde{\alpha}_j(\widetilde{x})\widetilde{T}^{n_j}\widetilde{y} - \widetilde{x}\|_{\widehat{\mathcal{X}}} &= \|\alpha_j(x)\,JT^{n_j}y - Jx\|_{\widehat{\mathcal{X}}}\\ &= \|J(\alpha_j(x)T^{n_j}y - x)\|_{\widehat{\mathcal{X}}} = \|\alpha_j(x)T^{n_j}y - x\|_{\mathcal{X}_\varphi}\end{aligned}$$

for some $x \in \mathcal{X}_\varphi$. Hence, since $y \in \mathcal{X}_\varphi$ is a supercyclic vector for $T \in \mathcal{B}[\mathcal{X}_\varphi]$ such that $\|\alpha_j(x)T^{n_j}y - x\|_{\mathcal{X}_\varphi} \to 0$, we get $\|\widetilde{\alpha}_j(\widetilde{x})\widetilde{T}^{n_j}\widetilde{y} - \widetilde{x}\|_{\widehat{\mathcal{X}}} \to 0$, and therefore $\widetilde{y} = Jy \in \widetilde{\mathcal{X}}$ is a supercyclic vector for $\widetilde{T} \in \mathcal{B}[\widetilde{\mathcal{X}}]$. This proves (1): for every $\widetilde{x} \in \widetilde{\mathcal{X}}$

$$\widetilde{\alpha}_j(\widetilde{x})\widetilde{T}^{n_j}\widetilde{y} \longrightarrow \widetilde{x}.$$

Proof of (2). Take an arbitrary $\widehat{x} \in \widehat{\mathcal{X}}$. Since $\widetilde{\mathcal{X}}^- = \widehat{\mathcal{X}}$, there is a sequence $\{\widetilde{x}_k\}_{k\geq 0}$, with $\widetilde{x}_k = \widetilde{x}_k(\widehat{x}) \in \widetilde{\mathcal{X}}$ for each k, such that $\widetilde{x}_k \to \widehat{x}$. Then, by the above convergence,

$$\widetilde{\alpha}_j(\widetilde{x}_k)\widetilde{T}^{n_j}\widetilde{y} \underset{j}{\longrightarrow} \widetilde{x}_k \underset{k}{\longrightarrow} \widehat{x}. \tag{$*$}$$

This ensures the existence of a sequence of nonzero numbers $\{\widehat{\alpha}_i(\widehat{x})\}_{i\geq 1}$ such that

$$\widehat{\alpha}_i(\widehat{x})\widehat{T}^{n_i}\widetilde{y} \longrightarrow \widehat{x} \tag{$**$}$$

for a sequence $\{\widehat{T}^{n_i}\}_{i\geq 1}$ with entries from $\{\widehat{T}^n\}_{n\geq 0}$, where each $\widehat{T}^n$ is the extension over completion of each T^n — equivalently, the extension by continuity of each $\widetilde{T}^n$.

Indeed, consider both convergences in ($*$). Take an arbitrary $\varepsilon > 0$. Thus there exists a positive integer k_ε such that $\|\widetilde{x}_k - \widehat{x}\|_{\widehat{\mathcal{X}}} \leq \frac{\varepsilon}{2}$ whenever $k \geq k_\varepsilon$. Moreover, for each k there exists a positive integer $j_{\varepsilon,k}$ such that $\|\widetilde{\alpha}_j(\widetilde{x}_k)\widetilde{T}^{n_j}\widetilde{y}-\widetilde{x}_k\|_{\widehat{\mathcal{X}}}\le\frac{\varepsilon}{2}$ whenever $j\ge j_{\varepsilon,k}$. Therefore, by taking $k=k_\varepsilon$,

$$\begin{aligned}\|\widetilde{\alpha}_j(\widetilde{x}_{k_\varepsilon})\widehat{T}^{n_j}\widetilde{y}-\widehat{x}\|_{\widehat{\mathcal{X}}} &= \|(\widetilde{\alpha}_j(\widetilde{x}_{k_\varepsilon})(\widehat{T}|_{\widetilde{\mathcal{X}}})^{n_j}\widetilde{y}-\widetilde{x}_{k_\varepsilon})+(\widetilde{x}_{k_\varepsilon}-\widehat{x})\|_{\widehat{\mathcal{X}}}\\ &\le \|\widetilde{\alpha}_j(\widetilde{x}_{k_\varepsilon})\widetilde{T}^{n_j}\widetilde{y}-\widetilde{x}_{k_\varepsilon}\|_{\widehat{\mathcal{X}}}+\|\widetilde{x}_{k_\varepsilon}-\widehat{x}\|_{\widehat{\mathcal{X}}}\le\varepsilon\end{aligned}$$

whenever $j\ge j_{\varepsilon,k_\varepsilon}$. Take an arbitrary integer $i\ge1$ and set $\varepsilon=\frac{1}{i}$. Consequently, set $k(i)=k_\varepsilon=k_{\frac{1}{i}}$ and $j(i)=j_{\varepsilon,k_\varepsilon}=j_{\frac{1}{i},k(i)}$ for every $i\ge1$. Thus for every integer $i\ge1$ there is an integer $j(i)\ge0$ such that, by setting $\widetilde{\alpha}_j(\widetilde{x}_{k(i)})=\widetilde{\alpha}_j(\widetilde{x}_{k_\varepsilon})=\widetilde{\alpha}_j(\widetilde{x}_{k_\varepsilon}(\widehat{x}))$, we get $\|\widetilde{\alpha}_j(\widetilde{x}_{k(i)})\widehat{T}^{n_j}\widetilde{y}-\widehat{x}\|_{\widehat{\mathcal{X}}}\le\frac{1}{i}$ whenever $j\ge j(i)$. Therfore, by taking $j=j(i)$,

$$\|\widetilde{\alpha}_{j(i)}(\widetilde{x}_{k(i)})\widehat{T}^{n_{j(i)}}\widetilde{y}-\widehat{x}\|_{\widehat{\mathcal{X}}}\le\tfrac{1}{i}\qquad\text{for every integer } i\ge1.$$

So for the sequence $\{\widetilde{\alpha}_{j(i)}(\widetilde{x}_{k(i)})\}_{i\ge1}$ we get $\|\widetilde{\alpha}_{j(i)}(\widetilde{x}_{k(i)})\widehat{T}^{n_{j(i)}}\widetilde{y}-\widehat{x}\|_{\widehat{\mathcal{X}}}\to0$. Hence

$$\widetilde{\alpha}_{j(i)}(\widetilde{x}_{k(i)})\widehat{T}^{n_{j(i)}}\widetilde{y}\longrightarrow\widehat{x}.$$

For each integer $i\ge1$, set $\widehat{\alpha}_i(\widehat{x})=\widetilde{\alpha}_{j(i)}(\widetilde{x}_{k(i)})=\widetilde{\alpha}_{j(i)}(\widetilde{x}_{k(i)}(\widehat{x}))$ and $\widehat{T}^{n_i}=\widehat{T}^{n_{j(i)}}$. Thus for every $\widehat{x}\in\widehat{\mathcal{X}}$ there exist a sequence of nonzero numbers $\{\widehat{\alpha}_i(\widehat{x})\}_{i\ge1}$ and a sequence $\{\widehat{T}^{n_i}\}_{i\ge1}$ with entries from $\{\widehat{T}^n\}_{n\ge0}$ such that $(**)$ holds. So $\widetilde{y}\in\widetilde{\mathcal{X}}\subseteq\widehat{\mathcal{X}}$ is a supercyclic vector for $\widehat{T}\in\mathcal{B}[\widehat{\mathcal{X}}]$. This proves (2), concluding the proof of Claim 1. □

But T is an isometry on the normed space $(\mathcal{X},\|\cdot\|_\varphi)$, and so is its extension $\widehat{T}$ on the completion $(\widehat{\mathcal{X}},\|\cdot\|_{\widehat{\mathcal{X}}})$ of $(\mathcal{X},\|\cdot\|_\varphi)$. Thus, by (a) and (b), the Banach-space isometry $\widehat{T}$ has a supercyclic vector. This, however, leads to a contradiction, since

an isometry on a Banach space is never supercyclic

[2, Proof of Theorem 2.1] (see also [34, Lemma 4.1(b)]). Outcome: there is no supercyclic vector $y\in\mathcal{X}$ for the operator T acting on the normed space $(\mathcal{X},\|\cdot\|)$. □

Theorem 5.2. [2] *If a power bounded operator T on a normed space $\mathcal{X}$ is supercyclic, then it is strongly stable.*

Proof. Suppose T is a power bounded operator and set $\beta=\sup_n\|T^n\|>0$.

Claim II. If $y\in\mathcal{X}$ is supercyclic for T, then $T^ny\to0$.

Proof of Claim II. Let $0\ne y\in\mathcal{X}$ be supercyclic for T. Equivalently, let $y\in\mathcal{X}$ be a supercyclic unit vector ($\|y\|=1$) for T. Suppose $T^ny\not\to0$. Thus by Claim 0 (cf. proof of Theorem 5.1) $0<\inf_n\|T^ny\|$; that is, there exists a $\delta>0$ for which

$$\delta<\|T^ny\| \tag{i}$$

for every integer n. Moreover, since T is power bounded and supercyclic, it follows from Theorem 5.1 that T is not of class $C_{1\cdot}$. So there exists a unit vector $v\in\mathcal{X}$ ($\|v\|=1$) such that $T^nv\to0$. But since y is a supercyclic vector for T, there exists a sequence $\{\alpha_k\}_{k\ge0}$ of nonzero numbers and a sequence $\{T^{n_k}\}_{k\ge0}$ with entries from $\{T^n\}_{n\ge0}$ such that $\alpha_kT^{n_k}y\to v$, and hence there exists a positive integer k_0 such that if $k\ge k_0$, then $\|\alpha_kT^{n_k}y-v\|<\frac{1}{2}$. So $|\,\|\alpha_kT^{n_k}y\|-1\,|=|\,\|\alpha_kT^{n_k}y\|-\|v\|\,|\le\|\alpha_kT^{n_k}y-v\|<\frac{1}{2}$, which implies $1-\|\alpha_kT^{n_k}y\|<\frac{1}{2}$, and therefore $\frac{1}{2}=1-\frac{1}{2}<\|\alpha_kT^{n_k}y\|\le|\alpha_k|\sup_n\|T^n\|\|y\|=\beta|\alpha_k|$. Hence for $k\ge k_0$

$$\tfrac{1}{2\beta}<|\alpha_k|. \tag{ii}$$

Now take any γ for which $0<\gamma<\frac{\delta}{2\beta^2}$. Since $\alpha_k T^{n_k}y\to v$, there is $k_1\ge k_0$ such that if $k\ge k_1$, then $\|\alpha_k T^{n_k}y-v\|<\frac{\gamma}{2}$. Since $\|\alpha_k T^{n_k+m}y-T^m v\|\le\|T^m\|\|\alpha_k T^{n_k}y-v\|$, we get for $k\ge k_1\ge k_0$ and $m\ge 0$,

$$\|\alpha_k T^{n_k+m}y-T^m v\|<\beta\tfrac{\gamma}{2}. \tag{iii}$$

Next, as $T^n v\to 0$, take m sufficiently large such that

$$\|T^m v\|<\beta\tfrac{\gamma}{2}. \tag{vi}$$

Thus according to the above four displayed inequalities, for k and m large enough

$$\begin{aligned}\tfrac{\delta}{2\beta}\;<\;\tfrac{\|T^{n_k+m}y\|}{2\beta}&<\|T^{n_k+m}y\|\,|\alpha_k|=\|\alpha_k T^{n_k+m}y\|\\ &\le\|\alpha_k T^{n_k+m}y-T^m v\|+\|T^m v\|<\beta\tfrac{\gamma}{2}+\beta\tfrac{\gamma}{2}=\beta\gamma<\tfrac{\delta}{2\beta},\end{aligned}$$

which is a contradiction. Hence if $y\in\mathcal{X}$ is supercyclic for T, then $T^n y\to 0$, which concludes the proof of Claim II. $\square$

Suppose T is supercyclic. As is readily verified, every nonzero vector in the projective orbit $\mathcal{O}_T([y])$ of any supercyclic vector y for T is again a supercyclic vector for T (see, e.g., [33, Lemma 5.1]). Hence

the set of all supercyclic vectors for T is dense in the normed space $\mathcal{X}$.

Therefore, according to Claim II, $T^n y\to 0$ for every y in a dense subset of $\mathcal{X}$. Thus, since T is power bounded, this implies that $T^n x\to 0$ for every $x\in\mathcal{X}$. Outcome: T is strongly stable. $\square$

Such detailed proofs of Theorems 5.1 and 5.2 are fundamental to supporting the forthcoming discussion in Section 6.

6. Weak Supercyclicity and Weak Stability

The problem considered in this section focuses on only three forms of supercyclicity, viz., (strong) supercyclicity, weak l-sequential supercyclicity, and weak supercyclicity. So consider the following implications taken from the diagram in Section 3:

SUPERCYCLIC $\Longrightarrow$ WEAKLY L-SEQUENTIALLY SUPERCYCLIC $\Longrightarrow$ WEAKLY SUPERCYCLIC,

where the converses fail. In fact, it was exhibited in [4, Example 3.6] a weakly supercyclic unitary operator on a Hilbert space, which actually is weakly l-sequentially supercyclic (cf. [4, p.10, Corollary to Example 3.6]), as we saw in Subsection 4.3. Thus, since there is no unitary supercyclic operator, because there is no supercyclic isometry on a Banach space, as we saw in the proof of Theorem 5.1 (cf. [2, Proof of Theorem 2,1] or [34, Lemma 4.1(b)]), it follows that weak l-sequential supercyclicity does not imply supercyclicity. Also, as we saw in Subsection 4.5, it was shown in [53, Corollary 1.3] that weak supercyclicity does not imply weak l-sequential supercyclicity — by exhibiting a weakly supercyclic unitary operator that is not weakly sequentially supercyclic, and so it is not weakly l-sequentially supercyclic.

Since supercyclicity implies strong stability, as we saw in Theorem 5.2, the next question arises quite naturally. For a power bounded operator on a normed space,

does weak supercyclicity imply weak stability?

This is a nontrivial question that remains unanswered. Pacing Pólia [47, p.114] when he says "can you imagine a more accessible related problem", we might replace weak supercyclicity in the above question by weak l-sequential supercyclicity. This again leads to another nontrivial question that also seems to remain unanswered so far. Regarding such a new question inquiring whether weak l-sequential supercyclicity implies weak stability, the following diagram summarises the above line of reasoning.

$$\begin{array}{ccc}
\text{SUPERCYCLICITY} & \Longrightarrow & \text{STRONG STABILITY} \\
\Downarrow & & \Downarrow \\
\text{WEAK L-SEQUENTIAL SUPERCYCLICITY} & \overset{?}{\Longrightarrow} & \text{WEAK STABILITY.}
\end{array}$$

There are, however, some affirmative answers for particular classes of operators. For instance, when weak l-sequential supercyclicity coincides with supercyclicity for a given class of operators, as is the case, for example, of compact operators. In these cases, Theorem 5.2 ensures a positive answer to the question for such classes. The connection between weak l-sequential supercyclicity and weak stability for the case when $\mathcal{X}$ is a Radon–Riesz space was explored in [34], [39], and [38]. Recall that a Radon–Riesz space is a normed space $\mathcal{X}$ for which an $\mathcal{X}$-valued sequence $\{x_n\}$ converges strongly if and only if it converges weakly and the sequence of norms $\{\|x_n\|\}$ converges to the norm of the limit (see, e.g., [42, Definition 2.5.26]). Hilbert spaces are Radon–Riesz spaces (cf. [22, Problem 20, p.13]).

Consider the proofs of Theorems 5.1 and 5.2. In an attempt to bring the arguments therein, concerning norm convergence, to weak convergence as far as possible, the following results have been established in [37, Theorem 3.1].

Theorem 6.1. [37] *Let T be a power bounded operator of class $C_{1\cdot}$ on a normed space $(\mathcal{X}, \|\cdot\|)$. Then*

(a) *there is a norm $\|\cdot\|_\varphi$ on $\mathcal{X}$ for which T is an isometry on $(\mathcal{X}, \|\cdot\|_\varphi)$.*

(b) *If a vector $y \in \mathcal{X}$ is weakly l-sequentially supercyclic for T when it acts on the normed space $(\mathcal{X}, \|\cdot\|)$, then the same vector $y \in \mathcal{X}$ is weakly l-sequentially supercyclic for T when it acts on the normed space $(\mathcal{X}, \|\cdot\|_\varphi)$.*

(c) *If T is weakly l-sequentially supercyclic when acting on $(\mathcal{X}, \|\cdot\|)$, then it has an extension $\widehat{T}$ on the completion $(\widehat{\mathcal{X}}, \|\cdot\|_{\widehat{\mathcal{X}}})$ of $(\mathcal{X}, \|\cdot\|_\varphi)$ which is a weakly l-sequentially supercyclic isometric isomorphism.*

(d) *If T on $(\mathcal{X}, \|\cdot\|)$ is weakly stable, then $\widehat{T}$ on $(\widehat{\mathcal{X}}, \|\cdot\|_{\widehat{\mathcal{X}}})$ is weakly stable.*

By comparing the statements in Theorem 6.1(a,b,c) with the proof of Theorem 5.1, we can gauge the extent to which the proof of Theorem 5.1 can survive the move from (strong) supercyclicity to weak l-sequential supercyclicity.

A crucial argument in the proof of Theorem 5.1 is the fact that *there is no supercyclic isometry on a Banach space*. This was necessary for concluding that *supercyclicity implies strong stability*. Differently from the argument in the proof of Theorem 5.1, there is no counterpart to such a fact in the context of Theorem 6.1. Indeed, as we have seen in Subsection 4.3, *there are weakly l-sequentially supercyclic unitary operators on a Hilbert space*. A comparison between the proofs of Theorem 6.1 as given in [37, Theorem 3.1] and that of Theorem 5.1 shows that both proofs share the same structure, although the weak counterpart of Theorem 5.2 (i.e., weak stability being implied by weak l-sequential supercyclicity) is not reached this way.

A way to smooth down weak stability leads to the notion of weak quasistability. Recall that an operator T on a normed space $\mathcal{X}$ is weakly stable if $\lim_n |f(T^n x)| = 0$

for every $x \in \mathcal{X}$, for every $f \in \mathcal{X}^*$. It is *weakly quasistable* if,

$$\liminf_n |f(T^n x)| = 0 \text{ for every } x \in \mathcal{X}, \text{ for every } f \in \mathcal{X}^*.$$

It is clear that weak stability implies weak quasistability, but the converse fails [40, Proposition 4.3]. Although the previous question remains open, the next result (which is from [38, Corollary 4.3]) says that, for power bounded operators,

$$\text{WEAK L-SEQUENTIAL SUPERCYCLICITY} \implies \text{WEAK QUASISTABILITY.}$$

Theorem 6.2. [38] *Every weakly l-sequentially supercyclic power bounded operator on a normed space is weakly quasistable.*

This shows that such a mild version of weak stability (viz., weak quasistability) is implied by weak l-sequential supercyclicity, as strong stability was implied by (strong) supercyclicity in Theorem 5.2. So weak l-sequential supercyclicity ensures that the origin is a weak limit point of the orbit of every vector [38, Corollary 4.4] — and so the origin is a weak accumulation point of it (cf. Subsection 3.1):

Corollary 6.3. [38] *If T is a power bounded operator on a normed space $\mathcal{X}$, then*

$$\mathcal{O}_T([y])^{-wl} = \mathcal{X} \text{ for some } y \in \mathcal{X} \quad \Longrightarrow \quad 0 \in \mathcal{O}_T(x)^{-wl} \text{ for every } x \in \mathcal{X}.$$

Let $y \in \mathcal{X}$ be a weakly l-sequentially supercyclic vector for $T \in \mathcal{B}[\mathcal{X}]$. Take an arbitrary $x \in \mathcal{X}\backslash\mathcal{O}([y])$. Then there exist a scalar sequence $\{\alpha_k\}$ and a subsequence $\{T^{n_k}\}$ of $\{T^n\}$ for which $\alpha_k T^{n_k} y \xrightarrow{w} x$. These are referred to as a scalar sequence $\{\alpha_k\}$ and a subsequence $\{T^{n_k}\}$ *of weak l-sequential supercyclicity* (of T for x with respect to y.) If $\{T^{n_k}\}$ is such that $\sup_k(n_{k+1} - n_k) < \infty$, then it is called *a boundedly spaced* subsequence of $\{T^n\}$. Theorem 6.4 below is from [40, Theorems 6.2 and 6.3], where item (a) is a consequence of Theorem 6.2 above.

Theorem 6.4.[40] *Let T be weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$.*

(a) *If T is power bounded, and if there is a boundedly spaced subsequence $\{T^{n_k}\}$ of weak l-sequential supercyclicity of T for each vector $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$, for every weakly l-sequentially supercyclic vector $y \in \mathcal{X}$, then T is weakly stable.*

(b) *If T is weakly stable, then all scalar sequences $\{\alpha_k\}$ of weak l-sequential supercyclicity of T are unbounded. In other words, $\{\alpha_k\}$ is unbounded for every $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$ and every weakly l-sequential supercyclic vector $y \in \mathcal{X}$.*

Theorems 6.1 and 6.2 lead to descriptions of weakly l-sequentially supercyclic isometries on a Hilbert space. The result below is from [37, Corollary 4.3] as an application of an inner-product-space version of Theorem 6.1 given in [37, Theorem 3.2].

Corollary 6.5. [37] *On a Hilbert space, if a weakly l-sequentially supercyclic operator is similar to an isometry, then it is similar to an unitary operator.*

This ensures that *if a weakly l-sequentially supercyclic operator on a Hilbert space is power bounded and power bounded below, then it is similar to unitary operator* [37, Remark 41] — we will return to this topic in Theorem 7.1(c) in the next section. On the other hand, as an application of Theorem 6.2 and of the upcoming Theorem 6.7, the next result (from [38, Corollary 5.5]) sets a suitable starting point, although not in a chronological order, for our discussion towards a characterisation of weakly l-sequentially supercyclic unitary operators.

Corollary 6.6. [38] *If an isometry on a Hilbert space is weakly l-sequentially supercyclic, then it is a weakly quasistable singular-continuous unitary operator.*

Being weakly quasistable, we may inquire, in general,

when is a unitary operator weakly stable?

Recall that (i) a scalar spectral measure for a normal operator is a positive finite measure equivalent to its spectral measure; (ii) a normal operator on a separable Hilbert space has a scalar spectral measure; (iii) any form of cyclicity implies separability — thus assume separability; (iv) a unitary operator is absolutely continuous, singular-discrete, or singular-continuous if its scalar spectral measure is absolutely continuous, singular-discrete, or singular-continuous with respect to normalised Lebesgue measure on the σ-algebra of Borel subsets of the unit circle, respectively; and (v) every unitary operator is the orthogonal direct sum of an absolutely continuous unitary, a singular-discrete unitary, and a singular-continuous unitary (where any part may be missing). Also recall that (vi) an absolutely continuous unitary is always weakly stable; (vii) a singular-discrete unitary is never weakly stable; and (viii) there exist singular-continuous unitaries that are either weakly stable or weakly unstable (see, e.g., [32, Section 3]). Therefore,

a unitary operator is weakly stable if and only if its singular-continuous part (if it exists) is weakly stable and its singular-discrete part does not exist.

Examples of weakly stable and weakly unstable singular-continuous unitary operators were given in [32, Proposition 3.2 and Proposition 3.3]. Also, along the lines discussed in [4, pp.10,12] (see [3, p.1383] as well) and [32, Proposition 4.1], it was shown in [32, Theorem 4.2] that singular-continuous unitary operators partially characterise weakly l-sequentially supercyclic unitary operators.

Theorem 6.7. [32] *Every weakly l-sequentially supercyclic unitary operator is singular-continuous.*

Theorem 6.7 prompts the question inquiring about the converse:

is every singular-continuous unitary operator on a separable Hilbert space weakly l-sequentially supercyclic?

Maybe not, but if yes, then we would get a power bounded weakly l-sequentially supercyclic (unitary) operator that is not weakly stable (but weakly quasistable by Theorem 6.2) since, as we saw above, there exist singular-continuous unitary operators that are not weakly stable. The next question also remains unanswered. Actually, in spite of Theorem 6.2, or because of it, and having in mind Theorem 5.2, the question below that closes this section is nearly the same one that has opened it.

Question 6.8. [34] *Is a weakly l-sequentially supercyclic power bounded operator weakly stabile?*

This was posed in [34, p.61]. Similarly (see, e.g., [32, Question 5.2]),

is every weakly l-sequentially supercyclic unitary operator weakly stable?

7. Spectrum of Weakly l-Sequentially Supercyclic Operators

We close the paper by giving a description of the spectrum of a weakly l-sequentially supercyclic operator.

Take the spectrum $\sigma(T)$ of an operator T acting on a complex Banach space $\mathcal{X}$, which is nonempty and compact, and consider its classical partition, namely, $\sigma(T) = \sigma_P(T) \cup \sigma_R(T) \cup \sigma_C(T)$, where $\sigma_P(T)$ is the point spectrum, $\sigma_R(T)$ is the residual spectrum, and $\sigma_C(T)$ is the continuous spectrum. Let $\mathbb{D}$ and $\mathbb{T}$ be the open unit disk and the unit circle in the complex plane $\mathbb{C}$. For any subset Λ of the complex plane $\mathbb{C}$, set $\Lambda^* = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \Lambda\}$, the set of all complex conjugates of all elements from Λ.

Perhaps the spectral saga of hypercyclicity has its origins in 1982, when Kitai [30, Theorems 2.7 and 2.8] has shown that if a Banach-space operator T is hypercyclic, then $\sigma(T) \cap \mathbb{T} \neq \varnothing$, as well as every component of $\sigma(T)$ meets $\mathbb{T}$. This has triggered the next results on weak hypercyclicity and, consequently, on weak supercyclicity. As briefly mentioned in Section 4 (Subsections 4.1 and 4.2), concerning weak hypercyclicity, Prăjitură [48, Theorem 2.2] has shown in 2005 that $\sigma_P(T^*) = \varnothing$ for a weakly hypercyclic operator T on a Hilbert space. This extended a previous result of Herero [24, Proposition 2.2] in 1991, where such an expression had been established for hypercyclic operators — see also [30, Corollary 2.4]. It was shown by Dilworth and Troitsky [14, Theorem 3] in 2003 that the expression $\sigma(T) \cap \mathbb{T} \neq \varnothing$ survives for a weakly hypercyclic operator T on a Banach space.

The above paragraph dealt with spectral properties of hypercyclic and weakly hypercyclic operators. Regarding supercyclicity, it was verified that $\#\sigma_P(T^*) \leq 1$ (i.e., the cardinality of $\sigma_P(T^*)$ is not greater than one) for supercyclic operators on a Hilbert space by Herrero [24, Proposition 3.1] in 1991, which was extended to operators on a normed space by Ansari and Bourdon [2, Theorem 3.2]) in 1997, and further extended to supercyclic operators on a locally convex space by Peris [46, Lemma 1, Theorem 4] in 2001. But the weak topology on a normed space is a locally convex subtopology of the locally convex norm topology (see, e.g., [42, Theorems 2.5.2 and 2.2.3]). Then the latter extension holds in particular on a normed space under the weak topology, thus including weakly supercyclic operators, and so weakly l-sequentially supercyclic operators, on normed spaces. This has also been obtained for weakly supercyclic operators on a Hilbert space by Prăjitură [48, Theorem 3.2] in 2005. Under additional assumptions of power boundedness and being of class $C_{1\cdot}$, it was shown by Kubrusly and Duggal [37, Corollary 4.1] in 2021 (as a corollary of Theorem 6.1) that $\#\sigma_P(T^*) = \#\sigma_P(T) = 0$ for every weakly l-sequentially supercyclic operators acting on Hilbert spaces.

We will proceed along these lines in Theorem 7.1 below for weakly l-sequentially supercyclic operators on a Hilbert space, but first we need the following result.

An invertible power bounded operator on a Hilbert space, whose inverse also is power bounded, is similar to a unitary operator. This is a classical result due to Sz. Nagy [55, Theorem 1] that will be required in the proof of the next theorem, which is similarly stated as follows: *a Hilbert-space operator is similar to a unitary operator if and only if it is power bounded and power bounded below*. A normed space version for operators similar to an isometry was later given by Koehler and Rosenthal [31, Theorem 2]: *a normed-space operator is similar to an isometry if and only if it is power bounded and power bounded below*.

Recall from Theorem 5.1: a power bounded operator of class $C_{1\cdot}$ is not supercyclic.

Theorem 7.1. *Let T be a power bounded operator of class $C_{1\cdot}$ on a Hilbert space $\mathcal{H}$.*

(a) *If T is weakly l-sequentially supercyclic, then*

$$\sigma(T)=\sigma_C(T)\subseteq\mathbb{D}^- \quad \textit{and} \quad \sigma(T)\cap\mathbb{T}\neq\varnothing.$$

(b) *If T is weakly l-sequentially supercyclic with closed range, then*

$$\sigma(T)=\sigma_C(T)\subseteq\mathbb{D}^-\backslash B_\varepsilon(0) \quad \textit{and} \quad \sigma(T)\cap\mathbb{T}\neq\varnothing.$$

for some open ball $B_\varepsilon(0)$ with radius $\varepsilon\in(0,1]$ centred at the origin of $\mathcal{H}$.

(c) *If T is weakly l-sequentially supercyclic and power bounded below, then T is similar to a unitary operator and*

$$\sigma(T)=\sigma_C(T)\subseteq\mathbb{T}.$$

(d) *If T is weakly l-sequentially supercyclic isometry, then T is a singular continuous unitary operator, so that*

$$\sigma(T)=\sigma_C(T)\subseteq\mathbb{T}$$

is the support of a finite positive measure that is singular-continuous with respect to normalised Lebesgue measure on the σ-algebra of Borel subsets of $\mathbb{T}$.

Proof. (a) If T is power bounded on a complex Banach space, then $\sigma(T)\subseteq\mathbb{D}^-$ (cf. Section 2). It was shown in [37, Corollary 4.1] (by using an inner-product-space version of Theorem 6.1) that if a power bounded operator T of class $C_{1\cdot}$ on a complex Hilbert space is weakly l-sequentially supercyclic, then

$$\sigma_P(T)=\sigma_P(T^*)=\varnothing,$$

where T^* stands for the Hilbert-space adjoint of T. Since for Hilbert-space operators

$$\sigma_R(T)=\sigma_P(T^*)^*\backslash\sigma_P(T),$$

it follows from the above two displayed expressions that

$$\sigma_P(T)=\sigma_R(T)=\varnothing,$$

and so $\sigma(T)=\sigma_C(T)\subseteq\mathbb{D}^-$. Moreover, as T is not strongly stable (because it is of class $C_{1\cdot}$), it is not uniformly stable, and so its spectral radius is not strictly less than one (see Section 2 again), which implies that its spectrum intersects the unit circle (i.e., $\sigma(T)\cap\mathbb{T}\neq\varnothing$, since $\sigma(T)\subseteq\mathbb{D}^-$). This concludes the proof of item (a).

(b) The range of a weakly l-sequentially supercyclic operator is dense (see, e.g., [38, Lemma 5.3]). Thus T is surjective if it has a closed range. Since T is injective (as $0\notin\sigma_P(T)$), it is invertible with a bounded inverse, and so $0\notin\sigma(T)$, which implies $\sigma(T)\subseteq\mathbb{D}^-\backslash B_\varepsilon(0)$ since $\sigma(T)$ is a closed set included in the closed unit disk by (a).

(c) If, besides being power bounded, T is power bounded below, then T is similar to a unitary operator (as we saw above). As similarity preserves the spectrum, and the spectrum of a unitary operator is included in the unit circle, we get (c) from (a).

(d) By Corollary 6.6, every weakly l-sequentially supercyclic isometry on a Hilbert space is a singular-continuous unitary. Recall that an isometry is trivially power bounded of class $C_{1\cdot}$, and a unitary operator is, in addition, power bounded below. As the spectrum of a singular-continuous unitary operator is the support of its singular-continuous spectral measure, the result in (c) comes from (b). ☐

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