Rendiconti del Circolo Matematico di Palermo, **73**(2) (2024) 663–673

WEAK L-SEQUENTIAL SUPERCYCLICITY AND WEAK QUASISTABILITY

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Abstract. It is known that supercyclicity implies strong stability. It is not known whether weak l-sequential supercyclicity implies weak stability. In this paper we prove that *weak l-sequential supercyclicity implies weak quasistability.* Corollaries concerning the characterisation of (i) weakly l-sequentially supercyclic vectors that are not (strongly) supercyclic, and (ii) weakly l-sequentially supercyclic isometries, are also proved.

1. Introduction

Supercyclicity implies strong stability. That is, if the projective orbit $\mathcal{O}_T([y])$ of a power bounded linear operator T on a normed space $\mathcal{X}$ is dense in $\mathcal{X}$ for some vector y in $\mathcal{X}$, then $T^n x \to 0$ for every $x \in \mathcal{X}$ (both density and convergence are in the norm topology). In other words, $\mathcal{O}_T([y])^- = \mathcal{X}$ implies $T^n \xrightarrow{s} O$. This is an important result from [1, Theorem 2.2]. It has been asked in [10] whether such an implication survives from norm topology to weak topology. In particular, whether a stronger version of weak supercyclicity (which is weaker than supercyclicity in the norm topology, but stronger than supercyclicity in the weak topology) implies weak stability (i.e., implies that $T^n x \xrightarrow{w} 0$ for every $x \in \mathcal{X}$). Such a stronger version of weak supercyclicity is called weak l-sequential supercyclicity, which means that the weak limit set $\mathcal{O}_T([y])^{-wl}$ of the projective orbit (i.e., the set of all weak limits of weakly convergent sequences in the projective orbit) coincides with $\mathcal{X}$. So it was asked whether $\mathcal{O}_T([y])^{-wl} = \mathcal{X}$ implies $T^n \xrightarrow{w} O$, where weak stability is defined by the expression: $\lim_n |f(T^n x)| = 0$ for every $x \in \mathcal{X}$, for every $f \in \mathcal{X}^*$. A weaker form of weak stability, called weak quasistability, is defined as: $\liminf_n |f(T^n x)| = 0$ for every $x \in \mathcal{X}$, for every $f \in \mathcal{X}^*$. In [10, Theorem 6.2] the authors gave a sufficient condition for a weakly l-sequentially supercyclic power bounded operator to be weakly stable. Here we improve that result by showing that, under the power boundedness assumption,

weak l-sequential supercyclicity implies weak quasistability.

Notation and terminology used above will be defined in the next section. The paper is organized into five sections. Notation and terminology are summarized in Section 2. Supplementary results on supercyclicity and strong stability required in the sequel are considered in Section 3. The main results are proved in Section 4, and synthesised in Corollary 4.3. These concern weak l-sequential supercyclicity and quasistability for power bounded operators acting on a normed space. Two applications are considered in Section 5, characterising weakly l-sequentially supercyclic operators that are not supercyclic (Corollary 5.2), and also characterising weakly l-sequentially supercyclic isometries on Hilbert spaces (Corollary 5.5).

2. Notation and Terminology

Linear spaces in this paper are all over a generic scalar field $\mathbb{F}$, which is either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{R}(L)$ and $\mathcal{N}(L)$ stand for range and kernel, respectively, of a linear

Date: June 13, 2023.
2010 *Mathematics Subject Classification.* Primary 47A16; Secondary 47A45.
Keywords. Weak stability/quasistability, weak l-sequential supercyclicity, weak supercyclicity.

transformation L between linear spaces. Let $\mathcal{B}[\mathcal{X}]$ denote the normed algebra of all bounded linear transformations of a normed space $\mathcal{X}$ into itself. Elements of $\mathcal{B}[\mathcal{X}]$ will be referred to as operators. Both norms, on $\mathcal{X}$ and the induced uniform norm on $\mathcal{B}[\mathcal{X}]$, will be denoted by $\|\cdot\|$. For each $T\in\mathcal{B}[\mathcal{X}]$ take its power sequence $\{T^n\}_{n\geq 0}$. An operator T is power bounded if $\sup_n\|T^n\|<\infty$, strongly stable if $\|T^n x\|\to 0$ (i.e., $T^n x\to 0$) for every $x\in\mathcal{X}$ (notation: $T^n\xrightarrow{s}O$), and of class $C_{1\cdot}$ if $\|T^n x\|\not\to 0$ (i.e., $T^n x\not\to 0$) for every nonzero $x\in\mathcal{X}$. It is weakly stable if $|f(T^n x)|\to 0$ for every f in the dual $\mathcal{X}^*$ of $\mathcal{X}$ (i.e., $T^n x\xrightarrow{w}0$) for every $x\in\mathcal{X}$ (notation: $T^n\xrightarrow{w}O$). Strong stability implies weak stability. If $\mathcal{X}$ is a Banach space, then weak stability implies power boundedness (by the Banach–Steinhaus Theorem). We say that an operator $T\in\mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$ is *weakly quasistable* if, for each $x\in\mathcal{X}$,

$$\liminf_n|f(T^n x)|=0 \quad \text{for every} \quad f\in\mathcal{X}^*.$$

In particular, every weakly stable operator is weakly quasistable.

The *weak limit set* A^{-wl} of a subset A of a normed space $\mathcal{X}$ is the set of all weak limits of weakly convergent A-valued sequences, that is,

$$A^{-wl}=\big\{x\in\mathcal{X}\colon x=w\text{-}\lim x_n \text{ with } x_n\in A\big\}.$$

Remark 2.1. Let A be a set in a normed space $\mathcal{X}$. Its closure (in the norm topology of $\mathcal{X}$) is denoted by A^- and its weak closure (in the weak topology of $\mathcal{X}$) is denoted by A^{-w}, so that A is dense (in the norm topology) or weakly dense (in the weak topology) if $A^-=\mathcal{X}$ or $A^{-w}=\mathcal{X}$, respectively. A set A is weakly sequentially closed if every A-valued weakly convergent sequence has its limit in A. Let the weak sequential closure of A be denoted by A^{-ws}, which is the smallest (i.e., the intersection of all) weakly sequentially closed subset of $\mathcal{X}$ including A. So A is weakly sequentially dense if $A^{-ws}=\mathcal{X}$. Consider the above definition of the weak limit set A^{-wl} of A (the set of all weak limits of weakly convergent A-valued sequences). We say that a set A is *weakly l-sequentially dense* if $A^{-wl}=\mathcal{X}$. It is known that

$$A^-\subseteq A^{-wl}\subseteq A^{-ws}\subseteq A^{-w},$$

and the inclusions may be proper in general (see, e.g., [15, pp.38,39], [3, pp.259,260], [9, pp.10,11]). Therefore if A is dense in the norm topology (i.e., $A^-=\mathcal{X}$), then it is dense with respect to all notions of denseness defined above. Recall that if a set A is convex, then $A^-=A^{-w}$ (see, e.g., [13, Theorem 2.5.16]). Thus if A is convex, then the above inclusions become a chain of identities, and so for a convex set all the above notions of denseness coincide.

The orbit $\mathcal{O}_T(y)$ of a vector $y\in\mathcal{X}$ under an operator $T\in\mathcal{B}[\mathcal{X}]$ is the set

$$\mathcal{O}_T(y)=\bigcup\nolimits_{n\geq 0}\{T^n y\}=\big\{T^n y\in\mathcal{X}\colon \text{for every integer } n\geq 0\big\}.$$

A vector y in $\mathcal{X}$ is a *cyclic vector* for T if the span of its orbit is dense in $\mathcal{X}$:

$$\big(\operatorname{span}\mathcal{O}_T(y)\big)^-=\mathcal{X}.$$

An operator T is a *cyclic operator* if it has a cyclic vector. Since $\operatorname{span}\mathcal{O}_T(y)$ is convex, $\big(\operatorname{span}\mathcal{O}_T(y)\big)^-=\big(\operatorname{span}\mathcal{O}_T(y)\big)^{-w}$, and so the notion of cyclicity is the same in the norm and in the weak topologies (i.e., *cyclicity coincides with weak cyclicity*).

Let $[x]=\operatorname{span}\{x\}$ denote the one-dimensional subspace of $\mathcal{X}$ spanned by a singleton $\{x\}$ at a vector $x\in\mathcal{X}$. The projective orbit of a nonzero vector $y\in\mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the orbit $\mathcal{O}_T([y]) = \bigcup_{n\geq 0} T^n([y])$ of the span of $\{y\}$:

$$\mathcal{O}_T([y]) = \big\{\alpha T^n y \in \mathcal{X}\colon \text{ for every } \alpha \in \mathbb{F} \text{ and every integer } n \geq 0\big\}.$$

A vector y in $\mathcal{X}$ is a *supercyclic vector* for T if its projective orbit $\mathcal{O}_T([y])$ is dense in $\mathcal{X}$; that is, if the orbit of the span of $\{y\}$ is dense in $\mathcal{X}$:

$$\mathcal{O}_T([y])^- = \mathcal{X}.$$

Since the norm topology is metrizable, a nonzero vector y in $\mathcal{X}$ is supercyclic for an operator T if and only if for every x in $\mathcal{X}$ there exists an $\mathbb{F}$-valued sequence $\{\alpha_k\}_{k\geq 0}$ (which depends on x for each y and consists of nonzero numbers if $x \neq 0$) such that for some sequence $\{T^{n_k}\}_{k\geq 0}$ with entries from the sequence $\{T^n\}_{n\geq 0}$, the $\mathcal{X}$-valued sequence $\{\alpha_k T^{n_k} y\}_{k\geq 0}$ converges to x (in the norm topology): for every $x \in \mathcal{X}$,

$$\alpha_k T^{n_k} y \to x \qquad (\text{i.e.,} \quad \|\alpha_k T^{n_k} y - x\| \to 0).$$

An operator T is a *supercyclic operator* if it has a supercyclic vector.

The weak version of the above convergence criterion leads to the notion of weak l-sequential supercyclicity. A nonzero y in $\mathcal{X}$ is a *weakly l-sequentially supercyclic vector* for T if for every x in $\mathcal{X}$ there is an $\mathbb{F}$-valued sequence $\{\alpha_k\}_{k\geq 0}$ such that for some sequence $\{T^{n_k}\}_{k\geq 0}$ with entries from $\{T^n\}_{n\geq 0}$, the $\mathcal{X}$-valued sequence $\{\alpha_k T^{n_k} y\}_{k\geq 0}$ converges weakly to x. In other words, if for every $x \in \mathcal{X}$,

$$\alpha_k T^{n_k} y \xrightarrow{\;w\;} x \qquad (\text{i.e.,} \quad f(\alpha_k T^{n_k} y - x) \to 0 \quad \text{for every} \quad f \in \mathcal{X}^*).$$

The $\mathbb{F}$-valued sequence $\{\alpha_k\}_{k\geq 0}$ depends on x for each weakly l-sequentially supercyclic vector y, and each $\alpha_k = \alpha_k(x)$ is nonzero whenever $x \neq 0$. The above displayed convergence means that there are $\{\alpha_k\}_{k\geq 0} = \{\alpha_k(x)\}_{k\geq 0}$ and $\{T^{n_k}\}_{k\geq 0} = \{T^{n_k(x)}\}_{k\geq 0}$ (depending on x for each y but not on f) such that, for each $x \in \mathcal{X}$,

$$\alpha_k(x) f(T^{n_k(x)} y) \to f(x) \quad \text{for every} \quad f \in \mathcal{X}^*.$$

Equivalently, the projective orbit $\mathcal{O}_T([y])$ of y under T is weakly l-sequentially dense in $\mathcal{X}$ in the sense that the weak limit set of $\mathcal{O}_T([y])$ coincides with $\mathcal{X}$:

$$\mathcal{O}_T([y])^{-wl} = \mathcal{X}.$$

An operator T is a *weakly l-sequentially supercyclic operator* if it has a weakly l-sequentially supercyclic vector.

The above notions are related as follows.

$$\text{SUPERCYCLIC} \Longrightarrow \text{WEAKLY L-SEQUENTIALLY SUPERCYCLIC} \Longrightarrow \text{CYCLIC}$$

(every supercyclic vector for T is a weakly l-sequentially supercyclic vector for T, which is a cyclic vector for T), and the converses fail. For a comparison between these and further notions of cyclicity (including hypercyclicity, weak hypercyclicity, weak sequential supercyclicity, and weak supercyclicity) see, for instance, [15, pp.38,39], [3, pp.259,260], [5, pp.159,232], [10, pp.50,51,54], [11, pp.372,373,374]. Weak l-sequential supercyclicity is the central theme in this paper. It has been considered in [4] (also in [2] implicitly), and discussed in [15], [10] and [12].

Remark 2.2. (a) Every nonzero vector is trivially supercyclic for every operator on a one-dimensional space, and all forms of cyclicity imply separability for any normed space $\mathcal{X}$. So we assume here that

all normed spaces are separable and have dimension greater than one.

(b) Take $0 \neq x \in \mathcal{X}$. If $x \in \mathcal{O}_T([y])$, then $\{T^{n_k}\}_{k\geq 0}$ can be viewed as a constant (infinite) sequence or, equivalently, as a one-entry (thus finite) subsequence of $\{T^n\}_{n\geq 0}$ (i.e., $x = \alpha_0 T^{n_0} y$ for some $\alpha_0 \neq 0$ and some $n_0 \geq 0$). In this case, both forms of supercyclicity (i.e., $\alpha_k T^{n_k} y \to x$ and $\alpha_k T^{n_k} y \xrightarrow{w} x$) coincide, where convergence of $\{T^{n_k} y\}_{k\geq 0}$ means eventually reached. Thus, in general, we use the expression

$$\{T^{n_k}\}_{k\geq 0} \textit{ is sequence with entries from } \{T^n\}_{n\geq 0}.$$

However, if $x \notin \mathcal{O}_T([y])$, this means $\{T^{n_k}\}_{k\geq 0}$ *is a subsequence of* $\{T^n\}_{n\geq 0}$.

(c) Let T be a weakly l-sequentially supercyclic operator, and let y be any weakly l-sequentially supercyclic vector for T. Then for every x there exists $\{\alpha_k(x)\}_{k\geq 0}$ for which $f(x) = \lim_k \alpha_k(x) f(T^{n_k(x)} y)$ for every f, for some sequence $\{T^{n_k(x)}\}_{k\geq 0}$ with entries from $\{T^n\}_{n\geq 0}$. Note that

if T is power bounded, then $\{\alpha_k(x)\}_{k\geq 0}$ cannot be uniformly bounded.

That is, if $\sup_n \|T^n\| < \infty$ (equivalently, if $\sup_n \|T^n x\| < \infty$ for every x in a Banach space by the Banach–Steinhaus Theorem), then it is not true that $\{\alpha_k(x)\}_{k\geq 0}$ is such that $\sup_x \sup_k |\alpha_k(x)| < \infty$. (Indeed, if T is power bounded and if $\{\alpha_k(x)\}_{k\geq 0}$ is bounded for each x, then set $\beta = \sup_n \|T^n\|$ and $\gamma(x) = \sup_k |\alpha_k(x)|$ for each x, so that $|f(x)| \leq \beta\, \gamma(x) \|f\|\, \|y\|$ for every f, and this leads to a contradiction if $\sup_x \gamma(x) < \infty$ whenever $|f(x)|$ is large enough, which can always be attained for an arbitrary f since linear functionals are surjective).

3. Supercyclicity and Strong Stability

The following result was proved in [1, Theorem 2.1].

Proposition 3.1. [1] *A power bounded supercyclic operator on a normed space is not of class $C_{1\cdot}$.*

This means that *if $T \in \mathcal{B}[\mathcal{X}]$ is such that* $\sup_n \|T^n\| < \infty$*, and if there exists a vector $y \in \mathcal{X}$ such that $\alpha_k(x) T^{n_k} y \to x$ for every $x \in \mathcal{X}$, for some $\mathbb{F}$-valued sequence $\{\alpha_k(x)\}_{k\geq 0}$* (i.e., some sequence of numbers that are nonzero whenever x is nonzero) *and some sequence $\{T^{n_k(x)}\}_{k\geq 0}$ with entries from $\{T^n\}_{n\geq 0}$* (both depending on x for each y), *then there exists a nonzero $z \in \mathcal{X}$ for which $T^n z \to 0$.*

Using the above proposition, more was proved in [1, Theorem 2.2], namely, *if $T \in \mathcal{B}[\mathcal{X}]$ is such that* $\sup_n \|T^n\| < \infty$*, and if there exists a vector $y \in \mathcal{X}$ such that $\alpha_k(x) T^{n_k} y \to x$ for every $x \in \mathcal{X}$, for some sequence of numbers $\{\alpha_k(x)\}_{k\geq 0}$ and some sequence $\{T^{n_k(x)}\}_{k\geq 0}$ with entries from $\{T^n\}_{n\geq 0}$, then $T^n z \to 0$ for every $z \in \mathcal{X}$.* In other words, the result in [1, Theorem 2.2] may be read as follows.

Proposition 3.2. [1] *A power bounded supercyclic operator on a normed space is strongly stable.*

Since supercyclicity implies weak l-sequential supercyclicity, and strong stability implies weak stability, Proposition 3.2 prompts the question: *does weak l-sequential supercyclicity imply weak stability for a power bounded operator*? The relationships around this question are summarized in the following diagram.

$$\begin{array}{ccc} \mathcal{O}_T([y])^- = \mathcal{X} & \Longrightarrow & T^n \xrightarrow{s} O \\ \Downarrow & & \Downarrow \\ \mathcal{O}_T([y])^{-wl} = \mathcal{X} & \overset{?}{\Longrightarrow} & T^n \xrightarrow{w} O\,. \end{array}$$

In other words,

$$\begin{array}{ccc} T \text{ is supercyclic} & \Longrightarrow & T \text{ is strongly stable} \\ \Downarrow & & \Downarrow \\ T \text{ is weakly l-sequentially supercyclic} & \overset{?}{\Longrightarrow} & T \text{ is weakly stable}\ . \end{array}$$

The above displayed question (originally posed in [10]) remains open in general. There are, however, affirmative answers for particular classes of operators. For instance, if an operator is compact, then weak l-sequential supercyclicity coincides with supercyclicity [11, Theorem 4.1], which implies strong stability (Proposition 3.2), which in turn implies weak stability — actually, in this case we get uniform stability [11, Theorem 4.2]. We prove in Corollary 4.3 below the following particular version of the above question for arbitrary power bounded operators.

$$T \text{ is weakly l-sequentially supercyclic} \quad \Longrightarrow \quad T \text{ is weakly quasistable.}$$

4. Main Results

From now on let $Y_T \subseteq \mathcal{X}$ denote the set of all weakly l-sequentially supercyclic vectors y for an operator T on a normed space $\mathcal{X}$,

$$Y_T = \big\{y \in \mathcal{X}\colon\, \mathcal{O}_T([y])^{-wl} = \mathcal{X}\big\} = \big\{y \in \mathcal{X}\colon\, \forall\, x,\ \exists\,\{\alpha_k\}\ \ni\ \alpha_k T^{n_k} y \xrightarrow{\ w\ } x\big\},$$

so that $0 \notin Y_T$, and T is weakly l-sequentially supercyclic if and only if $Y_T \neq \varnothing$. Observe that $Y_T \cup \{0\}$ is a T-invariant cone. (Indeed, if $y \in Y_T$, then $Ty \neq 0$ and every nonzero vector in $\mathcal{O}_T([y])$ lies in Y_T [9, Lemma 5.1]; in particular, Ty lies in Y_T, and $\alpha y \in Y_T$ for every nonzero $\alpha \in \mathbb{F}$ trivially.)

Theorem 4.1. *Suppose an operator T on a normed space $\mathcal{X}$ is weakly l-sequentially supercyclic. Take an arbitrary $y \in Y_T$.*

(a) *If there exists an $f_0 \in \mathcal{X}^*$ such that $\liminf_n |f_0(T^n y)| > 0$, then for every $x \in \mathcal{X}$ there exists a* bounded *sequence of scalars $\{\alpha_j(x)\}_{j\geq 0}$ such that $\alpha_j(x) f(T^{n_j(x)} y) \to f(x)$ for every $f \in \mathcal{X}^*$.*

(b) *If for some $x \in \mathcal{X}$ there exists a bounded scalar sequence $\{\alpha_j(x)\}_{j\geq 0}$ such that $\alpha_j(x) f(T^{n_j(x)} y) \to f(x)$ for every $f \in \mathcal{X}^*$, then there is a* convergent *scalar sequence $\{\alpha_k(x)\}_{k\geq 0}$ such that $\alpha_k(x) f(T^{n_k(x)} y) \to f(x)$ for every $f \in \mathcal{X}^*$.*

(c) *Suppose the operator T is power bounded. If for some nonzero $x \in \mathcal{X}$ there exists a convergent sequence of scalars $\{\alpha_k(x)\}_{k\geq 0}$, say $\lim_k \alpha_k(x) = \alpha(x) \in \mathbb{F}$, for which $\lim_k \alpha_k(x) f(T^{n_k(x)} y) = f(x)$ for every $f \in \mathcal{X}^*$, then $\alpha(x) \neq 0$ and $\alpha(x) \lim_k f(T^{n_k(x)} y) = f(x)$ for every $f \in \mathcal{X}^*$.*

(d) *Suppose there is a functional $\alpha\colon \mathcal{X} \to \mathbb{F}$ such that $\alpha(x) \neq 0$ whenever $x \neq 0$ and, for each $x \in \mathcal{X}$, $\alpha(x) \lim_k f(T^{n_k(x)} y) = f(x)$ for every $f \in \mathcal{X}^*$. Then $\liminf_n |f(T^n y)| = 0$ for every $f \in \mathcal{X}^*$.*

Proof. Let T be a weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$, and fix an arbritrary vector $y \in Y_T \neq \varnothing$.

(a) Take an arbitrary x in $\mathcal{X}$. By weak l-sequential supercyclicity there is a sequence $\{\alpha_j(x)\}_{j\geq 0}$ such that $\alpha_j(x) f(T^{n_j(x)} y) \to f(x)$, and so $|\alpha_j(x)|\,|f(T^{n_j(x)} y)| \to |f(x)|$, for every $f \in \mathcal{X}^*$. Suppose there is an $f_0 \in \mathcal{X}^*$ for which $\liminf_n |f_0(T^n y)| > 0$. Then for every ε in $(0, \liminf_n |f_0(T^n y)|)$ there is an integer $n_\varepsilon > 0$ such that $n \geq n_\varepsilon$ implies $\varepsilon < |f_0(T^n y)|$. Thus, since $\lim_j |\alpha_j(x)|\,|f_0(T^{n_j(x)} y)| = |f_0(x)|$ is a real number, $\limsup_j |\alpha_j(x)| < \infty$. (Indeed, $\varepsilon \limsup_j |\alpha_j(x)| < \limsup_j |\alpha_j(x)|\,|f_0(T^{n_j(x)} y)| = |f_0(x)|$ for $n_j(x)$ large enough — recall: $\{\alpha_j(x)\}_{j\geq 0}$ depends on x but not on f). So

$$\{\alpha_j(x)\}_{j\geq 0} \text{ is bounded.}$$

(b) If $\{\alpha_j(x)\}_{j\geq 0}$ is bounded, then (Bolzano–Weierstrass) there exists a subsequence $\{\alpha_k(x)\}_{k\geq 0} = \{\alpha_{j_k}(x)\}_{k\geq 0}$ of the sequence of scalars $\{\alpha_j(x)\}_{j\geq 0}$ such that

$$\{\alpha_k(x)\}_{k\geq 0} \text{ converges.}$$

Moreover, $\alpha_k(x) f(T^{n_k(x)} y) \to f(x)$. Indeed, since $\alpha_j(x) f(T^{n_j(x)} y) \to f(x)$ for every f in $\mathcal{X}^*$, the convergence holds for every subsequence of $\{\alpha_j(x) f(T^{n_j(x)} y)\}$; in particular, it holds for $\{\alpha_k(x) f(T^{n_k(x)} y)\}$, where $\{T^{n_k(x)}\}_{k\geq 0} = \{T^{n_{j_k}(x)}\}_{k\geq 0}$ is the associated subsequence of $\{T^{n_j}(x)\}_{j\geq 0}$.

(c) Take an arbitrary $x \in \mathcal{X}$. Let $\alpha(x) \in \mathbb{F}$ be the limit of a convergent sequence $\{\alpha_k(x)\}_{k\geq 0}$ of scalars for which $\alpha_k(x) f(T^{n_k(x)} y) \to f(x)$ for every $f \in \mathcal{X}^*$ (where $\{\alpha_k(x)\}_{k\geq 0}$ and $n_k(x)$ do not depend on f). From the Hahn–Banach Theorem, if $x \neq 0$, then there exists $f_1 \in \mathcal{X}^*$ such that $f_1(x) \neq 0$ (as $\|x\| = \sup_{\|f\|=1} |f(x)|$). Since T is power bounded, set $\beta = \sup_n \|T^n\|$. Thus if $x \neq 0$, then

$$\begin{aligned} 0 < |f_1(x)| &= \lim_k |\alpha_k(x)|\,|f_1(T^{n_k(x)} y)| \\ &\leq |\alpha(x)| \limsup_k |f_1(T^{n_k(x)} y)| \leq \beta\,|\alpha(x)|\,\|f_1\|\,\|y\| < \infty, \end{aligned}$$

and hence $\alpha(x) \neq 0$. Summing up:

$$x \neq 0 \quad \text{implies} \quad \alpha(x) \neq 0.$$

By definition, for $x \neq 0$ the sequence $\{\alpha_k(x)\}_{k\geq 0}$ is such that $\alpha_k(x) \neq 0$ for every $k \geq 0$. Thus if $x \neq 0$, then $0 \neq \alpha_k(x) \to \alpha(x) \neq 0$, and so $\alpha_k(x)^{-1} \to \alpha(x)^{-1}$. Since $\alpha_k(x) f(T^{n_k(x)} y) \to f(x)$, we get $f(T^{n_k(x)} y) = \frac{\alpha_k(x) f(T^{n_k(x)} y)}{\alpha_k(x)} \to \frac{f(x)}{\alpha(x)}$, and so

$$\alpha(x) \lim_k f(T^{n_k(x)} y) = f(x) \quad \text{for every} \quad f \in \mathcal{X}^*.$$

(d) We know from linear algebra that there is no injective linear functional on a linear space of dimension greater than one (recall: we are assuming $\dim \mathcal{X} > 1$). Take an arbitrary $0 \neq f \in \mathcal{X}^*$. Now take a nonzero $x_0 \in \mathcal{X}$ in the kernel $\mathcal{N}(f) \neq \{0\}$ of f.

(d_1) If $x_0 \in \mathcal{O}_T([y])$, then $x_0 = \alpha_0 T^{n_0} y$ for some scalar $\alpha_0 \neq 0$ and some nonnegative integer n_0. So $\alpha_0 f(T^{n_0} y) = f(x_0) = 0$. Then $f(T^{n_0} y) = 0$. But this cannot hold for every $f \in \mathcal{X}^*$ (since there is no $0 \neq z \in \mathcal{X}$ for which $f(z) = 0$ for every $f \in \mathcal{X}^*$.) Thus for each nonzero $f \in \mathcal{X}^*$ there exists a nonzero $x_0 \in \mathcal{N}(f)\backslash\mathcal{O}([y])$.

(d_2) Then suppose $x_0 \notin \mathcal{O}([y])$. Therefore $\alpha(x_0) \lim_k f(T^{n_k(x_0)} y) = f(x_0)$ for some subsequence $\{T^{n_k(x_0)}\}_{k\geq 0}$ of $\{T^n\}_{n\geq 0}$. Since $\alpha(x_0) \neq 0$, and since

$$\alpha(x_0) \lim_k f(T^{n_k(x_0)} y) = f(x_0) = 0,$$

we get $\lim_k f(T^{n_k(x_0)} y) = 0$. Then for an arbitrary $0 \neq f \in \mathcal{X}^*$ there is a subsequence of $\{f(T^n y)\}$ converging to zero. So $\liminf_n |f(T^n y)| = 0$ for every $f \in \mathcal{X}^*$. □

Before harvesting the corollaries we need the following lemma.

Lemma 4.2. *Let T be a power bounded weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$.*

(a) *T is weakly stable if and only if* $T^n y \xrightarrow{w} 0$ *for every* $y \in Y_T$
(*i.e., if and only if* $\lim_n |f(T^n y)| = 0$ *for every* $f \in \mathcal{X}^*$, for every $y \in Y_T$).

(b) *T is weakly quasistable if and only if* $\liminf_n |f(T^n y)| = 0$ *for every* $f \in \mathcal{X}^*$, *for every* $y \in Y_T$.

Proof. That the set of weakly supercyclic vectors is dense in the norm topology was shown in [14, Proposition 2.1]. This has been extended to the set of weakly l-sequentially supercyclic vectors in [9, Theorem 5.1], so that

$$Y_T \neq \varnothing \quad \Longrightarrow \quad Y_T^- = \mathcal{X} \quad \text{(and so } Y_T^{-wl} = \mathcal{X}\text{)}.$$

Take an arbitrary x in $\mathcal{X}$. Since $Y_T^- = \mathcal{X}$, there exists a Y_T-valued sequence $\{y_k\}$ such that $\|y_k - x\| \to 0$. If $\lim_n |f(T^n y)| = 0$ (or $\liminf_n |f(T^n y)| = 0$) for every f in $\mathcal{X}^*$ and for every y in Y_T, then $\lim_n |f(T^n y_k)| = 0$ (or $\liminf_n |f(T^n y_k)| = 0$) for every f in $\mathcal{X}^*$ and for each integer k. Thus, since T is power bounded, and since

$$|f(T^n x)| \leq |f(T^n(y_k - x))| + |f(T^n y_k)| \leq \|f\| \sup_n \|T^n\| \, \|y_k - x\| + |f(T^n y_k)|$$

for every $f \in \mathcal{X}^*$ and $x \in \mathcal{X}$, we get $\lim_n |f(T^n x)| \to 0$ (or $\liminf_n |f(T^n x)| = 0$) for every $f \in \mathcal{X}^*$ and every $x \in \mathcal{X}$. Hence $\lim_n |f(T^n y)| = 0$ (or $\liminf_n |f(T^n y)| = 0$) for every $f \in \mathcal{X}^*$ and every $y \in Y_T$ implies T is weakly stable (or weakly quasistable), and the converse (in both cases) is trivial. □

The result below consists of an improvement over [10, Theorem 6.2], ensuring that weak quasistability always happens for weakly l-sequentially supercyclic power bounded operators acting on arbitrary normed spaces.

Corollary 4.3. *Every weakly l-sequentially supercyclic power bounded operator on a normed space is weakly quasistable.*

Proof. Let T be a power bounded weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$. Take an arbitrary weakly l-sequentially supercyclic vector y in Y_T. Theorem 4.1 ensures that if there exists $f_0 \in \mathcal{X}^*$ such that $\liminf_n |f_0(T^n y)| > 0$, then $\liminf_n |f(T^n y)| = 0$ for every $f \in \mathcal{X}^*$, which is a contradiction. So if a power bounded operator T is weakly l-sequentially supercyclic, then $\liminf_n |f(T^n y)| = 0$ for every $f \in \mathcal{X}^*$, for every $y \in Y_T$. Thus T is weakly quasistable by Lemma 4.2(b). □

Corollary 4.4. *If T is a power bounded operator on a normed space, then*

$$\mathcal{O}_T([y])^{-wl} = \mathcal{X} \ \textit{for some} \ y \in \mathcal{X} \quad \Longrightarrow \quad 0 \in \mathcal{O}_T(x)^{-wl} \ \textit{for every} \ x \in \mathcal{X}.$$

Proof. Suppose $Y_T \neq \varnothing$ and take an arbitrary $y \in Y_T$. Now take an arbitrary $x \in \mathcal{X}$. If T is a power bounded operator on $\mathcal{X}$, then Corollary 4.3 says that $\mathcal{O}_T([y])^{-wl} = \mathcal{X}$ implies $\liminf_n |f(T^n x)| = 0$ for every $f \in \mathcal{X}^*$. This means that there is an $\mathcal{X}$-valued subsequence $\{T^{n_j} x\}_{j \geq 0}$ of $\{T^n x\}_{n \geq 0}$ (which actually is an $\mathcal{O}_T(x)$-valued sequence that does not depend on f) such that $\lim_j |f(T^{n_j} x)| = 0$ for every $f \in \mathcal{X}^*$. That is, $T^{n_j} x \xrightarrow{w} 0$, so that zero is a weak limit of the weakly convergent $\mathcal{O}_T(x)$-valued sequence $\{T^{n_j} x\}_{j \geq 0}$, which in turn means that $0 \in \mathcal{O}_T(x)^{-wl}$. □

5. Applications

A Radon–Riesz space is a normed space $\mathcal{X}$ for which the following property holds: an $\mathcal{X}$-valued sequence $\{x_n\}_{n \geq 0}$ converges strongly (i.e., converges in the norm topology) if and only if it converges weakly and the sequence of norms $\{\|x_n\|\}_{n \geq 0}$ converges to the norm of the limit (see, e.g., [13, Definition 2.5.26]). In oder words, a normed space $\mathcal{X}$ is a Radon–Riesz space if, for an arbitrary $\mathcal{X}$-valued sequence $\{x_n\}$,

$$x_n \to x \qquad \Longleftrightarrow \qquad \big\{x_k \xrightarrow{w} x \;\text{ and }\; \|x_n\| \to \|x\|\big\}.$$

Hilbert spaces are Radon–Riesz spaces (cf. [6, Problem 20, p.13]).

Lemma 5.1. *Let $\mathcal{X}$ be a Radon–Riesz space. If an $\mathcal{X}$-valued sequence $\{x_n\}_{n\ge0}$ converges weakly to $x \in \mathcal{X}$ but not strongly (i.e., $x_n \xrightarrow{w} x$ and $x_n \not\to x$), then*

$$\|x\| \le \liminf_n \|x_n\| < \limsup_n \|x_n\|.$$

Proof. If $\mathcal{X}$ is a normed space, then $x_n \xrightarrow{w} x$ implies $\|x\| \le \liminf_n \|x_n\|$ (see, e.g., [7, Proposition 46.1] — compare with [6, Problem 21, p.14] — norm is weakly lower semicontinuous). Thus if $x_n \not\to x$ and $\mathcal{X}$ is a Radon–Riesz space, then $\|x_n\| \not\to \|x\|$ by the above displayed equivalence, and this implies that $\{\|x_n\|\}_{n\ge0}$ does not converge, and hence $\liminf_n \|x_n\| < \limsup_n \|x_n\|$. This concludes the proof. □

As Corollary 4.3 supplied an improvement over [10, Theorem 6.2], the result below supplies an improvement over [10, Theorem 5.1], in the sense that it offers a sharper condition for distinguishing weak l-sequential supercyclicity from (strong) supercyclicity. Suppose T is a power bounded weakly l-sequentially supercyclic operator on a Radon–Riesz space $\mathcal{X}$ which is not supercyclic. Take an arbitrary y in Y_T so that, for each $x \in \mathcal{X}$, there is a sequence of scalars $\{\alpha_k(x)\}_{k\ge0}$ for which $\alpha_k(x)T^{n_k(x)}y \xrightarrow{w} x$ and $\alpha_k(x)T^{n_k(x)}y \not\to x$. Theorem 4.1(b,c,d) and Lemma 4.2(b) show that, if $\{\alpha_k(x)\}_{k\ge0}$ is bounded for each $x \in \mathcal{X}$, then T is weakly quasistable, and by Theorem 4.1(a) we saw in Corollary 4.3 that weak quasistability for T always holds. *Does boundedness (thus convergence) of $\{\alpha_k(x)\}_{k\ge0}$ always holds as well?*

Corollary 5.2. *Let T be a power bounded operator on a Radon–Riesz space. Suppose T is weakly l-sequentially supercyclic but not supercyclic. Take any vector y in Y_T so that for every x in $\mathcal{X}\backslash\mathcal{O}_T([y])$ there exists a sequence of scalars $\{\alpha_k(x)\}_{k\ge0}$ for which $\alpha_k(x)\,T^{n_k(x)}y \xrightarrow{w} x$ for some subsequence $\{T^{n_k(x)}\}_{k\ge0}$ of $\{T^n\}_{n\ge0}$. If $\{\alpha_k(x)\}_{k\ge0}$ is bounded for each $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$, then for every nonzero $f \in \mathcal{X}^*$*

$$\limsup_j |f(T^{n_j}y)| < \|f\| \limsup_j \|T^{n_j}y\| \tag{$*$}$$

for some subsequence $\{T^{n_j}\}_{j\ge0}$ of $\{T^n\}_{n\ge0}$.

Proof. Let T be an operator on a normed space $\mathcal{X}$. If it is weakly l-sequentially supercyclic but not supercyclic, then take any weakly l-sequentially supercyclic vector y in Y_T, and recall that it is not a supercyclic vector (there is no supercyclic vector). Take an arbitrary x in $\mathcal{X}\backslash\mathcal{O}_T([y])$. Thus there is a sequence of scalars $\{\alpha_j(x)\}_{j\ge0}$ such that $\alpha_j(x)\,T^{n_j(x)}y \xrightarrow{w} x$ for some subsequence $\{T^{n_j(x)}\}_{j\ge0}$ of $\{T^n\}_{n\ge0}$. Since $\{\alpha_j(x)\}_{j\ge0}$ is bounded for every $x \in \mathcal{X}$ and T is power bounded, Theorem 4.1(b,c) says that, in this case, there is a nonzero constant scalar sequence, say $\alpha_k(x) = \alpha(x)$ for every $k \ge 0$, for which $\alpha(x)\,T^{n_k(x)}y \xrightarrow{w} x$. Hence, for each $x \in \mathcal{X}\backslash\mathcal{O}_T([y])$,

$$|f(x)| = |\alpha(x)| \lim_k |f(T^{n_k(x)}y)| \quad \text{for every} \quad f \in \mathcal{X}^*. \tag{1}$$

As y is not supercyclic, there is an $x_0 \ne 0$ in $\mathcal{X}\backslash\mathcal{O}_T([y])$ such that $\alpha_i\,T^{n_i}y \not\to x_0$ for every sequence of scalars $\{\alpha_i\}_{i\ge0}$ and every subsequence $\{T^{n_i}\}_{i\ge0}$ of $\{T^n\}_{n\ge0}$. So

$$\alpha(x_0)\,T^{n_k(x_0)}y \xrightarrow{w} x_0 \quad \text{and} \quad \alpha(x_0)\,T^{n_k(x_0)}y \not\to x_0.$$

As $\mathcal{X}$ is a Radon–Riesz space, it follows by Lemma 5.1 that

$$\|x_0\| \le \liminf_k \|\alpha(x_0)\,T^{n_k(x_0)}y\| < \limsup_k \|\alpha(x_0)\,T^{n_k(x_0)}y\|. \tag{2}$$

Now suppose $(*)$ fails. That is, suppose there exists a nonzero $f_1 \in \mathcal{X}^*$ such that

$$\limsup_i |f_1(T^{n_i} y)| = \|f_1\| \limsup_i \|T^{n_i} y\| \tag{3}$$

for every subsequence $\{T^{n_i}\}_{i\geq 0}$ of $\{T^n\}_{n\geq 0}$. Then, by (1), (2), and (3),

$$\begin{aligned} |\alpha(x_0)| \limsup_k |f_1(T^{n_k(x_0)} y)| &= |\alpha(x_0)| \lim_k |f_1(T^{n_k(x_0)} y)| \\ &= |f_1(x_0))| \leq \|f_1\|\,\|x_0\| \\ &< \|f_1\| \limsup_k \|\alpha(x_0) T^{n_k(x_0)} y\| \\ &= |\alpha(x_0)| \limsup_k |f_1(T^{n_k(x_0)} y|, \end{aligned}$$

which is a contradiction. Thus $(*)$ holds. □

To proceed we need the following properties of weak l-sequential supercyclicity.

Lemma 5.3. *A weakly l-sequentially supercyclic operator has dense range.*

Proof. Take an operator $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$. If a vector y in $\mathcal{X}$ is a weakly l-sequentially supercyclic vector for T, then any vector in the projective orbit $\mathcal{O}_T(\operatorname{span}\{y\})$ is again weakly l-sequentially supercyclic [9, Lemma 5.1], and so is the vector Ty. Thus the projective orbit of Ty is included in the range of T:

$$\mathcal{O}_T(\operatorname{span}\{Ty\}) = \big\{\alpha T^n y \in \mathcal{X} \colon \alpha \in \mathbb{F},\, n \geq 1\big\} \subseteq \mathcal{R}(T) \subseteq \mathcal{X}.$$

As Ty is a weakly l-sequentially supercyclic vector for T, then $\mathcal{O}_T(\operatorname{span}\{Ty\})$ is weakly l-sequentially dense in $\mathcal{X}$, and so is $\mathcal{R}(T)$ (i.e., $\mathcal{R}(T)^{-wl} = \mathcal{X}$). Since $\mathcal{R}(T)$ is a linear manifold of $\mathcal{X}$, it is convex, and therefore it is dense (in the norm topology) in $\mathcal{X}$ (i.e., $\mathcal{R}(T)^- = \mathcal{X}$) according to Remark 2.1. □

Lemma 5.4. *A weakly l-sequentially supercyclic isometry on a Banach space is surjective. So a weakly l-sequentially supercyclic isometry on a Hilbert space is unitary.*

Proof. Isometries on Banach spaces have closed range (because they are bounded and bounded below). Thus apply Lemma 5.3 and get a surjective isometry. Surjective isometries on Hilbert spaces are precisely the unitary operators. □

It is known that there are weakly l-sequentially supercyclic unitary operators [2, Example 3.6], [2, pp.10,12], [15, Proposition 1.1, Theorem 1.2]). A unitary operator (acting on a Hilbert space) is singular-continuous if its scalar spectral measure is singular-continuous with respect to normalized Lebesgue measure on the σ-algebra of Borel subsets of the unit circle.

Corollary 5.5. *If an isometry on a Hilbert space is weakly l-sequentially supercyclic, then it is a weakly quasistable singular-continuous unitary operator.*

Proof. Take a weakly l-sequentially supercyclic isometry on a Hilbert space. Isometries are power bounded, so the conclusion of weak quasistability follows from Corollary 4.3. Lemma 5.4 says that the isometry must be unitary. But every weakly l-sequentially supercyclic unitary operator is singular-continuous [8, Theorem 4.2]. □

It is also known that there are singular-continuous unitary operators that are weakly stable, and singular-continuous unitary operators that are not weakly stable [8, Propositions 3.2 and 3.3]. We close the paper with a question.

Is there a singular-continuous unitary operator that is not weakly quasistable?

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