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FORMS OF BIISOMETRIC OPERATORS AND BIORTHOGONALITY

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Dedicated to the memory of our friend and colleague Nhan Levan (1936–2021)

Abstract. The paper proves two results involving a pair (A, B) of P-biisometric or (m,P)-biisometric Hilbert-space operators for arbitrary positive integer m and positive operator P. It is shown that if A and B are power bounded and the pair (A, B) is (m, P)-biisometric for some m, then it is a P-biisometric pair. The important case when P is invertible is treated in detail. It is also shown that if (A, B) is P-biisometric, then there are biorthogonal sequences with respect to the inner product $\langle\,\cdot\,;\cdot\,\rangle_P = \langle P\,\cdot\,;\cdot\,\rangle$ that have a shift-like behaviour with respect to this inner product.

1. Introduction

Let $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle)$ be a Hilbert space and let A and B be Hilbert-space operators. They are said to make a biisometric pair if $A^*B = I$, where I is the identity operator. This extends the notion of a Hilbert-space isometry: an operator A is an isometry if and only if $A^*A = I$. Biisometric pairs have been investigated in [9]. We show here that this is connected to the notion of (m, P)-biisometric pairs and, in particular, to the notion of P-biisometric pairs, where m is a positive integer and P is a positive operator.

The original results in this paper are stated and proved in Sections 4, 5 and 6, viz., Theorems 4.1, 5.1 and 6.1. Let P be a positive operator. In Theorem 4.1 we show that if (A,B) is an (m,P)-biisometric pair for some m, and A and B are power bounded, then (A,B) is a P-biisometric pair; so that if in addition P is invertible, then A and B are similar to a biisometric pair. In Theorem 5.1 we extend the results in [9], from biisometric pairs to an arbitrary P-biisometric pair, exhibiting a pair $\big\{\{\phi_n\}, \{\psi_n\}\big\}$ of biorthonormal sequences with values in the inner product space $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle_P)$, with inner product $\langle\,\cdot\,;\cdot\,\rangle_P = \langle P\,\cdot\,;\cdot\,\rangle$, which have a shift-like behaviour. Theorem 6.1 gives a comprehensive account of the case when the positive operator P is invertible (with a bounded inverse) for power bounded operators A and B.

All terms and notation used above will be defined in the next section. The paper is organised as follows. Basic notation and terminology are summarised in Section 2. Forms of biisometric operators, including P-biisometric and (m, P)-biisometric pairs, are considered in Section 3. Sections 4 and 5 contain the main results of the paper as discussed above. The particular case when the injective positive P is invertible (with a bounded inverse) closes the paper in Section 6.

2. Basic Notation and Terminology

Throughout this paper $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle)$ stands for a complex Hilbert space, equipped with an inner product $\langle\,\cdot\,;\cdot\,\rangle$ generating the norm $\|\cdot\|$. Let $A\colon \mathcal{H} \to \mathcal{H}$ be an operator

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(i.e., a bounded linear transformation) of $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle)$ into itself — referred to as a Hilbert-space operator, or an operator on $\mathcal{H}$. The normed algebra of all operators on a normed space $\mathcal{X}$ will denoted by $\mathcal{B}[\mathcal{X}]$, and so $\mathcal{B}[\mathcal{B}[\mathcal{X}]]$ stands for the normed algebra of all bounded linear transformations of $\mathcal{B}[\mathcal{X}]$ into itself (sometimes referred to as transformers). Let I stand for the identity operator and O for the null operator on any linear space. We use the same notation $\|\cdot\|$ for the induced uniform norm on $\mathcal{B}[\mathcal{X}]$ for any normed space $\mathcal{X}$. An operator A on a normed space $\mathcal{X}$ is power bounded if $\sup_n \|A^n\| < \infty$ (which means $\sup_n \|A^n x\| < \infty$ for every $x \in \mathcal{X}$ if $\mathcal{X}$ is a Banach space by the Banach–Steinhaus Theorem). The adjoint of an operator A on any (complex) Hilbert space $\mathcal{H}$ will be denoted by A^*. The kernel and range of an operator A on $\mathcal{H}$ will be denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, where $\mathcal{N}(A)$ is a subspace (i.e., a closed linear manifold) and $\mathcal{R}(A)$ is a linear manifold of $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle)$, respectively. Recall that $\mathcal{R}(A)^- = \mathcal{N}(A^*)^\perp$, where the superscripts $^-$ and $^\perp$ stand for closure and orthogonal complement in a Hilbert space $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle)$, respectively. A Hilbert-space operator P is self-adjoint if $P^* = P$ (equivalently, if $\langle Px\,;x\rangle$ is real for every $x \in \mathcal{H}$). A self-adjoint operator P on Hilbert space is nonnegative if $\langle Px\,;x\rangle \geq 0$ for every $x \in \mathcal{H}$, and positive if $\langle Px\,;x\rangle > 0$ for every nonzero $x \in \mathcal{H}$ (equivalently, if it is nonnegative and injective). An invertible positive operator (with a bounded inverse; i.e., nonnegative, injective and surjective after the Open Mapping Theorem) is sometimes referred to as a strictly positive operator. If P is positive, then (as it is self-adjoint and injective) it is left-invertible with a dense range (since $\mathcal{R}(P)^- = \mathcal{N}(P)^\perp = \{0\}^\perp = \mathcal{H}$), so has a left inverse $P^{-1}\colon \mathcal{R}(P) \to \mathcal{H}$, which is bounded if and only if it is surjective (i.e., if and only it is invertible). For any positive (i.e., injective nonnegative) P, the form defined by

$$\langle\,\cdot\,;\cdot\,\rangle_P = \langle P\,\cdot\,;\cdot\,\rangle\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

is an inner product generating a norm $\|\cdot\|_P$, so that $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle_P)$ is an inner product space. If P is strictly positive (i.e., invertible nonnegative), then $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle_P)$ is a Hilbert space (i.e., it remains complete). Let $\operatorname{span} S$ denotes the linear span of an arbitrary set $S \subseteq \mathcal{H}$ and let $\bigvee S = (\operatorname{span} S)^-$ denotes the closure of $\operatorname{span} S$.

Biorthogonal sequences were introduced in the context of basis for separable Banach spaces [13, Definition 1.4.1], [12, Definition 1.f.1] (and so $\mathcal{H}$ is *separable* if it is spanned by a sequence). In a Hilbert space setting (where dual pairs are identified with inner products after the Riesz Representation Theorem for Hilbert spaces), the notion of biorthogonality is defined in terms of the inner product. This is extended for the case of an inner product space $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle)$, as follows.

> Two sequences $\{f_n\}$ and $\{g_n\}$ of vectors in an inner product space $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle)$ are said to be *biorthogonal* (to each other) if $\langle f_m; g_n\rangle = \delta_{m,n}$ where δ stands for the Kronecker delta function. If $\{f_n\}$ is such that there exists a sequence $\{g_n\}$ for which $\{f_n\}$ and $\{g_n\}$ are biorthogonal, then it is said that $\{f_n\}$ *admits a biorthogonal sequence* (symmetrically, $\{g_n\}$ admits a biorthogonal sequence), and $\big\{\{f_n\}, \{g_n\}\big\}$ is referred to as a *biorthogonal pair* or a *biorthogonal system*.

If, in addition, $\|f_n\| = \|g_n\| = 1$ for all n, then $\big\{\{f_n\}, \{g_n\}\big\}$ might be said to be a biorthonormal pair. However, it has been shown in [9, Corollary 2.1] that *there is no distinct pair of biorthonormal sequences.* In other words, if two sequences $\{f_n\}$ and $\{g_n\}$ are biorthogonal and if $\|f_n\| = \|g_n\| = 1$ for all n, then $f_n = g_n$ for all n. Also, if an arbitrary pair $\{f_n\}$ and $\{g_n\}$ of biorthogonal sequences is such that $f_n = g_n$

for all n, then we get the usual definition of an orthonormal sequence, although in general neither $\{f_n\}$ nor $\{g_n\}$ are orthogonal (much less orthonormal) sequences.

A sequence $\{f_n\}$ spanning the whole space $\mathcal{H}$ is sometimes called *total*. This means $\bigvee\{f_n\} = \mathcal{H}$ (and $\mathcal{H}$ is separable in this case). It was pointed out in [17] that *if $\{f_n\}$ admits a biorthogonal sequence $\{g_n\}$, then $\{g_n\}$ is unique if and only if $\{f_n\}$ is total*. We will be concerned with total sequences in the proof of Corollary 5.1.

Replacing $\langle\cdot\,;\cdot\rangle$ with $\langle\cdot\,;\cdot\rangle_P$, for some invertible (or simply injective) nonnegative operator P, we get the definition of a *P-biorthogonal pair*.

3. Forms of Biisometric Operators

Let A be an operator on a Hilbert space $\mathcal{H}$, and let m be a positive integer. There is a myriad of equivalent definitions for a Hilbert-space isometry. The one that fits our needs here reads as follows: an operator A is an *isometry* if

$$A^*A = I \qquad \text{(i.e., } A^*A - I = O\text{)}.$$

By replacing "=" with "≤" we get another expression defining a contraction. Perhaps the above displayed form for an isometry has been popularised in [6]. It seems that the notion of m-isometry appeared in the last decade of the past century [1], and a considerable number of research papers dealing with several aspects of it has been noticed recently (see, e.g., [3, 15]). An operator A is an *m-isometry* if

$$\sum\nolimits_{j=0}^{m}(-1)^j\binom{m}{j}A^{*(m-j)}A^{m-j} = O$$

for some positive integer m. A 1-isometry is precisely a plain isometry. On the other hand, but still along the same line, there also is the notion of P-isometry with respect to an injective nonnegative operator P: An operator A is a *P-isometry* if

$$A^*PA = P \qquad \text{(i.e., } A^*PA - P = O\text{)},$$

reducing to a plain isometry if P is the identity operator. (For recent papers dealing with P-isometry — and its variations as, for instance, P-contractions, see, e.g., [14, 8]). The above two notions prompt the next one. An operator A is an (m, P)-*isometry* (for a positive integer m and a positive operator P) if

$$\sum\nolimits_{j=0}^{m}(-1)^j\binom{m}{j}A^{*(m-j)}PA^{m-j} = O.$$

Again, a $(1, P)$-isometry is P-isometry. (For (m, P)-isometries and their variations, such as (m, P)-expansive operators where the "=" sign is replaced by "≤", see, e.g., [4, 11]). The above notions are extended to a pair of operators as follows. Let B be another operator on $\mathcal{H}$. Operators A and B are said to make a *biisometric pair* [9] if

$$A^*B = I \qquad \text{(i.e., } A^*B - I = O\text{)}.$$

Equivalently, if $B^*A = I$. The pair (A, B) is said to be an *m-biisometric pair* if

$$\sum\nolimits_{j=0}^{m}(-1)^j\binom{m}{j}A^{*(m-j)}B^{m-j} = O$$

for a positive integer m (see, e.g., [5]). We say that A and B make a *P-biisometric pair* if, for a positive operator P,

$$A^*PB = P \qquad \text{(i.e., } A^*PB - P = O\text{)}.$$

The above two expressions naturally lead to the notion of (m,P)-*biisometric pair*:

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}A^{*(m-j)}PB^{m-j}=O,$$

where an (m,I)-biisometric is m-biisometric and a $(1,P)$-biisometric is P-biisometric. The present paper focuses on the last two notions.

4. On (m,P)-Biisometric and P-Biisometric Pairs

Given operators A,B in $\mathcal{B}[\mathcal{H}]$, let $L_A, R_B \in \mathcal{B}[\mathcal{B}[\mathcal{H}]]$ denote the operators of left multiplication by A and, respectively, right multiplication by B, given by

$$L_A(X)=AX \quad \text{and} \quad R_B(X)=XB \quad \text{for every} \quad X\in\mathcal{B}[\mathcal{H}].$$

Then set

$$\begin{aligned}\triangle^m_{A^*,B}(P) &= (L_{A^*}R_B-I)^m(P)\\ &= \Big(\sum_{j=0}^{m}(-1)^j\binom{m}{j}(L_{A^*}R_B)^{m-j}\Big)(P)\\ &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}A^{*(m-j)}PB^{m-j},\end{aligned}$$

so that the pair (A,B) is (m,P)-biisometric if and only if $\triangle^m_{A^*,B}(P)=0$. Since

$$\triangle^m_{A^*,B}(P)=(L_{A^*}R_B-I)\big(\triangle^{m-1}_{A^*,B}(P)\big)=L_{A^*}R_B\big(\triangle^{m-1}_{A^*,B}(P)\big)-\triangle^{m-1}_{A^*,B}(P),$$

if a pair (A,B) is (m,P)-biisometric, then

$$\begin{aligned}&\triangle^{m-1}_{A^*,B}(P)=A^*\triangle^{m-1}_{A^*,B}(P)B\\ \Longrightarrow\quad &\triangle^{m-1}_{A^*,B}(P)=A^*\triangle^{m-1}_{A^*,B}(P)B=A^{*2}\triangle^{m-1}_{A^*,B}(P)B^2\\ \Longrightarrow\quad &\triangle^{m-1}_{A^*,B}(P)=A^*\triangle^{m-1}_{A^*,B}(P)B=\cdots=A^{*n}\triangle^{m-1}_{A*,B}(P)B^n\end{aligned}$$

for all positive integers n. In particular,

a P-biisometric pair (A,B) is (m,P)-biisometric for positive integers m.

Does the converse hold? The answer given here, in the absence of any additional hypotheses, is an emphatic "no": for example, if $A=B=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and P is the positive operator $P=\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$, then A is a $(3,P)$-isometry but not a P-isometry (i.e., the pair (A,A) is $(3,P)$-isometric but not P-isometric). The following theorem says that

a necessary and sufficient condition for an (m,P)-biisometric pair (A,B) to be P-biisometric, for a positive $P\in B[\mathcal{H}]$, is that $L_{A^*}R_B$, given by $(L_{A^*}R_B)(P)=A^*PB$, is power bounded.

Remark 4.1. Take arbitrary operators A,B, let the L_A be the left multiplication by A, and R_B be the right multiplication by B, as defined above. The commutativity of L_A and R_B ensures that

$$(L_AR_B)^n=L_A^nR_B^n=L_{A^n}R_{B^n},$$

and so

$$(L_AR_B)^n(X)=(L_A^nR_B^n)(X)=(L_{A^n}R_{B^n})(X)=A^nXB^n$$

for every nonnegative integer n and every operator X. Since

$$\begin{aligned}\triangle^m_{A^n,B^n}(X) &= (L_{A^n}R_{B^n}-I)^m(X)=(L^n_AR^n_A-I)^m(X)\\ &= \Big[(L_AR_B-I)^m\sum_{j=0}^{m(n-1)}\alpha_j(L_AR_B)^{m(n-1)-j}\Big](X)\\ &= \sum_{j=0}^{m(n-1)}\alpha_j(L_AR_B)^{m(n-1)-j}(\triangle^m_{A,B}(X))\end{aligned}$$

for some scalars α_j, if (A^*,B) is (m,X)-biisometric, then (A^{*n},B^n) is (m,X)-biisometric for all positive integers n. Again, since

$$\|(L_AR_B)^n\|=\sup_{X\neq O}\tfrac{\|(L_AR_B)^n(X)\|}{\|X\|}=\sup_{X\neq O}\tfrac{\|A^nXB^n\|}{\|X\|}\leq \sup_n\|A^n\|\sup_n\|B^n\|,$$

L_AR_B is power bounded whenever both A and B are power bounded. Since an operator A is power bounded if and only if its adjoint A^* is power bounded, power boundedness of A and B implies power boundedness of $L_{A^*}R_B$, which is the condition of the next theorem.

Theorem 4.1. *Let A and B be operators on a Hilbert space $(\mathcal{H},\langle\,\cdot\,;\cdot\,\rangle)$ such that the pair (A,B) is (m,P)-biisometric for some positive integer m and positive operator P. If $L_{A^*}R_B$ is power bounded, then the pair (A,B) is P-biisometric.*

Proof. The easily proved (use an induction argument) identity

$$(a-1)^t=a^t-\sum_{j=0}^{t-1}\binom{t}{j}(a-1)^j$$

for all positive integers t implies

$$\begin{aligned}\triangle^m_{A^*,B}(P) &= (L_{A^*}R_B-I)^m(P)\\ &= (L_{A^*}R_B)^m(P)-\sum_{j=0}^{m-1}\binom{m}{j}\triangle^j_{A^*,B}(P)\\ &= A^{*m}PB^m-\sum_{j=0}^{m-1}\binom{m}{j}\triangle^j_{A^*,B}(P).\end{aligned}$$

Hence, if (A,B) is (m,P)-biisometric, then

$$\begin{aligned} O &= A^{*m}PB^m-\sum_{j=0}^{m-1}\binom{m}{j}\triangle^j_{A^*,B}(P)\\ \Longrightarrow\quad O &= A^{*(m+1)}PB^{m+1}-\sum_{j=0}^{m-1}\binom{m}{j}A^*\triangle^j_{A^*,B}(P)B\\ &= A^{*(m+1)}PB^{m+1}-\sum_{j=0}^{m-1}\binom{m}{j}\triangle^{j+1}_{A^*,B}(P)-\sum_{j=0}^{m-1}\binom{m}{j}\triangle^j_{A^*,B}(P)\\ &= A^{*(m+1)}PB^{m+1}-\binom{m}{m-1}\triangle^m_{A^*,B}(P)-\sum_{j=0}^{m-1}\binom{m+1}{j}\triangle^j_{A^*,B}(P)\\ &= A^{*(m+1)}PB^{m+1}-\sum_{j=0}^{m-1}\binom{m+1}{j}\triangle^j_{A^*,B}(P).\end{aligned}$$

An induction argument now leads us to the conclusion that

$$\begin{aligned} O &= A^{*n}PB^n-\sum_{j=0}^{m-1}\binom{n}{j}\triangle^j_{A^*,B}(P)\\ &= A^{*n}PB^n-\binom{n}{m-1}\triangle^{m-1}_{A^*,B}(P)-\sum_{j=0}^{m-2}\binom{n}{j}\triangle^j_{A^*,B}(P)\\ \Longleftrightarrow\quad A^{*n}PB^n &= \binom{n}{m-1}\triangle^{m-1}_{A^*,B}(P)+\sum_{j=0}^{m-2}\binom{n}{j}\triangle^j_{A^*,B}(P)\end{aligned}$$

for all integers $n\geq m$. Assume that $L_{A^*}R_B$ is power bounded, which means that there exists a positive scalar M such that $\|L_{A^{*n}}R_{B^n}\|\leq M$ for all n. (Recall that the power boundedness of $L_{A^*}R_B$ is guaranteed by the power boundedness of the operators A and B.) Then

$$\limsup_{n\to\infty} \|A^{*n}PB^n x\| \le M\|P\|\,\|x\|$$

and there exists a positive scalar M_j, dependent on j, such that

$$\limsup_{n\to\infty} \|\triangle^j_{A^*,B}(P)x\| \le M_j\|P\|\,\|x\|$$

for all $x\in\mathcal{H}$. Observe that $\binom{n}{m-1}$ is of the order of n^{m-1} and $\binom{n}{j}$, $0\le j\le m-2$, is of the order of n^{m-2}; hence, letting $n\to\infty$ in

$$\|\triangle^{m-1}_{A^*,B}(P)x\| \le \frac{1}{\binom{n}{m-1}}\Big(\|A^{*n}PB^n x\| + \sum\nolimits_{j=0}^{m-2}\binom{n}{j}\|\triangle^j_{A^*,B}(P)x\|\Big)$$

we have

$$\|\triangle^{m-1}_{A^*,B}(P)x\| = 0 \ \text{ for all } \ x\in\mathcal{H} \quad\iff\quad \triangle^{m-1}_{A^*,B}(P)=0.$$

Repeating the argument a finite number of times, this implies

$$\triangle_{A^*,B}(P)=0 \quad\iff\quad A^*PB=P. \qquad \square$$

Choosing $A=B$ in Theorem 4.1, with P positive (i.e., if A is a P-isometry), we have $A^*PA=(A^*P^{\frac12})(P^{\frac12}A)=P$ and A is an isometry with respect to the norm $\|\cdot\|_P$ (i.e., $\|Ax\|_P=\|x\|_p$ for every $x\in\mathcal{H}$ — see, e.g., [8, Proposition 4.1(b)]). So there exists an isometry V such that $A^*P^{\frac12}=P^{\frac12}V^*$. In other words, the operator A is a P-isometry in the sense that $\|P^{\frac12}Ax\|=\|P^{\frac12}x\|$ for all $x\in\mathcal{H}$. The P-isometric property of A for the case of a nonnegative P does not imply the left invertibility of A. For example, if $A=\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&0\end{smallmatrix}\right)$ and P is the nonnegative operator $P=\left(\begin{smallmatrix}1&1&0\\1&1&0\\0&0&0\end{smallmatrix}\right)$, then 0 is in the point spectrum of A and $A^*PA=P$. Such a situation can not, however, arise if P is a positive (i.e., nonnegative and injective) operator. For, in this case, if $\{x_n\}\subset\mathcal{H}$ is a sequence of unit vectors such that $\lim_{n\to\infty}\|(A-\lambda)x_n\|=0$ and A is a P-isometry (i.e., the pair (A,A) is (m,P)-isometric), then

$$\lim_{n\to\infty}\langle\triangle^m_{A^*,A}(P)x_n\,;x_n\rangle=(|\lambda|^2-1)^m\lim_{n\to\infty}\langle Px_n\,;x_n\rangle=0$$

implies $(|\lambda|^2-1)=0$; that is, the approximate point spectrum of A lies in the unit circle (hence, A is left invertible).

5. P-Biisometric Operators and Biorthogonal Sequences

A P-biisometric pair of operators was defined in Section 3 and biorthogonal sequences were defined in Section 2.

Theorem 5.1. *Let A and B be operators on a Hilbert space $(\mathcal{H},\langle\,\cdot\,;\cdot\,\rangle)$. Suppose they make a P-biisometric pair whose adjoints are noninjective,*

(i) $$\mathcal{N}(A^*)\ne\{0\} \quad\textit{and}\quad \mathcal{N}(B^*)\ne\{0\},$$

and there exists an injective nonnegative (i.e., positive) operator P on $\mathcal{H}$ for which

(ii) $$A^*PB=P, \qquad \mathcal{N}(A^*)\cap\mathcal{R}(P)\ne\{0\}, \quad\textit{and}\quad \mathcal{N}(B^*)\cap\mathcal{R}(P)\ne\{0\}.$$

Consider the inner product space $(\mathcal{H},\langle\,\cdot\,;\cdot\,\rangle_P)$ with inner product given by $\langle\,\cdot\,;\cdot\,\rangle_P=\langle P\,\cdot\,;\cdot\,\rangle$. Take arbitrary nonzero vectors

$$v\in\mathcal{N}(A^*)\cap\mathcal{R}(P) \quad\textit{and}\quad w\in\mathcal{N}(B^*)\cap\mathcal{R}(P)$$

and, for each nonnegative integer n*, consider the vectors*

$$\phi_n = A^n P^{-1} w \quad \textit{and} \quad \psi_n = B^n P^{-1} v$$

in $\mathcal{H}$*. We claim that there exist* $v \in \mathcal{N}(A^*) \cap \mathcal{R}(P)$ *and* $w \in \mathcal{N}(B^*) \cap \mathcal{R}(P)$ *such that the sequences* $\{\phi_n\}$ *and* $\{\psi_n\}$ *are biorthogonal on* $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle_P)$*. Moreover,*

$$A\phi_n = \phi_{n+1} \quad \textit{and} \quad B\psi_n = \psi_{n+1},$$

$$A^*(P\psi_{n+1}) = P\psi_n \quad \textit{and} \quad B^*(P\phi_{n+1}) = P\phi_n.$$

Proof. Let $\langle\,\cdot\,;\cdot\,\rangle$ be an inner product on $\mathcal{H}$, where $(\mathcal{H}, \langle\,\cdot\,;\cdot\,\rangle)$ is a Hilbert space. First suppose A^* and B^* are noninjective (i.e., $\mathcal{N}(A^*) \neq \{0\}$ and $\mathcal{N}(B^*) \neq \{0\}$, which is equivalent to saying that A and B have nondense ranges; see, e.g., [7, Propositions 5.12 and 5.76]). Next suppose there exists an injective nonnegative (thus self-adjoint) operator P on $\mathcal{H}$ for which

$$A^*PB = P \qquad (\text{equivalently, } B^*PA = P),$$

so that a trivial induction shows that, for every nonnegative integer n,

$$A^{*n}PB^n = P \qquad (\text{equivalently, } B^{*n}PA^n = P).$$

Moreover, suppose

$$\mathcal{N}(A^*) \cap \mathcal{R}(P) \neq \{0\} \quad \text{and} \quad \mathcal{N}(B^*) \cap \mathcal{R}(P) \neq \{0\}.$$

As P is an injective nonnegative operator, the form $\langle\,\cdot\,;\cdot\,\rangle_P \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ given by

$$\langle\,\cdot\,;\cdot\,\rangle_P = \langle P\,\cdot\,;\cdot\,\rangle$$

is another inner product in $\mathcal{H}$. Take arbitrary nonzero vectors

$$v \in \mathcal{N}(A^*) \cap \mathcal{R}(P) \quad \text{and} \quad w \in \mathcal{N}(B^*) \cap \mathcal{R}(P).$$

Let $P^{-1} \colon \mathcal{R}(P) \to \mathcal{H}$ be the left inverse of P and set

$$y = P^{-1}v \neq 0 \quad \text{and} \quad z = P^{-1}w \neq 0$$

in $\mathcal{H}$. Now take ϕ_n and ψ_n in $\mathcal{H}$ defined for every nonnegative integer n by

$$\phi_n = A^n z = A^n P^{-1} w \quad \text{and} \quad \psi_n = B^n y = B^n P^{-1} v.$$

Another trivial induction shows that

$$A\,\phi_n = A^{n+1} z = \phi_{n+1} \quad \text{and} \quad B\psi_n = B^{n+1} y = \psi_{n+1}.$$

Therefore, since $A^*PB = P$, we also get

$$A^*(P\psi_{n+1}) = A^*PB\psi_n = P\psi_n \quad \text{and} \quad B^*(P\phi_{n+1}) = B^*PA\,\phi_n = P\phi_n.$$

Now let m, n be a pair of nonnegative integers. If $m < n$, then

$$\langle \phi_m\,;\psi_n\rangle_P = \langle PA^m z\,;B^n y\rangle = \langle z\,;A^{*m}PB^mB^{n-m}y\rangle = \langle Pz\,;B^{n-m}y\rangle = \langle B^{*(n-m)}w\,;y\rangle$$

and so $\langle \phi_m\,;\psi_n\rangle_P = 0$ since $w \in \mathcal{N}(B^*)$ implies $w \in \mathcal{N}(B^{*(m-n)})$. Symmetrically,

$$\langle \phi_m\,;\psi_n\rangle_P = \langle B^{*n}PA^nA^{m-n}z\,;y\rangle = \langle PA^{m-n}z\,;y\rangle = \langle z\,;A^{*(m-n)}v\rangle = 0$$

if $n < m$, since $v \in \mathcal{N}(A^*)$. Moreover, for $m = n$,

$$\langle \phi_n\,;\psi_n\rangle_P = \langle PA^n z\,;B^n y\rangle = \langle z\,;A^{*n}PB^n y\rangle = \langle z\,;Py\rangle = \langle z\,;v\rangle = \langle P^{-1}w\,;v\rangle.$$

Summing up.

$$\langle \phi_m\,;\psi_n\rangle_P = 0 \;\text{ whenever }\; m \neq n \quad \text{and} \quad \langle \phi_n\,;\psi_n\rangle_P = \langle P^{-1}w\,;v\rangle.$$

Next we proceed to show that there are $v \in \mathcal{N}(A^*) \cap \mathcal{R}(P)$ and $w \in \mathcal{N}(B^*) \cap \mathcal{R}(P)$ such that $\langle P^{-1}w\,;v\rangle \neq 0$, and so (as $\mathcal{N}(A^*) \cap \mathcal{R}(P)$ and $\mathcal{N}(B^*) \cap \mathcal{R}(P)$ are linear spaces), there exist a pair (v, w) with $v \in \mathcal{N}(A^*) \cap \mathcal{R}(P)$ and $w \in \mathcal{N}(B^*) \cap \mathcal{R}(P)$ such that $\langle P^{-1}w\,;v\rangle = 1$. In this case (that is, for such a pair (v, w) of vectors in $\mathcal{N}(A^*) \cap \mathcal{R}(P) \times \mathcal{N}(B^*) \cap \mathcal{R}(P))$ we get

$$\langle \phi_m\,;\psi_n\rangle_P = 0 \;\text{ if }\; m \neq n \quad \text{ and } \quad \langle \phi_n\,;\psi_n\rangle_P = 1,$$

so that $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal sequences on $(\mathcal{H}, \langle\,\cdot\,;\,\cdot\,\rangle_P)$. That there is such a pair (v, w) for which $\langle P^{-1}w\,;v\rangle \neq 0$ is a consequence of the following result.

Claim. $$\mathcal{N}(A^*) \cap \mathcal{R}(P) \not\perp P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)).$$

Proof of Claim. Since $A \neq O$ (as $A^*PB = P \neq O$), take $0 \neq u \in \mathcal{R}(A)$ so that $u = Ax$ for some $0 \neq x \in \mathcal{H}$. Suppose $u \in P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P))$. Then $u = P^{-1}w$ for some $w \in \mathcal{R}(P)$ for which $B^*w = 0$. Thus $Ax = u = P^{-1}w$, and so $PAx = PP^{-1}w = w$ (as $w \in \mathcal{R}(P)$). Then $Px = P^*x = B^*PAx = B^*w = 0$, so that $x = 0$ (as $\mathcal{N}(P) = \{0\}$), which is a contradiction. Hence, $\mathcal{R}(A) \cap P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)) = \{0\}$, and so $\mathcal{R}(A)^- \cap P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)) = \{0\}$. Equivalently (as $\mathcal{R}(A)^- = \mathcal{N}(A^*)^\perp$),

$$\mathcal{N}(A^*)^\perp \cap P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)) = \{0\}.$$

Suppose $P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)) \perp \mathcal{N}(A^*)$. Then $P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)) \subseteq \mathcal{N}(A^*)^\perp$. So

$$\begin{aligned} P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)) &= P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)) \cap P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)) \\ &\subseteq \mathcal{N}(A^*)^\perp \cap P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)) = \{0\} \end{aligned}$$

by the above identity, so that $\mathcal{N}(B^*) \cap \mathcal{R}(P) = \{0\}$, which is a contradiction. Hence

$$\mathcal{N}(A^*) \not\perp P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)).$$

Thus there exist $v' \in \mathcal{N}(A^*)$ and $u \in P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P))$ such that $\langle v'\,;u\rangle \neq 0$. Since $\mathcal{R}(P)$ is dense in $\mathcal{H}$, and since the inner product is continuous, there exists a vector $v \in \mathcal{N}(A^*) \cap \mathcal{R}(P)$ such that $\langle v\,;u\rangle \neq 0$. Therefore

$$\mathcal{N}(A^*) \cap \mathcal{R}(P) \not\perp P^{-1}(\mathcal{N}(B^*) \cap \mathcal{R}(P)). \quad \square$$

The above claim ensures that there are $w \in \mathcal{N}(B^*) \cap \mathcal{R}(P)$ and $v \in \mathcal{N}(A^*) \cap \mathcal{R}(P)$ such that $\langle P^{-1}w\,;v\rangle \neq 0$, concluding the proof of the theorem. □

If the positive P is surjective (i.e., if it is invertible), then the fact that $\mathcal{R}(P) = \mathcal{H}$ simplifies condition (ii). The above theorem generalises [9, Theorem 3.1] (also [10, Theorem 3.1]). Indeed, by setting $P = I$ we get the result in [9] as a particular case.

The next corollary shows that a P-biisometric pair has a shift-like property regarding biorthogonal sequences with respect to the inner product $\langle\,\cdot\,;\,\cdot\,\rangle_P$ (i.e., with respect to P-biorthogonal sequences).

Corollary 5.1. *Let A and B be operators on a Hilbert space $\mathcal{H}$ for which there exists an injective nonnegative operator P on $\mathcal{H}$ such that*

$$A^*PB = P,$$

and consider the biorthogonal sequences $\{\phi_n\}$ and $\{\psi_n\}$ defined in Theorem 5.1 in terms of nonzero vectors $v \in \mathcal{N}(A^*) \cap \mathcal{R}(P)$ and $w \in \mathcal{N}(B^*) \cap \mathcal{R}(P)$. In addition, *suppose these biorthogonal sequences span $\mathcal{H}$, and also suppose a vector $x \in \mathcal{H}$ has series expansion in terms of $\{\phi_n\}$ and $\{\psi_n\}$. Then this x can be expressed as*

$$x = \sum\nolimits_{k=0}^{\infty} \langle x\,;\psi_k\rangle_P\,\phi_k = \sum\nolimits_{k=0}^{\infty} \langle x\,;\phi_k\rangle_P\,\psi_k,$$

and its image with respect to A, A^*, B, and B^* can be expressed as

$$Ax = \sum\nolimits_{k=0}^{\infty} \langle x\,;\psi_k\rangle_P\,\phi_{k+1} \quad \textit{and} \quad Bx = \sum\nolimits_{k=0}^{\infty} \langle x\,;\phi_k\rangle_P\,\psi_{k+1},$$

$$A^*x = \sum\nolimits_{k=0}^{\infty} \langle x\,;\phi_{k+1}\rangle_P\,\psi_k \quad \textit{and} \quad B^*x = \sum\nolimits_{k=0}^{\infty} \langle x\,;\psi_{k+1}\rangle_P\,\phi_k.$$

Proof. Take an arbitrary $x \in \mathcal{H}$. To begin with, note that if a sequence in a biorthogonal pair spans $\mathcal{H}$, then it does not necessarily follow that all elements in $\mathcal{H}$ have an expansion as the limit of a linear combination of elements of the sequence (cf. [2, Example 5.4.6]). Thus first suppose the biorthogonal sequences $\{\phi_n\}$ and $\{\psi_n\}$ span $\mathcal{H}$ (i.e., $\bigvee\{\phi_n\} = \bigvee\{\psi_n\} = \mathcal{H}$). According to [17, p.537], $\{\phi_n\}$ admits a biorthogonal sequence if and only if none of its elements is the limit of a linear combination of the others, and in this case the biorthogonal sequence $\{\psi_n\}$ is uniquely determined if and only $\bigvee\{\phi_n\} = \mathcal{H}$. Thus assume first that the biorthogonal pair $\big\{\{\phi_n\},\{\psi_n\}\big\}$ is unique in the above sense. In addition, also suppose that

$$x = \sum\nolimits_{k=0}^{\infty} \alpha_k \phi_k = \sum\nolimits_{k=0}^{\infty} \beta_k \psi_k$$

for some pair of sequences of scalars $\{\alpha_n\}$ and $\{\beta_n\}$. Then

$$\alpha_n = \langle x\,;\psi_n\rangle_P \quad \text{and} \quad \beta_n = \langle x\,;\phi_n\rangle_P$$

for every $n \geq 0$. Indeed, by the continuity of the inner product, and recalling from Theorem 5.1 that $\{\phi_k\}$ and $\{\psi_k\}$ are biorthogonal with respect to the inner product $\langle\,\cdot\,;\cdot\,\rangle_P$, we get

$$\langle x\,;\psi_n\rangle_P = \sum\nolimits_{k=0}^{\infty} \alpha_k \langle \phi_k\,;\psi_n\rangle_P = \alpha_n \quad \text{and} \quad \langle x\,;\phi_n\rangle_P = \sum\nolimits_{k=0}^{\infty} \beta_k \langle \psi_k\,;\phi_n\rangle_P = \beta_n$$

for every $n \geq 0$. Recall again from Theorem 5.1 that for each $n \geq 0$

$$A\,\phi_n = \phi_{n+1} \quad \text{and} \quad B\psi_n = \psi_{n+1}.$$

Using the above identities only (and the continuity of the inner product), apply A and B to the expansions of x in terms of $\{\phi_n\}$ and $\{\psi_n\}$, respectively, and apply A^* and B^* to the expansions of x in terms of $\{\psi_n\}$ and $\{\phi_n\}$, respectively, to get

$$Ax = \sum\nolimits_{k=0}^{\infty} \langle x\,;\psi_k\rangle_P\,A\,\phi_k = \sum\nolimits_{k=0}^{\infty} \langle x\,;\psi_k\rangle_P\,\phi_{k+1},$$

$$Bx = \sum\nolimits_{k=0}^{\infty} \langle x\,;\phi_k\rangle_P\,B\psi_k = \sum\nolimits_{k=0}^{\infty} \langle x\,;\phi_k\rangle_P\,\psi_{k+1},$$

$$A^*x = \sum\nolimits_{k=0}^{\infty} \langle A^*x\,;\phi_k\rangle_P\,\psi_k = \sum\nolimits_{k=0}^{\infty} \langle x\,;A\,\phi_k\rangle_P\,\psi_k = \sum\nolimits_{k=0}^{\infty} \langle x\,;\phi_{k+1}\rangle_P\,\psi_k,$$

$$B^*x = \sum\nolimits_{k=0}^{\infty} \langle B^*x\,;\psi_k\rangle_P\,\phi_k = \sum\nolimits_{k=0}^{\infty} \langle x\,;B\psi_k\rangle_P\,\phi_k = \sum\nolimits_{k=0}^{\infty} \langle x\,;\psi_{k+1}\rangle_P\,\phi_k. \qquad \square$$

6. (m,P)-Biisometric Pair for a Strictly Positive P

Throughout the paper the operator $P\in\mathcal{B}[\mathcal{H}]$ has been assumed positive (i.e., nonnegative and injective) so that $\langle\,\cdot\,;\cdot\,\rangle_P=\langle P\,\cdot\,;\cdot\,\rangle$ is an inner product generating the norm $\|\cdot\|_P=\|P^{\frac12}\cdot\|^{\frac12}$. If P is, in addition, surjective, so that it is invertible (i.e., strictly positive), then the inner product space $(\mathcal{H},\langle\,\cdot\,;\cdot\,\rangle_P)$ becomes a Hilbert space whenever $(\mathcal{H},\langle\,\cdot\,;\cdot\,\rangle)$ is a Hilbert space.

If (A,B) is an (m,P)-biisometric pair and the positive P is invertible, then assuming that $L_{A^*}R_B$ is power bounded we get

$$\triangle_{A^*,B}(P)=0 \quad\iff\quad A^*(PBP^{-1})=(P^{-1}A^*P)B=I,$$

so that both A and B are left invertible. Furthermore, if either of A^* and B^* is injective, then both A and B are invertible: in particular,

$$\mathcal{N}(A^*)\neq\{0\} \quad\iff\quad \mathcal{N}(B^*)\neq\{0\},$$

and A is invertible if and only if B is invertible. The following theorem shows that a stronger result than Theorem 5.1 is possible in the case in which the operators A and B are power bounded. But before that, some notation and terminology is in order. The numerical range $W(A)$ of an operator A is the set

$$W(A)=\big\{\lambda\in\mathbb{C}\colon\,\lambda=\langle Ax\,;x\rangle,\ x\in\mathcal{H},\ \|x\|=1\big\},$$

and the numerical radius $w(A)$ of A is

$$w(A)=\sup\{|\lambda|\colon\,\lambda\in W(A)\}.$$

The spectral radius $r(A)$ of A is

$$r(A)=\sup\{|\lambda|\colon\,\lambda\in\sigma(A)\}=\lim_{n\to\infty}\|A^n\|^{\frac1n}.$$

A is *normaloid* if $r(A)=\|A\|$ (which implies $r(A)=w(A)=\|A\|$), A is *convexoid* if the closure $\overline{W(A)}$ of the numerical range of A equals the convex hull $\operatorname{conv}\sigma(A)$ of the spectrum of A, and A is *spectraloid* if $r(A)=w(A)$ [6, Problem 219]. It is well known that the classes consisting of normaloid and convexoid operators are independent of each other, and that both these classes are contained in the class of spectraloid operators.

We close the paper by showing that for an invertible positive P, a pair (A,B) of power bounded (m,P)-biisometric operators is such that either A and B are both similar to the same unitary operator, or they satisfy the conclusion of Theorem 5.1.

Theorem 6.1. *If $\triangle^m_{A^*,B}(P)=0$ (i.e., (A,B) is an (m,P)-biisometric pair) for some invertible positive (i.e., strictly positive) P, and A,B are power bounded, then*

(a) *either*

 (i) *there exists a unitary operator U such that A and B are similar to U,*

 or

 (ii) *A and B satisfy the conclusions of Theorem 5.1 (with some obvious changes, since now $\mathcal{R}(P)=\mathcal{H}$).*

(b) *Furthermore, in case* (a-i)*, if A and A^{-1} (or B and B^{-1}) are either normaloid or convexoid or spectraloid (all combinations are allowed), then A is unitary (respectively, B is unitary) and $B=P^{-1}AP$ (respectively, $A=PBP^{-1}$).*

Proof. The power boundedness of A and B implies (the power boundedness of $L_{A^*}R_B$, and hence)

$$\triangle_{A^*,B}(P)=0 \iff A^*=PB^{-1}P^{-1} \iff A=P^{-1}B^{*-1}P$$

and, for all $x\in\mathcal{H}$ and positive integers n,

$$\begin{aligned} A^*PB=P &\Longrightarrow A^{*n}PB^nP^{-1}=I \\ &\Longrightarrow \|x\|\le\|P^{-1}B^{*n}P\|\,\|A^nx\|\le M_1\|A^nx\|\le M_1M_2\|x\| \end{aligned}$$

for some positive scalars M_1 and M_2. But then

$$\tfrac{1}{M_1}\|x\|\le\|A^nx\|\le M_2\|x\| \quad \text{for all} \quad x\in\mathcal{H}$$

implies the existence of an invertible operator S and an isometry V such that $SA=VS$ [8, Proposition 4.2]. Thus, since $A^*S^*SA=S^*S$, there exists an invertible positive operator P_1, $P_1^2=S^*S$, and an isometry U such that $A^*P_1=P_1U^*$, or, $P_1A=UP_1$. Similarly, since

$$\|x\|\le\|P^{-1}A^{*n}P\|\,\|B^nx\|\le M_{11}\|B^nx\|\le M_{12}\|x\|$$

for some positive scalars M_{11},M_{12} and all $x\in\mathcal{H}$, there exists an invertible positive operator P_2 and an isometry V_2 such that $P_2B=V_2P_2$.

(a) The operators A and B being left invertible, if neither of A^* and B^* is injective, then $\mathcal{N}(A^*)$ and $\mathcal{N}(B^*)$ are nonzero, and since $\mathcal{R}(P)=\mathcal{H}$, the argument of the proof of Theorem 5.1 goes through to prove (a-ii). If one of A^* and B^* is injective, then both A and B are invertible, $A=P_1^{-1}UP_1$ with the isometry U being a unitary operator and $B=P^{-1}A^{*-1}P=P^{-1}P_1UP_1^{-1}P=Q^{-1}UQ$ for an invertible operator Q. This proves (a-i).

(b) Assume now that A and A^{-1} are either normaloid or convexoid or spectraloid. Then, since $\sigma(A^{\pm1})$ is a subset of the boundary $\partial\mathbb{D}$ of the unit disc $\mathbb{D}$,

$$r(A^{\pm1})=w(A^{\pm1})=1,$$

and hence

$$W(A^{\pm1})\subseteq \operatorname{conv}\sigma(A^{\pm1}).$$

This [16, Theorem 1] implies that A is a normal operator. Since

$$A^*P_1=P_1U^* \iff AP_1=P_1U$$

by the Putnam-Fuglede commutativity theorem ([6, p.104]),

$$A^*P_1^2=P_1U^*P_1=P_1^2A^*\Longrightarrow A^*P_1=P_1A^*\Longrightarrow P_1U^*=P_1A^*\iff U=A.$$

Trivially, $B^{-1}=P^{-1}A^*P=P^{-1}U^*P$ implies $B=P^{-1}UP$. Since a similar argument works for the case in which B is normaloid or convexoid or spectraloid, the proof is complete. □

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