IMA Journal of Mathematical Control and Information **12** (1995) 325–329

ON MEAN-SQUARE STABLE BILINEAR SYSTEMS*

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Abstract. It has been shown in a previous paper [6] that an infinite-dimensional stochastic discrete bilinear system is mean-square stable if and only if the spectral radii of two Hilbert-space operators transformations are both less than one. The present paper investigates conditions to be imposed on the model operators in order to ensure that such spectral radii coincide. Several examples are presented and the main result establishes the spectral radius indentity for models with compact operators.

Key words. Discrete bilinear systems, stability, operator theory.

AMS (MOS) Subject Classification. 47N70, 93C25, 93C55.

September 1994

* Research partially supported by CNPq, FAPESP and CAPES.

1. INTRODUCTION

Throughout this paper H will denote a separable nontrivial complex Hilbert space, and $B[X]$ will stand for the Banach algebra of all bounded linear transformations of a Banach space X into itself. Both the norm in X and the induced uniform norm in $B[X]$ will be denoted by $\|\cdot\|$, and $r(\cdot)$ will stand for the spectral radius in $B[X]$. An upper star * will mean adjoint in $B[H]$ as usual. Let $\{A_k \in B[H];\ k \geq 0\}$ be a bounded sequence of operators, and let $\{\gamma_k \in \mathbb{C};\ k \geq 0\}$ be a nonnegative sequence in ℓ_1. For each $n \geq 0$ let $\mathcal{F}_n \in B[B[H]]$ be defined as $\mathcal{F}_n(Q) = \Sigma_{k=0}^{n}\gamma_k A_k Q A_k^*$ for all $Q \in B[H]$. Note that $\{\mathcal{F}_n \in B[B[H]];\ n \geq 0\}$ converges uniformly (indeed $\sup_{\nu\geq 0}\|\mathcal{F}_{n+\nu} - \mathcal{F}_n\| = \sup_{\nu\geq 0}\sup_{\|Q\|=1}\|\mathcal{F}_{n+\nu}(Q) - \mathcal{F}_n(Q)\| \leq \sup_{k\geq 0}\|A_k\|^2\Sigma_{k=n+1}^{\infty}\gamma_k \to 0$ as $n \to \infty$). Let $\mathcal{F} \in B[B[H]]$ be the limit of $\{\mathcal{F}_n;\ n \geq 0\}$ so that

$$\mathcal{F}(Q) = \sum_{k=0}^{\infty}\gamma_k A_k Q A_k^*$$

for all $Q \in B[H]$. Similarly define $\mathcal{F}^{\#} \in B[B[H]]$ as

$$\mathcal{F}^{\#}(Q) = \sum_{k=0}^{\infty}\gamma_k A_k^* Q A_k$$

for all $Q \in B[H]$, where the above convergences are in the uniform topology of $B[B[H]]$.

Such a pair $(\mathcal{F}, \mathcal{F}^{\#})$ of bounded linear transformations from the Banach space $B[H]$ into itself appears in the stability analysis for infinite-dimensional stochastic discrete bilinear systems (see e.g. [1], [5] and [6]). For instance, consider a discrete bilinear model whose H-valued state sequence $\{x_i;\ i \geq 0\}$ evolves as follows.

$$x_{i+1} = \left[A_0 + \sum_{k=0}^{\infty}\langle w_i; e_k\rangle A_k\right] x_i \ + \ u_{i+1}\ , \qquad\qquad x_0 = u_0;$$

where $\{w_i;\ i \geq 0\}$ and $\{u_i;\ i \geq 0\}$ (the multiplicative and additive input sequences, respectively) are zero-mean independent H-valued second-order random sequences which are independent of each other. Suppose $\{w_i;\ i \geq 0\}$ is stationary in correlation so that it has a constant correlation operator, say $S \in B_1^+[H]$. Here $B_1[H]$ denotes the class of all nuclear operators (i.e. the trace class) from $B[H]$, and $B_1^+[H]$ stands for the class of all nonnegative nuclear operators. Moreover, let $\{e_k;\ k \geq 1\}$ be an orthonormal basis for H made up of eigenvectors of S. The existence of such an orthonormal basis is ensured by the Spectral Theorem for compact normal operators (see e.g. [7, p.12]). The above described discrete-time model is *mean-square stable* if the state correlation sequence converges in $B[H]$ to a correlation operator (i.e. to an operator in $B_1^+[H]$) whenever the addive input correlation sequence does so. (For continuous-time versions see e.g. [3] and [9]). By setting $\gamma_0 = 1$ and $\gamma_k = \langle Se_k; e_k\rangle$ for each $k \geq 1$ (i.e. the eigenvalues of S), it has been shown in [6] that $r(\mathcal{F}^{\#}) < 1$ is a necessary condition for mean-square stability which, together with its dual $r(\mathcal{F}) < 1$, is sufficient as well. Therefore, if $r(\mathcal{F}^{\#}) = r(\mathcal{F})$, then $r(\mathcal{F}) < 1$ is a necessary and sufficient condition for mean-square stability.

The purpose of the present paper is to investigate conditions to be imposed on the operators $\{A_k;\ k \geq 0\}$ that ensure the spectral radii identity $r(\mathcal{F}^{\#}) = r(\mathcal{F})$. Note that this trivially holds if $\{A_k;\ k \geq 0\}$ is a sequence of self-adjoint operators. We shall verify that it also holds whenever the operators $\{A_k;\ k \geq 0\}$ are normal and commute. On the other hand, a class of operators for which the inequality $r(\mathcal{F}^{\#}) < r(\mathcal{F})$ necessarily holds will be exhibited as well. Our main theorem establishes the identity $r(\mathcal{F}^{\#}) = r(\mathcal{F})$ for any sequence $\{A_k;\ k \geq 0\}$ of compact operators.

2. PRELIMINARIES

The results in the remaining paper are all based on the following property whose proof can be found in [6].

LEMMA. *For any bounded sequence $\{A_k;\ k \geq 0\}$ the identity $\|\mathcal{F}^i\| = \|\mathcal{F}^i(I)\|$ holds for every $i \geq 0$.*

Given a sequence of operators $\{A_k \in B[H];\ k \geq 0\}$ let $I\!\!K$ denote the set of all nonnegative integers such that $A_k \neq O$. If $A_k = A$ for every $k \in I\!\!K$, then $\mathcal{F}^i(Q) = (\Sigma_{k \in \boldsymbol{K}} \gamma_k)^i A^i Q A^{*i}$ for all $Q \in B[H]$ and every $i \geq 0$. Hence $\|\mathcal{F}^i(I)\| = (\Sigma_{k \in \boldsymbol{K}} \gamma_k)^i \|A^i\|^2$ for each $i \geq 0$. By using the above lemma, and according to the Beurling-Gelfand formula for the spectral radius (i.e. $r(T) = \lim_{i \to \infty} \|T^i\|^{1/i}$ for every $T \in B[X]$), it follows that $r(\mathcal{F}) = (\Sigma_{k \in \boldsymbol{K}} \gamma_k) r(A)^2$. Since $r(A) = r(A^*)$ we may conclude:

If the nonzero elements of $\{A_k;\ k \geq 0\}$ are constant, then $r(\mathcal{F}^{\#}) = r(\mathcal{F})$.

In particular, if the bilinear model described in Section 1 is reduced to a linear one (e.g. by setting $A_k = O$ for every $k \geq 1$), then $r(\mathcal{F}^{\#}) = r(\mathcal{F}) = r(A_0)^2$. This settles the mean-square stability problem for infinite-dimensional discrete linear systems.

If A_j and A_k are normal operators that commute, then each of them also commutes with the other's adjoint. (This is Fuglede's theorem – see e.g. [7, p.20].) Thus $A_j \mathcal{F}(I) = \mathcal{F}(I) A_j$ for every $j \geq 0$, so that $\mathcal{F}^i(I) = \mathcal{F}(I)^i$ for every $i \geq 0$ by induction on i, whenever $\{A_k;\ k \geq 0\}$ is a sequence of commuting normal operators. Applying the preceding lemma and the Beurling-Gelfand formula for the spectral radius again we get $r(\mathcal{F}) = r(\mathcal{F}(I))$. However, for any sequence of normal operators $\{A_k;\ k \geq 0\}$, we have $\mathcal{F}^{\#}(I) = \mathcal{F}(I)$. Conclusion:

If $\{A_k;\ k \geq 0\}$ is a sequence of commuting normal operators, then $r(\mathcal{F}^{\#}) = r(\mathcal{F})$.

The commuting condition can be relaxed if the nonzero normal operators have a constant absolute value. That is, if $A_k A_k^* = A_k^* A_k = R$ for every $k \in I\!\!K$. In such a case it is readily verified by induction that $\mathcal{F}^i(I) = \mathcal{F}^{\#i}(I) = (\Sigma_{k\in\boldsymbol{K}}\gamma_k)^i R^i$ for every $i \geq 0$. Hence, by using the same arguments (previous lemma and Beurling-Gelfand formula), we have:

If the nonzero elements of $\{A_k;\ k \geq 0\}$ are normal operators with a constant absolute value (particular case: unitary operators), then $r(\mathcal{F}^{\#}) = r(\mathcal{F})$.

Next we shall characterize a class of operators for which $r(\mathcal{F}^{\#}) < r(\mathcal{F})$. First note that: *if $\{A_k^* A_k;\ k \geq 0\}$ is a sequence of (orthogonal) projections that are orthogonal to each other, then $r(\mathcal{F}^{\#}) \leq (\Sigma_{k\in\boldsymbol{K}}\gamma_k^2)^{\frac{1}{2}}$.* (Indeed $\|\mathcal{F}^{\#}(I)x\|^2 = \Sigma_{k\in\boldsymbol{K}}\gamma_k^2\|A_k^* A_k x\|^2 \leq \Sigma_{k\in\boldsymbol{K}}\gamma_k^2\|x\|^2$ for every $x \in H$ so that $r(\mathcal{F}^{\#}) \leq \|\mathcal{F}^{\#}\| = \|\mathcal{F}^{\#}(I)\| \leq (\Sigma_{k\in\boldsymbol{K}}\gamma_k^2)^{\frac{1}{2}}$ by the previous lemma.) Moreover, *if the nonzero elements of $\{A_k;\ k \geq 0\}$ are coisometries (i.e. $A_k A_k^* = I$), then $r(\mathcal{F}) = \Sigma_{k\in\boldsymbol{K}}\gamma_k$.* (In fact $\mathcal{F}(I) = (\Sigma_{k\in\boldsymbol{K}}\gamma_k)I$ so that $\mathcal{F}^i(I) = (\Sigma_{k\in\boldsymbol{K}}\gamma_k)^i I$ for every $i \geq 0$ and hence the result follows – use the previous lemma and the Beurling-Gelfand formula.) Conclusion:

If the nonzero elements of $\{A_k;\ k \geq 0\}$ are coisometries such that $A_k^* A_k$ are (orthogonal) projections that are orthogonal to each other, then $r(\mathcal{F}^{\#}) \leq (\Sigma_{k\in\boldsymbol{K}}\gamma_k^2)^{\frac{1}{2}} \leq \Sigma_{k\in\boldsymbol{K}}\gamma_k = r(\mathcal{F})$.

Note that the second of the above inequalities in fact is Jensen's inequality. Thus $r(\mathcal{F}^{\#}) < r(\mathcal{F})$ whenever the ℓ_1-sequence $\{\gamma_k \geq 0;\ \ k \in \mathbb{K}\}$ has at least two nonzero elements. This generalizes the concrete example given in [2] (i.e. by setting $H = \ell_2$, $A_0 x = (\xi_2, \xi_4, \xi_6, \cdots)$ and $A_1 x = (\xi_1, \xi_3, \xi_5, \cdots)$ for all $x = (\xi_1, \xi_2, \xi_3, \cdots) \in \ell_2$, $A_k = O$ for every $k \geq 2$ and $\gamma_1 = \gamma_0 = 1$, it follows that $r(\mathcal{F}) = 2r(\mathcal{F}^{\#}) = 2$).

3. CONCLUSION

In this final section we shall verify that $r(\mathcal{F}^{\#}) = r(\mathcal{F})$ for any bounded sequence $\{A_k;\ k \geq 0\}$ of compact operators. In particular, for any bounded sequence of operators defined on finite-dimensional spaces, so that this also settles the mean-square stability problem for finite-dimensional discrete bilinear systems.

THEOREM. *If $\{A_k;\ k \geq 0\}$ is a sequence of compact operators, then $r(\mathcal{F}^{\#}) = r(\mathcal{F})$.*

Proof. Let $B_\infty[H]$ be the class of all compact operators from $B[H]$. Suppose $A_k \in B_\infty[H]$ for every $k \geq 0$ so that $\mathcal{F}_n(I) = \Sigma_{k=0}^n \gamma_k A_k A_k^*$ is compact for every $n \geq 0$ (for $B_\infty[H]$ is a two-sided ideal of $B[H]$ – see e.g. [8, p.132]). Since $\mathcal{F}_n(I) \to \mathcal{F}(I)$ in $B[H]$ as $n \to \infty$ it follows that $\mathcal{F}(I) \in B_\infty[H]$ (because $B_\infty[H]$ is closed in $B[H]$ – see e.g. [8, p.132]). Let $\{E_m \in B[H];\ m \geq 1\}$ be an increasing sequence of finite-rank (orthogonal) projections that converges strongly to the identity. Since $\mathcal{F}(I)$ is compact it follows that (see e.g. [8, p.136])

$$E_m \mathcal{F}(I) E_m \to \mathcal{F}(I) \qquad \text{in } B[H] \qquad \text{as } m \to \infty.$$

Now recall that $B_1[H]$ (the class of all nuclear operators) contains the finite-rank operators so that $E_m \in B_1[H]$ for each $m \geq 1$. Let $\|\cdot\|_1$ be the standard (trace) norm in $B_1[H]$

(i.e. $\|T\|_1$ is the trace of $(T^*T)^{\frac{1}{2}}$ for every $T \in B_1[H]$). Since $B_1[H]$ is a two-sided ideal of $B[H]$ such that $\max\{\|TL\|_1, \|LT\|_1\} \leq \|L\| \;\; \|T\|_1$ for every $L \in B[H]$ and $T \in B_1[H]$ (see e.g. [8, p.173]), if follows that $E_m\mathcal{F}(I)E_m \in B_1[H]$. Indeed

$$\|E_m\mathcal{F}(I)E_m\|_1 \;\leq\; \|E_m\|_1^2 \;\|\mathcal{F}(I)\| \;\;<\;\; \infty$$

for each $m \geq 1$. Since $E_m\mathcal{F}(I)E_m \in B_1[H]$ we get

$$\left\|\mathcal{F}^i\left(E_m\mathcal{F}(I)E_m\right)\right\| \;\;\leq\;\; \|E_m\mathcal{F}(I)E_m\|_1 \;\;\|\mathcal{F}^{\#i}\|$$

for each $m \geq 1$ and every $i \geq 0$ (actually $\|\mathcal{F}^i(Q)\| \leq \|\mathcal{F}^i(Q)\|_1 \leq \|Q\|_1\|\mathcal{F}^{\#i}\|$ for every $i \geq 0$ whenever $Q \in B_1[H]$ – cf. [6]). Therefore, by the preceding lemma,

$$\begin{aligned}
\left\|\mathcal{F}^{i+1}\right\| = \left\|\mathcal{F}^{i+1}(I)\right\| &= \left\|\mathcal{F}^i(\mathcal{F}(I))\right\| \\
&= \left\|\mathcal{F}^i\left(E_m\mathcal{F}(I)E_m + \mathcal{F}(I) - E_m\mathcal{F}(I)E_m\right)\right\| \\
&\leq \left\|\mathcal{F}^i\left(E_m\mathcal{F}(I)E_m\right)\right\| + \left\|\mathcal{F}^i\left(\mathcal{F}(I) - E_m\mathcal{F}(I)E_m\right)\right\| \\
&\leq \|E_m\|_1^2\,\|\mathcal{F}\| \;\left\|\mathcal{F}^{\#i}\right\| + \left\|\mathcal{F}^i\right\| \;\|\mathcal{F}(I) - E_m\mathcal{F}(I)E_m\|
\end{aligned}$$

for each $m \geq 1$ and every $i \geq 0$. Moreover, since $\|\mathcal{F}(I) - E_m\mathcal{F}(I)E_m\| \to 0$ as $m \to \infty$, take an integer M large enough such that $\|\mathcal{F}(I) - E_M\mathcal{F}(I)E_M\| \leq \frac{1}{2}$. Thus

$$\left\|\mathcal{F}^{i+1}\right\| - \tfrac{1}{2}\left\|\mathcal{F}^i\right\| \;\leq\; \|E_M\|_1^2\,\|\mathcal{F}\| \;\;\|\mathcal{F}^{\#i}\|$$

for every $i \geq 0$. Adding up from zero to an arbitrary positive integer N we get

$$\tfrac{1}{2}\textstyle\sum_{i=0}^N \left\|\mathcal{F}^i\right\| + \left\|\mathcal{F}^{N+1}\right\| - \tfrac{1}{2} \;\;\leq\;\; \|E_M\|_1^2\,\|\mathcal{F}\|\textstyle\sum_{i=0}^N \|\mathcal{F}^{\#i}\|.$$

Hence $\Sigma_{i=0}^{\infty}\|\mathcal{F}^i\| < \infty$ whenever $\Sigma_{i=0}^{\infty}\|\mathcal{F}^{\#i}\| < \infty$. Similarly (just replace A_k by A_k^*) we can prove the converse so that, in fact,

$$\sum_{i=0}^{\infty}\|\mathcal{F}^{\#i}\| < \infty \qquad \Leftrightarrow \qquad \sum_{i=0}^{\infty}\left\|\mathcal{F}^i\right\| < \infty.$$

Equivalenty (see e.g. [4]),

$$r(\mathcal{F}^{\#}) < 1 \qquad \Leftrightarrow \qquad r(\mathcal{F}) < 1.$$

Finally note that, for any $\alpha > 0$, $(\alpha^{-1}\mathcal{F})(Q) = \Sigma_{k=0}^{\infty}(\alpha^{-\frac{1}{2}}A_k)Q(\alpha^{-\frac{1}{2}}A_k^*)$ for all $Q \in B[H]$. Since $\alpha^{-\frac{1}{2}}A_k$ is compact whenever A_k is, the above equivalence leads to

$$r\big(\mathcal{F}^{\#}\big) < \alpha \qquad \Leftrightarrow \qquad r(\mathcal{F}) < \alpha$$

for every $\alpha > 0$ (reason: $\alpha^{-1}r(\mathcal{F}) = r(\alpha^{-1}\mathcal{F})$). Thus, if one of the above spectral radii is less than the other, then it is less than itself. Therefore

$$r\big(\mathcal{F}^{\#}\big) \;\;=\;\; r\left(\mathcal{F}\right).$$

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