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WEAK STABILITY AND QUASISTABILITY

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Abstract. It is known that weak l-sequential supercyclicity implies weak quasistability, and it is still unknown whether weak l-sequential supercyclicity implies weak stability, much less whether weak supercyclicity implies weak stability (although it has been known for a long time that strong supercyclicity implies strong stability). It is shown that weak l-sequential supercyclicity implies weak stability under the assumption of boundedly spaced subsequences. It is given an example of a power bounded weakly quasistable operator that is not weakly stable.

1. **Introduction**

Let $\mathcal{X}$ be a normed space, and let $\mathcal{X}^*$ be its dual. An $\mathcal{X}$-valued sequence $\{x_n\}$ converges weakly if the scalar-valued sequence $\{f(x_n)\}$ converges for every $f \in \mathcal{X}^*$. Take an operator T on $\mathcal{X}$. It is weakly stable if $\lim_n |f(T^n x)| = 0$, and weakly quasistable if $\liminf_n |f(T^n x)| = 0$, for every $x \in \mathcal{X}$, for every $f \in \mathcal{X}^*$. It is clear that weak stability implies weak quasistability. It is shown that the converse fails by exhibiting a power bounded weakly quasistable operator that is not weakly stable (Proposition 4.3).

Let A be any nonempty set. A subsequence $\{a_{n_k}\}$ of an A-valued sequence $\{a_n\}$ is boundedly spaced if $\sup_k (n_{k+1} - n_k) < \infty$. A central result in this paper shows that if a weakly quasistable operator has a boundedly spaced subsequence of weak quasistability for every vector, then it is weakly stable (Theorem 5.3).

Let $\mathcal{O}_T([y])$ be the projective orbit of a nonzero vector $y \in \mathcal{X}$ under an operator T. The operator T is weakly l-sequentially supercyclic if every $x \in \mathcal{X}$ is the weak limit of an $\mathcal{O}_T([y])$-valued sequence. It is unknown whether weak l-sequential supercyclicity implies weak stability. An application of Theorem 5.3 gives a sufficient condition for weak l-sequential supercyclicity to imply weak stability (Theorem 6.2).

On the other hand, it is also given a necessary condition for weak stability of weakly l-sequentially supercyclic operators (Theorem 6.3), which together with Theorem 6.2, shows what happens when weak l-sequential supercyclicity holds without boundedly spaced subsequences (Corollary 6.4).

The paper is organised as follows. Basic notation and terminology are summarised in Section 2. The notions of uniform, strong, and weak stabilities, and their quasistability counterparts, are considered in Section 3. It is shown in Section 4 that the notions of stability and quasistability coincide for the strong and uniform cases, but not for the weak case. Boundedly spaced subsequences are considered in Section 5, and an application to weak l-sequential supercyclicity is considered in Section 6.

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2. **Basic notation and terminology**

Throughout this paper, all linear spaces are over the same scalar field $\mathbb{F}$, which is either $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{X}$ denotes an infinite-dimensional normed space. Particular cases of inner product, Banach, and Hilbert spaces are discussed accordingly. Let $\mathcal{B}[\mathcal{X}] = \mathcal{B}[\mathcal{X}, \mathcal{X}]$ stand for the collection of all bounded linear transformations of $\mathcal{X}$ into itself. That is, $\mathcal{B}[\mathcal{X}]$ stands for the normed algebra of all operators on $\mathcal{X}$. The linear manifold $\mathcal{R}(T) = T(\mathcal{X})$ of $\mathcal{X}$ is the range of T in $\mathcal{B}[\mathcal{X}]$. The Banach space $\mathcal{X}^* = \mathcal{B}[\mathcal{X}, \mathbb{F}]$ is the dual of $\mathcal{X}$, and T^* in $\mathcal{B}[\mathcal{X}^*]$ denotes the normed-space adjoint of T in $\mathcal{B}[\mathcal{X}]$ (i.e., $T^*f = fT$ for every $f \in \mathcal{X}^*$). We use the same notation for the Hilbert-space adjoint $T^* \in \mathcal{B}[\mathcal{X}]$ of $T \in \mathcal{B}[\mathcal{X}]$ on a Hilbert space $\mathcal{X}$, where the concepts of dual and adjoint are shaped after the Riesz Representation Theorem in Hilbert space — in a Hilbert space setting, adjoints are always Hilbert-spaces's.) An $\mathcal{X}$-valued sequence $\{x_n\}$ converges weakly to $x \in \mathcal{X}$ if $\lim_n f(x_n) \to f(x)$ for every $f \in \mathcal{X}^*$, equivalently, $\lim_n f(x_n - x) \to 0$ for every $f \in \mathcal{X}^*$. (This definition is standard — see, e.g., [22, Definition 1.13.2]). Alternative and usual notation for weak convergence that will be used here: $x_n \xrightarrow{w} x$ and $x = w\text{-}\lim_n x_n$. If $\mathcal{X}$ is a Hilbert space with inner product $\langle\,\cdot\,;\cdot\,\rangle$, then (again, by the Riesz Representation Theorem in Hilbert space) weak convergence of $\{x_n\}$ to x means $\langle x_n - x\,; z\rangle \to 0$ for every $z \in \mathcal{X}$. Norms on $\mathcal{X}$ and the induced uniform norm on $\mathcal{B}[\mathcal{X}]$ (and the norm on $\mathcal{X}^*$) will be denoted by the same symbol $\|\cdot\|$. An $\mathcal{X}$-valued sequence $\{x_n\}$ is strongly convergent (or converges in the norm topology) to $x \in \mathcal{X}$ if $\|x_n - x\| \to 0$. Alternative notation: $x_n \xrightarrow{s} x$; it is clear that strong convergence implies weak convergence. For an arbitrary operator $T \in \mathcal{B}[\mathcal{X}]$, take its power sequence $\{T^n\}$. According to the above definitions of convergence in $\mathcal{X}$, an operator T is said to be weakly or strongly stable if $T^n x \xrightarrow{w} 0$ or $T^n x \xrightarrow{s} 0$ for every $x \in \mathcal{X}$, respectively. An operator T is power bounded if $\sup_n \|T^n\| < \infty$ (equivalently, if $\sup_n \|T^n x\| < \infty$ for every $x \in \mathcal{X}$ by the Banach–Steinhaus Theorem, if $\mathcal{X}$ is a Banach space).

3. **Stability and quasistability**

Take $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$ and consider the $\mathcal{B}[\mathcal{X}]$-valued power sequence $\{T^n\}$. The operator T is uniformly stable if $\{T^n\}$ converges uniformly (i.e., if the sequence $\{T^n\}$ converges in the norm topology on $\mathcal{B}[\mathcal{X}]$) to the null operator. Thus an operator T is uniformly, strongly, or weakly stable if

$$\lim_n \|T^n\| = 0, \quad \text{or} \quad \lim_n \|T^n x\| \to 0 \;\text{ for every }\; x \in \mathcal{X}, \quad \text{or}$$

$$\lim_n |f(T^n x)| = 0 \;\text{ for every }\; x \in \mathcal{X}, \;\text{ for every }\; f \in \mathcal{X}^*,$$

respectively. Alternative notations for strong and weak stability were given in the previous section and, accordingly, if $\mathcal{X}$ is a Hilbert space, then weak stability for T means $\langle T^n x\,; z\rangle \to 0$ for every $x, z \in \mathcal{X}$ — if the Hilbert space $\mathcal{X}$ is complex, this is equivalent to $\langle T^n x\,; x\rangle \to 0$ for every $x \in \mathcal{X}$. It is clear that uniform stability implies strong stability, which implies weak stability, which in turn implies power boundedness (if $\mathcal{X}$ is a Banach space), and the converses fail. We say that an operator $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$ is *uniformly quasistable*, or *strongly quasistable*, or *weakly quasistable* if the above limits are replaced by limits inferior. That is, if

$$\liminf_n \|T^n\| = 0, \quad \text{or} \quad \liminf_n \|T^n x\| \to 0 \;\text{ for every }\; x \in \mathcal{X}, \quad \text{or}$$

$$\liminf_n |f(T^n x)| = 0 \;\text{ for every }\; x \in \mathcal{X}, \;\text{ for every }\; f \in \mathcal{X}^*,$$

respectively. It is clear that each form of stability implies its respective form of quasistability, and each form of quasistability implies the existence of a subsequence of the power sequence that converges to zero. In other words,

(i) uniform quasistability means that there exists a subsequence $\{T^{n_k}\}$ of $\{T^n\}$ that converges in $\mathcal{B}[\mathcal{X}]$ to the null operator, that is, $\|T^{n_k}\| \to 0$;

(ii) strong quasistability means that for every $x \in \mathcal{X}$ there exists a subsequence $\{n_k\} = \{n_k(x)\}$ of the sequence of the positive integers (which depends on x) such that the $\mathcal{X}$-valued sequence $\{T^{n_k}x\}$ converges in the norm topology to the origin of $\mathcal{X}$. That is, $T^{n_k}x \xrightarrow{s} 0$ for every $x \in \mathcal{X}$, for some subsequence $\{T^{n_k}\}$ of $\{T^n\}$ — equivalent notation: $\|T^{n_k}x\| \to 0$ for every $x \in \mathcal{X}$;

(iii) weak quasistability means that for each $x \in \mathcal{X}$ there is a subsequence $\{n_k\} = \{n_k(x)\}$ of the sequence of the positive integers (which depends on x) such that $T^{n_k}x \xrightarrow{w} 0$. Equivalently, for each vector $x \in \mathcal{X}$ there exists a subsequence $\{T^{n_k}\} = \{T^{n_k(x)}\}$ of $\{T^n\}$ such that the scalar-valued sequences $\{f(T^{n_k}x)\}$ converge to zero for every functional $f \in \mathcal{X}^*$. We summarise this by saying that there is a subsequence $\{T^{n_k}\}$ of $\{T^n\}$ such that $|f(T^{n_k}x)| \to 0$ for every $x \in \mathcal{X}$, for every $f \in \mathcal{X}^*$.

We will be particularly concerned with weak quasistability: for every x in $\mathcal{X}$, $T^{n_k}x \xrightarrow{w} 0$ for some subsequence $\{T^{n_k}\}$ of $\{T^n\}$. Thus an operator $T \in \mathcal{B}[\mathcal{X}]$ is weakly quasistable if, for every $x \in \mathcal{X}$, there exists a subsequence $\{T^{n_k}\}$ of $\{T^n\}$ for which $\lim_k f(T^{n_k}x) = 0$ (equivalently, $\lim_k |f(T^{n_k}x)| = 0$) for every $f \in \mathcal{X}^*$ (where, for each $x \in \mathcal{X}$, the subsequence $\{T^{n_k(x)}\}$ depends on x but not on f).

4. **Auxiliary results**

To begin with, we verify that the notions of uniform and strong stabilities coincide with the motions of uniform and strong quasistabilities, respectively. The notion of weak stability, however, does not coincide with the notion of weak quasistability. As we saw above, uniform and strong stabilities imply uniform and strong quasistability, respectively and trivially. Thus we verify next only the converses for normed-space operators (for a Hilbert-space version, see [20, proof of Theorem 3]).

Proposition 4.1. *For every operator T on a normed space $\mathcal{X}$, uniform quasistability is equivalent to uniform stability. Indeed,*

$$\inf_n \|T^n\| < 1 \quad \Longrightarrow \quad \lim_n \|T^n\| = 0,$$

and hence

$$\liminf_n \|T^n\| = 0 \quad \Longleftrightarrow \quad \lim_n \|T^n\| = 0.$$

Proof. Let T be an operator on a normed space. Recall that $\lim_n \|T^n\|^{1/n}$ exists as a real number and, by setting $r(T) = \lim_n \|T^n\|^{1/n}$, also recall that

$$r(T)^n = r(T^n) \leq \|T^n\| \leq \|T\|^n$$

for every nonnegative integer n (see, e.g., [13, Lemma 6.8] — in a Banach space setting, which is not necessarily the case here, $r(T)$ coincides with the spectral radius of T, and uniform stability is equivalent to $r(T) < 1$.) However, the implication

$$r(T) = \lim_n \|T^n\|^{1/n} < 1 \quad \Longrightarrow \quad \lim_n \|T^n\| \to 0$$

holds in a normed space as well (see, e.g., [13, proof of Proposition 6.22]). Now, if $\inf_n \|T^n\| < 1$, then either there is an n_0 such that $\|T^{n_0}\| < 1$, or $\liminf_n \|T^n\| < 1$, which also ensures that there exists an n_0 such that $\|T^{n_0}\| < 1$. Hence

$$r(T)^{n_0} = r(T^{n_0}) \leq \|T^{n_0}\| < 1,$$

and therefore $r(T) < 1$, which implies $\lim_n \|T^n\| = 0$. Thus

$$\inf_n \|T^n\| < 1 \quad \Longrightarrow \quad \lim_n \|T^n\| = 0.$$

So $\liminf_n \|T^n\| = 0$ implies $\lim_n \|T^n\| = 0$, leading to the stated equivalence. □

Note that $\lim_n \|T^n\| = 0$ naturally implies power boundedness for T.

Proposition 4.2. *If T is a power bounded operator on a normed space $\mathcal{X}$, then strong quasistability is equivalent to strong stability*:

$$\liminf_n \|T^n x\| = 0 \;\; \textit{for every} \;\; x \in \mathcal{X} \quad \Longleftrightarrow \quad \lim_n \|T^n x\| = 0 \;\; \textit{for every} \;\; x \in \mathcal{X}.$$

Proof. Take an arbitrary $x \in \mathcal{X}$. If $\liminf_n \|T^n x\| = 0$, then there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$ (which depends on $x \in \mathcal{X}$) for which $\lim_k \|T^{n_k}x\| = 0$. Thus, if T is power bounded,

$$\|T^n x\| \leq \|T^{n-n_k}\| \, \|T^{n_k}x\| \leq \sup_n \|T^n\| \|T^{n_k}x\| \quad \text{whenever} \quad n \geq n_k.$$

By the above inequality,

$$\lim_k \|T^{n_k}x\| = 0 \;\Leftrightarrow\; \limsup_k \|T^{n_k}x\| = 0 \;\Rightarrow\; \limsup_n \|T^n x\| = 0 \;\Leftrightarrow\; \lim_n \|T^n x\| = 0.$$

Therefore, if T is power bounded,

$$\liminf_n \|T^n x\| = 0 \;\; \textit{for every}\; x \in \mathcal{X} \;\;\Rightarrow\;\; \lim_n \|T^n x\| = 0 \;\; \textit{for every}\; x \in \mathcal{X}. \qquad \square$$

However, although weak stability implies weak quasistability, the converse fails:

$$\liminf_n |f(T^n x)| = 0 \quad \text{for every} \quad x \in \mathcal{X}, \quad \text{for every} \quad f \in \mathcal{X}^*$$
$$\not\Longrightarrow \; \lim_n |f(T^n x)| = 0 \;\; \text{for every} \;\; x \in \mathcal{X}, \;\; \text{for every} \;\; f \in \mathcal{X}^*.$$

We show in Proposition 4.3 below that the Foguel operator, which is power bounded and acts on a separable Hilbert space, is weakly quasistable but not weakly stable. Let $\mathcal{X}$ be a separable Hilbert space, and consider the Foguel operator

$$F = \begin{pmatrix} S^* & P \\ O & S \end{pmatrix}$$

acting on the separable Hilbert space $\mathcal{X} \oplus \mathcal{X}$ (where $\oplus$ means orthogonal direct sum) [10, 11]. Here S is a unilateral shift of multiplicity one on $\mathcal{X}$, which shifts an orthonormal basis $\{e_k\}_{k \geq 0}$ for $\mathcal{X}$, and $P\colon \mathcal{X} \to \mathcal{X}$ is the orthogonal projection onto $\mathcal{R}(P) = \operatorname{span}\{e_j : j \in \mathbb{J}\}^-$ (closure of span of $\{e_j : j \in \mathbb{J}\}$), where $\mathbb{J}$ is a sparse infinite set of positive integers with the following property: if $i, j \in \mathbb{J}$ and $i < j$, then $2i < j$ (e.g., $\mathbb{J} = \{j \geq 1 \colon\; j = 3^k;\; k \geq 0\}$, the set of all integral powers of 3).

Proposition 4.3. *The Foguel operator is a power bounded operator that is weakly quasistable but not weakly stable.*

Proof. Let F be the Foguel operator defined above. It is well known that F is power bounded [11, p.791] and not weakly stable (see, e.g., [12, Remark 8.7]). Note that

$$F^n = \begin{pmatrix} S^{*n} & P_n \\ O & S^n \end{pmatrix} \quad \text{for every} \quad n \geq 0,$$

with $P_n\, \mathcal{X} \to \mathcal{X}$ given by $P_{n+1} = \sum_{i=0}^{n} S^{*\,n-i} P S^i$ and $P_0 = O$ (the null operator). Therefore, for every $x = (x_1, x_2) \in \mathcal{X} \oplus \mathcal{X}$ and every $z = (z_1, z_2) \in \mathcal{X} \oplus \mathcal{X}$,

$$\langle F^{n+1}x\,; z\rangle = \langle S^{*n+1}x_1,\; z_1\rangle + \langle P_{n+1}x_2\,; z_1\rangle + \langle S^{n+1}x_2, z_2\rangle.$$

Since S is weakly stable (and so is S^*), we get

$$\liminf_n|\langle F^n x\,;z\rangle| = \liminf_n|\langle P_n x_2\,;z_1\rangle| \tag{§}$$

for every $x=(x_1,x_2)\in\mathcal{X}\oplus\mathcal{X}$, for every $z=(z_1,z_2)\in\mathcal{X}\oplus\mathcal{X}$, where

$$\langle P_{n+1}x_2\,;z_1\rangle = \sum\nolimits_k \langle x_2;e_k\rangle\langle P_{n+1}e_k\,;z_1\rangle,$$

with

$$P_{n+1}e_k = \sum\nolimits_{i=0}^{n} S^{*\,n-i}Pe_{k+i} = \sum\nolimits_{i=k}^{k+n} S^{*\,k+n-i}Pe_i.$$

If $i\not\in\mathbb{J}$, then $Pe_i=0$; if $i\in\mathbb{J}$, then $Pe_i=e_i$. Hence

$$S^{*\,k+n-i}Pe_i = \begin{cases} e_{2i-(k+n)} & \text{if } \; k+n\le 2i, \;\; i\in\mathbb{J},\\ 0 & \text{otherwise}\,.\end{cases}$$

Given a pair of nonnegative integers (k,n), consider the set

$$\mathbb{J}_{k,n} = \big\{j\in\mathbb{J} \quad k\le j\le k+n\le 2j\big\} = \mathbb{J}\cap\big[\max\{k,\tfrac{k+n}{2}\}\,,\,k+n\big],$$

which has at most one element (i.e., $\#\mathbb{J}_{k,n}\le 1$). (Indeed, if $i,j\in\mathbb{J}_{k,n}$ with $i<j$, then $2i<j$ (for $i,j\in\mathbb{J}$) and $j\le k+n\le 2i$, which is a contradiction.) Then

$$P_{n+1}e_k = \begin{cases} \sum_{i\in\mathbb{J}_{k,n}} e_{2i-(k+n)} \;=\; e_{2j-(k+n)} & \text{for } \; j\in\mathbb{J}_{k,n} \;\text{ whenever } \; \mathbb{J}_{k,n}\ne\varnothing,\\ 0 & \text{otherwise}\,.\end{cases}$$

Now take an arbitrary pair $(x_1,z_2)\in\mathcal{X}\oplus\mathcal{X}$. Since P is an orthogonal projection, P_{n+1} is self-adjoint, thus we get $\langle P_{n+1}x_1\,;z_2\rangle = \sum_k\langle x_1\,;P_{n+1}e_k\rangle\langle e_k\,;z_2\rangle$, and so

$$\langle P_{n+1}x_1\,;z_2\rangle = \begin{cases} \sum_k\langle x_1\,;e_{2j-(k+n)}\rangle\langle e_k\,;z_2\rangle & \text{for } \; j\in\mathbb{J}_{k,n} \;\text{ whenever } \; \mathbb{J}_{k,n}\ne\varnothing,\\ 0 & \text{otherwise}\,.\end{cases}$$

Then we can take an infinite subsequence $\{P_{n_j}\}$ of $\{P_n\}$ for which $\langle P_{n_j}x_1\,;z_2\rangle=0$ for every j, and so $\lim_j\langle P_{n_j}x_1\,;z_2\rangle=0$, for every $x_1,z_2\in\mathcal{X}$. For instance, with $n_j=2j+1$ for $j\in\mathbb{J}$ we get $\liminf_n|\langle P_n x_1\,;z_2\rangle|=0$ for every $x_1,z_2\in\mathcal{X}$, which implies $\liminf_n|\langle F^n x,z\rangle|=0$ for every $x,z\in\mathcal{X}\oplus\mathcal{X}$ by (§). □

Observe that $\{n_j\}$ is a rather sparse subsequence of the nonnegative integers, whose increments $n_{j+1}-n_j$ increase unboundedly as $j\to\infty$ (by the construction of the set $\mathbb{J}$). Such a property was fundamental for proving that F is not weakly stable (see, e.g., [12, Proposition 8.3 and Remark 8.7]).

5. **Boundedly spaced subsequences**

Let $\{n\}_{n\ge1}$ stand for the self-indexing of the set of all positive integers equipped with the natural order and regard it as the positive-integer-valued identity sequence. A subsequence $\{n_k\}_{k\ge1}$ of $\{n\}_{n\ge1}$ is of *bounded increments* (or has bounded gaps) if $\sup_{k\ge1}(n_{k+1}-n_k)<\infty$. Let A be an arbitrary nonempty set. We say that a subsequence $\{a_{n_k}\}$ of an arbitrary A-valued sequence $\{a_n\}$ is *boundedly spaced* if it is indexed by a subsequence $\{n_k\}_{k\ge1}$ of bounded increments (i.e., $\{a_{n_k}\}$ is boundedly spaced if $\sup_{k\ge1}(n_{k+1}-n_k)<\infty$). These notions (bounded increments and boundedly spaced) have been applied, in the present context, in [20, 21]. The next proposition will be required in the sequel. (A standard result, for the particular case of scalar sequences, with an easy proof, which we include for sake of completeness.)

Proposition 5.1. *If* $\{\alpha_{n_k}\}$ *is a boundedly spaced subsequence of an* $\mathbb{F}$*-valued sequence* $\{\alpha_n\}$*, then* $\alpha_{n_k+j}\to\alpha$ *as* $k\to\infty$ *for every* $j\ge 1$ *implies* $\alpha_n\to\alpha$ *as* $n\to\infty$*.*

Proof. For each $j\ge 1$, suppose the sequence $\{\alpha_{n_k+j}\}$ converges to α as $k\to\infty$. In addition, suppose j lies in a bounded set J, and take an arbitrary $\varepsilon>0$. Thus there is an integer k_ε such that for all $j\in[1,\max j]$, with $\max j=\max\{j: j\in J\}$,

$$k\ge k_\varepsilon \quad\Longrightarrow\quad |\alpha_{n_k+j}-\alpha|<\varepsilon,$$

(Indeed, for any $\varepsilon>0$ and each j take $k_{\varepsilon,j}$ such that if $k\ge k_{\varepsilon,j}$ then $|\alpha_{n_k+j}-\alpha|<\varepsilon$, and consider the largest integer $k_\varepsilon=\max_{j\in[1,\max j]}\{k_{\varepsilon,j}\}$). Next observe that

> if $\{n_k\}_{k\ge 0}$ is of bounded increments, say with $M=\sup_k(n_{k+1}-n_k)$, then $\bigcup_{j\in[0,M]}\{n_k+j\}_{k\ge 1}=\{n\}_{n\ge 1}$.

So every positive integer n is written as $n=n_k+j$ for some n_k and some $j\in[1,M]$. Thus by the above displayed implication (with $J=[1,M]$ so that $\max j=M$), for every $\varepsilon>0$ there exists an integer $n_\varepsilon=n_{k_\varepsilon}+M$ such that

$$n\ge n_\varepsilon \quad\Longrightarrow\quad |\alpha_n-\alpha|<\varepsilon.$$

Hence $\alpha_n\to\alpha$ as $n\to\infty$. The converse is trivial. □

As shown next, $T^{n_k}x\xrightarrow{w}0\Longrightarrow T^n x\xrightarrow{w}0$ whenever $\{T^{n_k}\}$ is boundedly spaced.

Lemma 5.2. *Suppose* $\{T^{n_k}\}$ *is a boundedly spaced subsequence of* $\{T^n\}$*. Take* $x\in\mathcal{X}$*.*

If $\{T^{n_k}x\}$ *converges weakly to zero, then* $\{T^n x\}$ *converges weakly to zero.*

Proof. Take a vector $x\in\mathcal{X}$ and a subsequence $\{T^{n_k}\}_k$ of $\{T^n\}_n$. Suppose $\{T^{n_k}x\}_k$ converges weakly to zero. That is, $\lim_k f(T^{n_k}x)=0$ for every $f\in\mathcal{X}^*$. Thus, in particular, for an arbitrary g in $\mathcal{X}^*$ (so that $gT^j=T^{*j}g$ lies in $\mathcal{X}^*$ for every nonnegative integer j), $\lim_k(T^{*j}g)(T^{n_k}x)=0$. Now suppose $\{T^{n_k}\}_k$ is a boundedly spaced subsequence of $\{T^n\}_n$. Then $\{(gT^j)(T^{n_k}x)\}_k$ is a boundedly spaced subsequence of the $\mathbb{F}$-valued sequence $\{(gT^j)(T^n x)\}_n$ for each $j\ge 0$. Since $\lim_k g(T^{n_k+j}x)=\lim_k(gT^j)(T^{n_k}x)=0$ for every $j\ge 0$ (i.e., for every $gT^j=T^{*j}g\in\mathcal{X}^*$), Proposition 5.1 ensures that $\lim_n g(T^n x)=0$. As this holds for an arbitrary $g\in\mathcal{X}^*$, it follows that $\lim_n g(T^n x)=0$ for every $g\in\mathcal{X}^*$, and so $\{T^n x\}_n$ converges weakly to zero. □

Let $T\in\mathcal{B}[\mathcal{X}]$ be a weakly quasistable operator on a normed space $\mathcal{X}$. This means $\liminf_n|f(T^n x)|=0$ for every $x\in\mathcal{X}$, for every $f\in\mathcal{X}^*$. Equivalently, for each $x\in\mathcal{X}$ there is at least one subsequence $\{T^{n_k(x)}\}$ of $\{T^n\}$ for which $\lim_k|f(T^{n_k(x)}x)|=0$ for every $f\in\mathcal{X}^*$ (where, for each $x\in\mathcal{X}$, the sequence of integers $\{n_k(x)\}$ depends on x but not on f). Such a subsequence $\{T^{n_k(x)}\}$ of $\{T^n\}$ will be referred to as *a subsequence of weak quasistability of* T *for* $x\in\mathcal{X}$.

Theorem 5.3 below is a central result.

Theorem 5.3. *If* $T\in\mathcal{B}[\mathcal{X}]$ *has a boundedly spaced subsequence of weak quasistability for every* $x\in\mathcal{X}$*, then* T *is weakly stable.*

Proof. Take an arbitrary $x\in\mathcal{X}$. Suppose T has a boundedly spaced subsequence of weak quasistability $\{T^{n_k(x)}\}$ for such an x. That is, $\{T^{n_k(x)}x\}$ converges weakly to zero, and the subsequence $\{T^{n_k(x)}\}$ of $\{T^n\}$ is boundedly spaced. Then $\{T^n x\}$ converges weakly to zero according to Lemma 5.2. If this holds for every $x\in\mathcal{X}$, then T is weakly stable. □

Proposition 4.3 and Theorem 5.3 together ensure the following nonimplication. Take any $x \in \mathcal{X}$. If $\lim_k f(T^{n_k}x) = 0$ for every $f \in \mathcal{X}^*$, for some subsequence $\{T^{n_k}\}$ of $\{T^n\}$, then it does not follow that there exists a subsequence of bounded increments $\{m_k\}$ of $\{n\}$ for which $\lim_k f(T^{m_k}x) = 0$ for every $f \in \mathcal{X}^*$.

6. Application to weak l-sequential supercyclicity

The orbit $\mathcal{O}_T(y)$ of a vector $y \in \mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the set

$$\mathcal{O}_T(y) = \bigcup\nolimits_{n\ge 0}\{T^n y\} = \big\{T^n y \in \mathcal{X}\colon \text{for every integer } n \ge 0\big\}.$$

Let $[x] = \operatorname{span}\{x\}$ denote the one-dimensional subspace of $\mathcal{X}$ spanned by a singleton $\{x\}$ at a vector $x \in \mathcal{X}$. The projective orbit of a nonzero vector $y \in \mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the orbit $\mathcal{O}_T([y]) = \bigcup_{n\ge 0} T^n([y])$ of the span of $\{y\}$:

$$\mathcal{O}_T([y]) = \big\{\alpha T^n y \in \mathcal{X}\colon \text{ for every } \alpha \in \mathbb{F} \text{ and every integer } n \ge 0\big\}.$$

A vector y in $\mathcal{X}$ is a (*strongly*) *supercyclic vector* for T if its projective orbit $\mathcal{O}_T([y])$ is dense in $\mathcal{X}$. Since the norm topology is metrizable, a nonzero vector y in $\mathcal{X}$ is supercyclic for an operator T if and only if for every x in $\mathcal{X}$ there exists an $\mathbb{F}$-valued sequence $\{\alpha_k\}_{k\ge 0}$ such that for some sequence $\{T^{n_k}\}_{k\ge 0}$ with entries from $\{T^n\}_{n\ge 0}$, the $\mathcal{X}$-valued sequence $\{\alpha_k T^{n_k} y\}_{k\ge 0}$ converges to x (in the norm topology):

$$\alpha_k T^{n_k} y \xrightarrow{\;s\;} x \;\text{ for every }\; x \in \mathcal{X} \qquad (\text{i.e., } \;\; \|\alpha_k T^{n_k} y - x\| \to 0 \;\text{ for every }\; x \in \mathcal{X}).$$

An operator T is a (*strongly*) *supercyclic operator* if it has a supercyclic vector.

The weak version of the above convergence criterion leads to the notion of weak l-sequential supercyclicity. A nonzero vector y in $\mathcal{X}$ is a *weakly l-sequentially supercyclic vector* for T if for every x in $\mathcal{X}$ there is an $\mathbb{F}$-valued sequence $\{\alpha_k\}_{k\ge 0}$ such that, for some sequence $\{T^{n_k}\}_{k\ge 0}$ with entries from $\{T^n\}_{n\ge 0}$, the $\mathcal{X}$-valued sequence $\{\alpha_k T^{n_k} y\}_{k\ge 0}$ converges weakly to x. In other words, if

$$\alpha_k T^{n_k} y \xrightarrow{\;w\;} x \;\text{ for every }\; x \in \mathcal{X}$$

$$(\text{i.e., } \;\; f(\alpha_k T^{n_k} y - x) \to 0 \;\text{ for every }\; x \in \mathcal{X}, \;\text{ for every }\; f \in \mathcal{X}^*).$$

Fix a weakly l-sequentially supercyclic vector y. The $\mathbb{F}$-valued sequence $\{\alpha_k\}_{k\ge 0}$ depends on x (but not on f), and each $\alpha_k = \alpha_k(x)$ is nonzero whenever $x \ne 0$. Similarly, the sequence $\{T^{n_k}\}_{k\ge 0}$ also depends on x (but not on f) — and so (as before) we sometimes write $T^{n_k} = T^{n_k(x)}$. An operator T is a *weakly l-sequentially supercyclic operator* if it has a weakly l-sequentially supercyclic vector.

Remark 6.1. (a) Note that, as remarked in [15], weak l-sequential supercyclicity is not a topological notion. In fact, as we have defined above, weak l-sequential supercyclicity comes as a weak version of a norm convergence criterion, but it does not mean denseness of the projective orbit in any topology. In particular, weak l-sequential supercyclicity does not mean denseness of the projective orbit in weak topology or in weak sequential topology, so weak l-sequential supercyclicity is not a weak topology notion. Therefore, it is not the case to argue in terms of weak topology techniques when dealing with weak l-sequential supercyclicity — see [26] for a discussion along this line. Also note that, according to the definition of weak convergence in Section 2, weak l-sequential supercyclicity does not mean convergence of the $\mathcal{B}[\mathcal{X}]$-valued sequence $\{\alpha_k T^{n_k}\}$ in the weak operator topology on $\mathcal{B}[\mathcal{X}]$.

(b) Take a nonzero $x \in \mathcal{X}$. If $x \in \mathcal{O}_T([y])$ for some vector y, then $\{T^{n_k}\}_{k\geq 0}$ can be viewed as a constant infinite sequence; equivalently, as a single-entry finite subsequence of $\{T^n\}_{n\geq 0}$ — i.e., $x = \alpha_0 T^{n_0} y$ for some $\alpha_0 \neq 0$ and some $n_0 \geq 0$. In this case, the notions of $\alpha_k T^{n_k} y \xrightarrow{s} x$ and $\alpha_k T^{n_k} y \xrightarrow{w} x$, related to plain supercyclicity and weak l-sequential supercyclicity, coincide. In this case, convergence of the constant or finite sequence $\{\alpha_k T^{n_k} y\}$ means that a limit is eventually reached. Therefore, in general, we use the expression

$$\{T^{n_k}\}_{k\geq 0} \textit{ is a sequence with entries from } \{T^n\}_{n\geq 0}.$$

However, if $x \notin \mathcal{O}_T([y])$, this means $\{T^{n_k}\}_{k\geq 0}$ *is a subsequence of* $\{T^n\}_{n\geq 0}$.

For further weak forms of cyclicity see, e.g., [2], [6], [7], [8], [23], [24], [25]. Weak l-sequential supercyclicity is the central theme in this section, and it has been considered, for instance, in [5], [3], [14], and discussed in [26], [16], [18]. Observe that weak l-sequential supercyclicity is not weak sequential supercyclicity. For a comparison between weakly l-sequentially supercyclicity and other notions of weak cyclicity, including weak hypercyclicity, weak sequential supercyclicity, and weak supercyclicity (plain weak cyclicity coincides with plain cyclicity) see, for instance, [26, pp.38,39], [4, pp.259,260], [9, pp.159,232], [16, pp.50,51,54], [17, pp.372,373,374].

It has been known for a long time that *a power bounded supercyclic operator on a normed space is strongly stable* [1, Theorem 2.2]. Along this line, the following question has been posed in [16]:

Is a power bounded weakly l-sequentially supercyclic operator weakly stable ?

This is a nontrivial question that, as far as we are aware, remains unanswered. The next results give partial answers to it.

Let $Y_T \subseteq \mathcal{X}$ denote the set of all weakly l-sequentially supercyclic vectors y for an operator T on a normed space $\mathcal{X}$,

$$Y_T = \big\{y \in \mathcal{X} \colon \text{ for every } x \in \mathcal{X} \text{ there exists } \{\alpha_k\} \text{ such that } x = w\text{-}\lim_n a_k T^{n_k} y\big\},$$

so that T is weakly l-sequentially supercyclic if and only if $Y_T \neq \varnothing$. In this case, take an arbitrary $y \in Y_T$. So, for every $x \in \mathcal{X}$, there is at least one sequence $\{T^{n_k}\}$ with entries from $\{T^n\}$, and an associated scalar sequence $\{\alpha_k\}$, for which $\lim_k f(\alpha_k T^{n_k} y) = f(x)$ for every $f \in \mathcal{X}^*$. By Remark 6.1(b), if $x \in \mathcal{O}_T([y])$, then the above sequences can be viewed as constant sequences, say $x = \alpha_k T^{n_k} y$ with $\alpha_k = \alpha_0$ and $n_k = n_0$ for every $k \geq 0$, where the constant sequence $\{n_k\}$ is not a subsequence of the nonnegative integers, and so it makes no sense to talk of boundedly spaced subsequences in this case. We will, however, be concerned with boundedly spaced subsequences $\{T^{n_k}\}$ of $\{T^n\}$ when x lies in $\mathcal{X}\backslash\mathcal{O}_T([y])$ for an arbitrary y in Y_T.

Let $T \in \mathcal{B}[\mathcal{X}]$ be a weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$. Let y be any vector in Y_T. For each x in $\mathcal{X}\backslash\mathcal{O}_T([y])$ there is at least one subsequence $\{T^{n_k(x)}\}$ of $\{T^n\}$ for which $\lim_k \alpha_k(x) f(T^{n_k(x)} y) = f(x)$ for every $f \in \mathcal{X}^*$, where the $\mathbb{F}$-valued sequence $\{\alpha_k(x)\}$ and the sequence of integers $\{n_k(x)\}$ depend on x but not on f. This subsequence $\{T^{n_k(x)}\}$ of $\{T^n\}$ is referred to as *a subsequence of weak l-sequential supercyclicity of T for $x \in \mathcal{X}\backslash\mathcal{O}([y])$* with respect to $y \in Y_T$.

Theorem 6.2. *Let T be a power bounded weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$. If there exists a boundedly spaced subsequence of weak l-sequential supercyclicity for each $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$, for every $y \in Y_T$, then T is weakly stable.*

Proof. It has recently been proved in [19, Corollary 4.3] that

$$\text{\textit{every power bounded weakly l-sequentially supercyclic operator on a normed space is weakly quasistable.}} \qquad (*)$$

We sketch the proof below, both for sake of completeness and also because part of its argument will be required later in this proof.

Suppose T a weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$, and take an arbitrary $y \in Y_T$. It was shown in [19, Theorem 4.1] that if this T is power bounded, then there is no $f \in \mathcal{X}^*$ such that $\liminf_n |f(T^n y)| > 0$. Consequently,

$$\liminf_n |f(T^n y)| = 0 \quad \text{for every} \quad y \in Y_T,$$

for every $f \in \mathcal{X}^*$. Also, we know that if $Y_T \neq \varnothing$, then Y_T is dense in $\mathcal{X}$ [15, Theorem 5.1]. Then for an arbitrary $x \in \mathcal{X}$, take (by density) any Y_T-valued sequence$\{y_k\}$ converging (in the norm topology) to x, so that (since T is power bounded)

$$|f(T^n x)| \le |f(T^n(y_k - x))| + |f(T^n y_k)| \le \|f\| \, (\sup_m \|T^m\|) \, \|y_k - x\| + |f(T^n y_k)|$$

for each integer $n \ge 0$, for every $f \in \mathcal{X}^*$. Hence, as $\liminf_n |f(T^n y)| = 0$ for every y in Y_T, $\liminf_n |f(T^n y_k)| = 0$ for every k, and so (as x was arbitrarily taken from $\mathcal{X}$)

$$\liminf_n |f(T^n x)| = 0 \quad \text{for every} \quad x \in \mathcal{X},$$

for every $f \in \mathcal{X}^*$, proving the result in $(*)$.

Now we split the proof into two parts. Take $y \in Y_T$ and $x \in \mathcal{X} \backslash \mathcal{O}_T([y])$ arbitrarily. Thus T has a subsequence of weak l-sequential supercyclicity $\{T^{n_k(x)}\}$ for x (with respect to y) so that, for some scalar sequence $\{\alpha_k(x)\}$,

$$\lim_k \alpha_k(x) f(T^{n_k(x)} y) = f(x) \quad \text{for every} \quad f \in \mathcal{X}^*,$$

where the above limit holds for every subsequence of $\{\alpha_k(x) f(T^{n_k(x)} y)\}$.

Part 1. Suppose x is such that $\{\alpha_k(x)\}$ is unbounded. So $\limsup_k |\alpha_k(x)| = \infty$. Since $|f(x)| \in \mathbb{R}$, the above displayed identity ensures that

$$\liminf_k |f(T^{n_k(x)} y)| = 0 \quad \text{for every} \quad f \in \mathcal{X}^*.$$

If T has a boundedly spaced subsequence of weak l-sequential supercyclicity for such an $x \in \mathcal{X} \backslash \mathcal{O}_T([y])$ and such a y, then take it so that this same $\{T^{n_k(x)}\}$ is a boundedly spaced subsequence of weak quasistability for y as well. So $T^n y \xrightarrow{w} 0$ according to Lemma 5.2. If this holds for every $y \in Y_T$, then $T^n y \xrightarrow{w} 0$ for every $y \in Y_T$. Since Y_T is dense in $\mathcal{X}$, we get (as we saw above) $T^n x \xrightarrow{w} 0$ for every $x \in \mathcal{X}$.

Part 2. Suppose x is such that $\{\alpha_k(x)\}$ is bounded. If T has a boundedly spaced subsequence $\{T^{n_k(x)}\}$ of weak l-sequential supercyclicity for such an $x \in \mathcal{X} \backslash \mathcal{O}_T([y])$ and such a y, then take it so that the corresponding $\mathbb{F}$-valued sequence $\{\alpha_k(x)\}$ is boundedly spaced as well. Since $\{\alpha_k(x)\}$ is bounded, there is a convergent subsequence $\{\alpha_{k_j}(x)\}$ of $\{\alpha_k(x)\}$ (by the Bolzano–Weierstrass Theorem), which can be taken to be boundedly spaced as well. Hence $\{T^{n_{k_j}(x)}\}$ is a boundedly spaced subsequence of weak l-sequential supercyclicity for x, so that $\lim_j \alpha_{k_j}(x) f(T^{n_{k_j}(x)} y) = f(x)$ for every $f \in \mathcal{X}^*$, with $\lim_j \alpha_{k_j}(x) = \alpha(x)$. If T is power bounded, then it was shown in [19, Theorem 4.1(c)] that $\alpha(x) \lim_j f(T^{n_{k_j}(x)} y) = f(x)$ for every $f \in \mathcal{X}^*$. But this implies that $\liminf_j |f(T^{n_{k_j}(x)} y)| = 0$ (cf. [19, Theorem 4.1(d)]). Thus, as in Part 1, $\{T^{n_{k_j}}\}$ is a boundedly spaced subsequence of weak quasistability for y. Therefore the same argument as in Part 1 ensures $T^n x \xrightarrow{w} 0$ for every $x \in \mathcal{X}$. $\square$

Theorem 6.2 does not ensure the existence of a boundedly spaced subsequence of weak l-sequential supercyclicity. But as we have seen in the proof of Theorem 6.2 (cf. [19, Corollary 4.3]), weak quasistability holds under power boundedness: *if T is a weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$, then*

(a) $\qquad T$ *power bounded* $\Longrightarrow$ T weakly quasistable.

We show next that

(b) $\qquad \{\alpha_k(x)\}$ *bounded for some* $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$ $\Longrightarrow$ T weakly unstable.

Theorem 6.3. *Let T be a weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$. Take an arbitrary vector $y \in Y_T$ so that for every vector $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$ there are a sequence of scalars $\{\alpha_k(x)\}$ and a subsequence $\{T^{n_k(x)}\}$ of $\{T^n\}$ such that $\alpha_k(x)f(T^{n_k(x)}y) \to f(x)$ for every $f \in \mathcal{X}^*$ (i.e., such that $\alpha_k(T^{n_k}y) \xrightarrow{w} x$).*

If for some $y \in Y_T$ the sequence of scalars $\{\alpha_k(x)\}$ is bounded for some $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$, then T is not weakly stable.

In other words, if T is weakly stable, then all scalar sequences $\{\alpha_k\}$ are unbounded.

Proof. Take an arbitrary $y \in Y_T$ and an arbitrary nonzero $x_0 \in \mathcal{X}\backslash\mathcal{O}_T([y])$. Although there is no injective linear functional on a linear space of dimension greater than one (we are assuming $\dim\mathcal{X} > 1$), note that there exists $f_0 \in \mathcal{X}^*$ such that $f_0(x_0) \neq 0$ (since there is no $0 \neq z \in \mathcal{X}$ for which $f(z) = 0$ for every $f \in \mathcal{X}^*$ by the Hahn–Banach Theorem.) Thus, since T is weakly l-sequentially supercyclic, there exist a sequence of scalars $\{\alpha_k(x_0)\}$ and a subsequence $\{T^{n_k(x_0)}\}$ of $\{T^n\}$ such that

$$|\alpha_k(x_0)|\,|f_0(T^{n_k(x_0)}y)| \to |f_0(x_0)| \in \mathbb{R}\backslash\{0\}.$$

If T is weakly stable, then $\lim_n |f(T^n y)| = 0$, so that $\lim_k |f(T^{n_k(x_0)}y)| = 0$, for every $f \in \mathcal{X}^*$; in particular, for such an $f_0 \in \mathcal{X}^*$. Therefore the above displayed expression ensures that the sequence $\{\alpha_k(x_0)\}$ is unbounded for such an arbitrary x_0 in $\mathcal{X}\backslash\mathcal{O}_T([y])$. Consequently, if for some $y \in Y_T$ the sequence of scalars $\{\alpha_k(x)\}$ is bounded for some $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$, then the operator T is not weakly stable. □

Take an arbitrary $y \in Y_T$. Recall that an operator T is said to have a *boundedly spaced subsequence of weak l-sequential supercyclicity* for a given $x \in \mathcal{X}\backslash\mathcal{O}([y])$ if $\alpha_k(x)f((T^{n_k(x)}y) \to f(x)$ for every $f \in \mathcal{X}^*$ and the sequence $\{T^{n_k(x)}\}$ is boundedly spaced (equivalently, the sequence of integers $\{n_k(x)\}$ is of bounded increments).

Corollary 6.4. *Let T be a power bounded weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$. Then either*

(a) *T does not have a boundedly spaced subsequence of weak l-sequential supercyclicity for some $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$ and some $y \in Y_T$,*

or

(b) *the sequence of scalars $\{\alpha_k(x)\}$ is unbounded for every $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$, for every $y \in Y_T$.*

Proof. If T is weakly unstable, then Theorem 6.2 ensures the alternative (a), and if T is weakly stable, then Theorem 6.3 ensures the alternative (b), exclusively. □

We close the paper with a question.

Is the Foguel operator in Proposition 4.3 weakly l-sequentially supercyclic?

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