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STRONG STABILITY FOR COHYPONORMAL OPERATORS*

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Abstract: A cohyponormal operator is strongly stable if and only if it is a completely nonunitary contraction. An elementary geometric proof for the above equivalence comes out by identifying the essential properties that ensure it for the normal case.

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1. INTRODUCTION

Let T be an operator on a Hilbert space $\mathcal{H}$ (i.e. a bounded linear transformation of $\mathcal{H}$ into itself). By a subspace of $\mathcal{H}$ we mean a closed linear manifold of it. $\mathcal{N}(T)$ and $\mathcal{R}(T)$ will denote the null subspace and range of T, respectively. Recall that T is hyponormal if $O \leq T^*T - TT^*$ (equivalently, if $||T^*x|| \leq ||Tx||$ for all $x \in \mathcal{H}$), and cohyponormal if its adjoint T^* is hyponormal. T is normal if it is both hyponormal and cohyponormal, and seminormal if it is either hyponormal or cohyponormal. Any seminormal operator is normaloid (i.e. it is such that $r(T) = ||T||$, where $r(T)$ denotes the spectral radius of T). An operator is completely nonunitary if the restriction of it to any nonzero reducing subspace is not unitary. We shall say that an operator T is strongly stable if the power sequence $\{T^n;\ n \geq 0\}$ converges strongly to O as $n \to \infty$ (i.e. $T^n \xrightarrow{s} O$).

Clearly a strongly stable operator is completely nonunitary (actually a strongly stable operator is completely nonisometric); but it may not be a contraction (not even similar to a contraction - e.g. see [2]). However a strongly stable normaloid operator is a contraction. Indeed, if T is strongly stable then it is power bounded (i.e. $\sup_n ||T^n|| < \infty$, according to the Banach-Steinhaus theorem), and a power bounded operator T is such that $r(T) \leq 1$ (by the Beurling-Gelfand formula for the spectral radius). Summing up: *if a normaloid operator is strongly stable, then it is a completely nonunitary contraction.*

Strong stability can be readily characterized for normal operators: *if a normal operator is a completely nonunitary contraction, then it is strongly stable.* A rather elementary and essentially geometric proof for the above statement will be given below. Moreover such a characterization for strong stability can be extended to cohyponormal operators as originally verified by Putnam [4]: *if a cohyponormal operator is a completely nonunitary contraction, then it is strongly stable.* Opposite to the normal case the proof of the above statement, as given in [4], is essentially analytic involving a nonelementary measure theoretical approach.

In the present paper we shall give a new and elementary geometric proof for the characterization of strong stability for cohyponormal operators. By “elementary” we mean that all proofs in this paper use only standard results of single operator theory.

2. PRELIMINARIES

If T is a contraction (i.e. $||T|| \leq 1$) then $\{T^{*n}T^n;\ n \geq 0\}$ is a monotone self-adjoint bounded sequence (in fact a nonincreasing nonnegative sequence of contractions), so that $\{T^{*n}T^n;\ n \geq 0\}$ converges strongly. Let the operator A on $\mathcal{H}$ be its (strong) limit. It is trivially verified that A is a nonnegative contraction (actually $O \leq A \leq I$) such that

$$(1) \qquad ||T^n x|| \;\longrightarrow\; ||A^{\frac{1}{2}}x|| \qquad \text{as} \quad n \to \infty$$

for every $x \in \mathcal{H}$ (i.e. $T^{*n}T^n \xrightarrow{s} A \geq O$), and

$$(2) \qquad ||A^{\frac{1}{2}}T^n x|| \;=\; ||A^{\frac{1}{2}}x||$$

for all $x \in \mathcal{H}$ and every $n \geq 0$ (i.e. $T^{*n}AT^n = A \geq O$ for every $n \geq 0$). According to property (1) *T is strongly stable if and only if $A = O$.* Since $\mathcal{N}(A) = \mathcal{N}(A^{\frac{1}{2}})$ property (1) actually leads to

$$\mathcal{N}(A) = \{x \in \mathcal{H} : \quad T^n x \to 0 \qquad \text{as} \quad n \to \infty\}.$$

Recall that $\mathcal{N}(I - A) = \{x \in \mathcal{H} : \; ||A^{\frac{1}{2}}x|| = ||x||\}$, for $||(I - A)^{\frac{1}{2}}x||^2 = \langle (I - A)x; x\rangle = ||x||^2 - ||A^{\frac{1}{2}}x||^2$ for all $x \in \mathcal{H}$. Moreover, since $A^{\frac{1}{2}}$ and T are contractions, $\{x \in \mathcal{H};\ ||T^n x|| = ||x|| \;\; \forall n \geq 0\} = \{x \in \mathcal{H} : \; ||A^{\frac{1}{2}}x|| = ||x||\}$ by properties (1) and (2). Therefore

$$(3) \qquad \mathcal{N}(I - A) = \{x \in \mathcal{H} : \quad ||T^n x|| = ||x|| \qquad \forall n \geq 0\}.$$

Since T^* is a contraction whenever T is, let the operator A_* on $\mathcal{H}$ be the strong limit of $\{T^n T^{*n};\ n \geq 0\}$. Thus, by property (3),

$$\mathcal{N}(I - A) \cap \mathcal{N}(I - A_*) = \{x \in \mathcal{H} : \; ||T^n x|| = ||T^{*n}x|| = ||x|| \quad \forall n \geq 0\}.$$

Now, according to Nagy-Foias-Langer decomposition for contractions (cf. [5, p.9]), $\mathcal{N}(I - A) \cap \mathcal{N}(I - A_*)$ is the largest reducing subspace of T on which T is unitary; so that *a contraction is completely nonunitary if and only if* $\mathcal{N}(I - A) \cap \mathcal{N}(I - A_*) = \{0\}$. Further properties on the operators A and A_* have been investigated in [1] and [3].

Remark. If T is a normal contraction then $A_* = A = A^2$. Indeed the normality of T implies that $T^{*n}T^n = T^n T^{*n} = (T^*T)^n$ for every $n \geq 0$. Thus $A = A_*$ by their very definition, and so $\mathcal{N}(I - A) = \mathcal{N}(I - A_*)$. Therefore, if the normal contraction T is completely nonunitary,

$$\mathcal{N}(I - A) = \{0\}.$$

Moreover $A = A^2$ because it is the (strong) limit of a power sequence (viz. $\{(T^*T)^n : \; n \geq 0\}$), so that $\mathcal{R}(A) = \mathcal{N}(I - A)$. Hence $A = O$ whenever $\mathcal{N}(I - A) = \{0\}$. Outcome: *a normal completely nonunitary contraction is strongly stable.*

3. CONCLUSION

A key property for achieving strong stability in the above remark was the idempotency of A (i.e. $A = A^2$). As we shall see below any cohyponormal contraction is endowed with such a property, and so a cohyponormal contraction will be strongly stable provided that $\mathcal{N}(I - A) = \{0\}$.

Lemma. *Let T be a contraction and let A be the strong limit of $\{T^{*n}T^n;\ n \geq 0\}$. If T is cohyponormal then $A = A^2$.*

Proof. Let T be a cohyponormal operator so that $\|T^*x\|^2 = \langle TT^*x;\ x\rangle \leq ||TT^*x||\ ||x|| \leq ||T^{*2}x||\ ||x||$ for all $x \in \mathcal{H}$. Thus, by induction,

$$\|T^*x\|^n \;\leq\; \|T^{*n}x\|\ \|x\|^{n-1}$$

for all $x \in \mathcal{H}$ and every $n \geq 1$ (indeed the assertion is tautological for $n = 1$, holds for $n = 2$ and, if it holds for some $n \geq 2$, then it holds for $n+1$ since $||T^*x||^{2n} \leq ||T^{*2}x||^n||x||^n \leq ||T^{*n+1}x||\ ||T^*x||^{n-1}||x||^n$ for all $x \in \mathcal{H}$). Now suppose the cohyponormal operator T is a contraction and take $y \notin \mathcal{N}(A)$ arbitrary. The above inequality leads to

$$\left(\frac{||T^{n+1}y||}{||T^ny||}\right)^n \;\leq\; \left(\frac{||T^*T^ny||}{||T^ny||}\right)^n \;\leq\; \frac{||T^{*n}T^ny||}{||T^ny||} \;\leq\; 1$$

for every $n \geq 1$. Recall that $||T^ny|| \to ||A^{\frac{1}{2}}y|| \neq 0$ by property (1), and note that $||T^{*n}T^ny|| \to ||Ay|| \neq 0$ by the very definition of A. Moreover it is easy to show that

$$\limsup_{n\to\infty} \left(\frac{||T^{n+1}y||}{||T^ny||}\right)^n = 1$$

for every contraction T (in fact $\limsup_{n\to\infty}(\beta_{n+1}/\beta_n)^n = 1$ for any positive nonincreasing real sequence $\{\beta_n;\ n \geq 1\}$ that converges to a nonzero limit). Therefore

$$1 \;\leq\; \frac{||Ay||}{||A^{\frac{1}{2}}y||} \;\leq\; 1$$

so that $||(A - A^2)^{\frac{1}{2}}y||^2 = ||A^{\frac{1}{2}}y||^2 - ||Ay||^2 = 0$, and hence $Ay = A^2y$. Thus $Ax = A^2x$ for all $x \in \mathcal{H}$. □

Next we shall verify that $\mathcal{N}(I - A) \subseteq \mathcal{N}(I - A_*)$ whenever T is a cohyponormal contraction, and so any cohyponormal completely nonunitary contraction is such that $\mathcal{N}(I - A) = \{0\}$.

Theorem. *A cohyponormal completely nonunitary contraction is strongly stable.*

Proof. Let T be a cohyponormal operator (i.e. $||Tx|| \leq ||T^*x||$ for every $x \in \mathcal{H}$, or equivalently $O \leq TT^* - T^*T$) so that $||(TT^* - T^*T)^{\frac{1}{2}}x||^2 = ||T^*x||^2 - ||Tx||^2$ for all

$x \in \mathcal{H}$. Suppose T is a contraction and take $y \in \mathcal{N}(I - A)$ arbitrary. By property (3) we get $||y|| = ||Ty|| \leq ||T^*y|| \leq ||y||$, and hence $(TT^* - T^*T)^{\frac{1}{2}} y = 0$. Thus

$$\mathcal{N}(I - A) \subseteq \mathcal{N}(TT^* - T^*T).$$

According to the above inclusion, and since $\mathcal{N}(I - A)$ is invariant under T (cf. property (3) again), it is readily verified by induction that

$$T^*T^n y = T^n T^* y$$

for every $n \geq 0$. Hence $||y|| = ||T^{n+1}y|| \leq ||T^*T^n y|| = ||T^n T^* y|| \leq ||T^*y|| \leq ||y||$, so that $||T^n T^* y|| = ||T^* y||$, for every $n \geq 0$. Thus $T^* y \in \mathcal{N}(I - A)$ and $||Ty|| = ||T^*y|| = ||y||$. Conclusion: $\mathcal{N}(I - A)$ is a reducing subspace for T on which T is unitary. Therefore, if the cohyponormal contraction T is completely nonunitary,

$$\mathcal{N}(I - A) = \{0\}.$$

However $\mathcal{R}(A) = \mathcal{N}(I - A)$ by the previous lemma. Hence $A = O$. □

REFERENCES

1. E. Durszt, Contractions as restricted shifts, *Acta Sci. Math.* **48** (1985) 129–134.

2. C.S. Kubrusly, Strong stability does not imply similarity to a contraction, *Syst. Control Lett.* **14** (1990) 349–352.

3. V. Pták and P. Vrbová, An abstract model for compressions, *Casopis Pest. Mate.* **113** (1988) 252–259.

4. C.R. Putnam, Hyponormal contractions and strong power convergence, *Pacific J. Math.* **57** (1975) 531–538.

5. B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Spaces*, North-Holland, Amsterdam (1970).

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