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RICCATI EQUATION FOR INFINITE-DIMENSIONAL DISCRETE BILINEAR SYSTEMS*

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Abstract. The quadratic optimal feedback control problem for infinite-dimensional discrete bilinear systems is associated with an algebraic Riccati-like operator equation. The existence and uniqueness of nonnegative solution to such a Riccati-like operator equation is established as the strong limit of a sequence of solutions to Lyapunov-like operator equations.

Key words. Discrete bilinear systems; infinite-dimensional systems; operator theory; optimal control.

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1. NOTATIONAL PRELIMINARES

Let X and X' be Banach spaces, and let $B[X, X']$ be the Banach space of all bounded linear transformations of X into X'. We shall write $B[X]$ for $B[X, X]$, and $G[X]$ for the group of all invertible operators from $B[X]$ (i.e. those which are also bounded below and map X onto itself). The norms in X, in X' and the induced uniform norm in $B[X, X']$ will all be denoted by $||\cdot||$, and $r(\cdot)$ will stand for the spectral radius in the Banach algebra $B[X]$. Throughout the paper H, H' and H'' will stand for separable nontrivial complex Hilbert spaces. Inner product in any of them will be denoted by $\langle\cdot;\cdot\rangle$, and an upper star * will stand for adjoint as usual. Let $B^+[H]$ be the weakly closed convex cone of all (self-adjoint) nonnegative operators in $B[H]$, and set $G^+[H] = B^+[H] \cap G[H]$. The class of all nuclear operators (i.e. the trace-class) from $B[H]$ will be denoted by $B_1[H]$. Set $B_1^+[H] = B_1[H] \cap B^+[H]$. The trace of $T \in B_1[H]$ is defined as $\mathrm{tr}(T) = \sum_{k=0}^{\infty} \langle Te_k; e_k\rangle$ which does not depend on the choice of the orthonormal basis $\{e_k;\ k \geq 0\}$ for H. For each $T \in B_1[H]$ set $||T||_1 = \mathrm{tr}((T^*T)^{\frac{1}{2}})$ so that $||T||_1 = \mathrm{tr}(T)$ whenever $T \in B_1^+[H]$. Indeed, $||\cdot||_1$ is a norm in $B_1[H]$ and $(B_1[H], ||\ \ ||_1)$ is a Banach space which is a two-sided ideal of $B[H]$ such that $\max\{||TL||_1, ||LT||_1\} \leq ||L||\ \ ||T||_1$ and $\mathrm{tr}(TL) = \mathrm{tr}(LT)$ for every $L \in B[H]$ and $T \in B_1[H]$. Since $|\mathrm{tr}(T)| \leq ||T||_1$ for every $T \in B_1[H]$, $\mathrm{tr} : B_1[H] \to \mathbb{C}$ is a bounded linear functional. For any $f, g \in H$ let $(f \circ g) \in B_1[H]$ be defined as $(f \circ g)h = \langle h; g\rangle f$ for all $h \in H$, so that $(f \circ f) \in B_1^+[H]$. Let $\ell_1(B_1[H])$ be the Banach space of all $B_1[H]$-valued $||\cdot||_1$-summable sequences. We shall use the same symbol $||\cdot||_1$ to denote the norm in $\ell_1(B_1[H])$, so that $||\mathbf{T}||_1 = \sum_{i=0}^{\infty} ||T_i||_1 < \infty$ for every $\mathbf{T} = \{T_i \in B_1[H]; i \geq 0\} \in \ell_1(B_1[H])$. The class of all $B_1^+[H]$-valued sequences from $\ell_1(B_1[H])$ will be denoted by $\ell_1(B_1^+[H])$.

Given a fixed probability space (Ω, Σ, μ), where Σ is a σ-algebra of subsets of a nonempty set Ω and μ is a probability measure on Σ, set $\mathcal{H} = L_2(\Omega, \mu; H)$ for any Hilbert space H. Thus $\mathcal{H}$ is the Hilbert space made up of equivalence classes of H-valued measurable maps x defined (almost everywhere with respect to the measure μ) on Ω such that $||x||^2_{\mathcal{H}} := \varepsilon||x||^2 := \int_\Omega ||x(\omega)||^2 d\mu(\omega) < \infty$. This is the so-called second-order property and such a norm in $\mathcal{H}$ is generated by the inner product $\langle x; y\rangle_{\mathcal{H}} := \varepsilon\langle x; y\rangle := \int_\Omega \langle x(\omega); y(\omega)\rangle d\mu(\omega)$ for all $x, y \in \mathcal{H}$. Hence $\mathcal{H}$ is the Hilbert space of all H-valued second-order random variables on (Ω, Σ, μ). Here ε stands for the expectation of the underlying scalar-valued random variables on (Ω, Σ, μ). For any $x \in \mathcal{H}$ its expectation, correlation and covariance (defined as usual - see e.g. [5]) will be denoted by $Ex \in H, \mathcal{E}(x \circ x) \in B_1^+[H]$ and $\mathcal{E}(x \circ x) - (Ex \circ Ex) \in B_1^+[H]$, respectively. Let $\ell_2(\mathcal{H})$ be the Hilbert space of all H-valued second-order random sequences $\mathbf{x} = \{x_i \in \mathcal{H};\ i \geq 0\}$ such that $||\mathbf{x}||^2_{\ell_2(\mathcal{H})} := \sum_{i=0}^{\infty} ||x_i||^2_{\mathcal{H}} < \infty$. Recall that $||x||^2_{\mathcal{H}} = ||\mathcal{E}(x \circ x)||_1$ for every $x \in \mathcal{H}$ (see e.g. [5]). Hence $\mathbf{x} \in \ell_2(\mathcal{H})$ if and only if $\{\mathcal{E}(x_i \circ x_i);\ i \geq 0\} \in \ell_1(B_1^+[H])$.

2. INTRODUCTION

Quadratic optimal control problems and the associated algebraic Riccati operator equation for infinite-dimensional discrete linear systems, operating either in a deterministic or in a stochastic environment, have been investigated over the past two decades (see e.g. [8],

[10], [3] and [12]). For infinite-dimensional discrete bilinear systems operating in a stochastic environment, which comprise the models we shall be addressing in the present paper, this has previously been investigated in [11]. We shall consider here an operator theoretic approach that differs from that of [11] in which it relies on the properties of the operators $\mathcal{F}$ and $\mathcal{F}^{\#}$ in $B[B[H]]$ (see Section 3) that have recently been established in [7]. Moreover the criterion to be minimized here is motivated from the H_∞- control problem; the main difference relying on the admissible class of additive input sequences. We shall work under independence and zero-mean assumptions which considerably simplify the problem and allows for an exact solution to be obtained. Furthermore the solution to the quadratic stochastic optimal control problem will be stated under detectability hypothesis, thus mirroring its linear counterpart. Existence and uniqueness of a nonnegative solution to the associated Riccati-like equation will be established in Section 7 by using the auxiliary results developed in Section 6.

Consider a discrete bilinear system with a linear feedback loop operating in a stochastic environment, whose model is given by the following infinite-dimensional difference equation.

$$(1) \qquad x_{i+1} = \left[A_0 + \sum_{k=1}^{\infty} \langle w_i ; e_k \rangle A_k \right] x_i - BKx_i + Dv_{i+1}, \qquad x_0 = Dv_0,$$

where $\{A_k \in B[H];\ k \geq 0\}$ is a bounded sequence, $B \in B[H', H]$, $D \in B[H'', H]$, $K \in B[H, H']$, $\{e_k;\ k \geq 1\}$ is an orthonormal basis for H, $\{x_i \in \mathcal{H};\ i \geq 0\}$ (the state sequence) and $\{w_i \in \mathcal{H};\ i \geq 0\}$ (the multiplicative input sequence) are second-order H-valued random sequences, and $\mathbf{v} = \{v_i \in \mathcal{H}'';\ i \geq 0\}$ (the additive input sequence) is a second-order H''-valued random sequence. Conditions ensuring convergence in $\mathcal{H}$ for the above infinite-dimensional model have been established in [6]. We shall precise the assumptions on the stochastic environment that will suffice our needs in the next section. Now take $M \in B^+[H]$ and $N \in G^+[H']$, and consider the output sequence $\mathbf{z} = \{z_i \in \mathcal{H} \oplus \mathcal{H}';\ i \geq 0\}$ given by

$$z_i = \begin{pmatrix} M^{\frac{1}{2}} \\ N^{\frac{1}{2}} K \end{pmatrix} x_i$$

for every integer $i \geq 0$. The quadratic stochastic optimal control problem associated with the above discrete model that we shall be addressing in this paper is that of finding the state feedback stabilizing controller K that minimizes the impact of the additive input disturbance $\mathbf{v}$ on the output $\mathbf{z}$. Formally, find $K \in \Gamma_B$ that minimizes

$$(2.a) \qquad \sup_{\mathbf{0} \neq \mathbf{v} \in \mathcal{V}_{\mathbf{w}}} \frac{||\mathbf{z}||_{\ell_2(\mathcal{H} \oplus \mathcal{H}')}}{||\mathbf{v}||_{\ell_2(\mathcal{H}'')}},$$

where the admissible classes $\mathcal{V}_{\mathbf{w}} \subset \ell_2(\mathcal{H}'')$ and $\Gamma_B \subset B[H, H']$ will be specified next so that the above criterion is well defined. Indeed, it will be shown that the solution $K \in \Gamma_B$ to the above problem actually minimizes $||\mathbf{z}||_{\ell_2(\mathcal{H} \oplus \mathcal{H}')}$ over Γ_B for any $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$ (and, in particular, for $v_i = 0$ for every $i \geq 1$, which characterizes an additive-input-free model – i.e. one with $\mathbf{v} = \mathbf{0}$).

3. THE CLASS $\mathcal{V}_{\mathbf{w}}$

Let us make the following assumptions on the stochastic environment. Suppose $\{w_i \in \mathcal{H};\ i \geq 0\}$ is an independent random sequence that is stationary in expectation and correlation, so that $Ew_i = s \in H$ and $\mathcal{E}(w_i \circ w_i) = S \in B_1^+[H]$ for every $i \geq 0$. Set $C = S-(s\circ s) \in B_1^+[H]$ and let $\mathcal{V}_{\mathbf{w}}$ be the class of all zero-mean independent random sequences from $\ell_2(\mathcal{H}'')$ that are independent of $\{w_i \in \mathcal{H};\ i \geq 0\}$. Suppose $\mathbf{v} = \{v_i \in \mathcal{H}'';\ i \geq 0\} \in \mathcal{V}_{\mathbf{w}}$ and set $R_i = \mathcal{E}(v_i \circ v_i) \in B_1^+[H]$ for every $i \geq 0$. It has been shown in [6] that, under the above assumptions (which in fact are essentially the same assumptions made in [7]), the state correlation sequence $\{Q_i = \mathcal{E}(x_i \circ x_i) \in B_1^+[H];\ i \geq 0\}$ evolves according to the following linear model.

$$(3.a) \qquad Q_{i+1} = \mathcal{F}_{BK}(Q_i) + DR_{i+1}D^*, \qquad Q_0 = DR_0D^*,$$

so that

$$(3.b) \qquad Q_i = \sum_{j=0}^{i} \mathcal{F}_{BK}^{i-j}(DR_jD^*),$$

for every $i \geq 0$ (recall: $\mathcal{E}(Dv_i \circ Dv_i) = D\mathcal{E}(v_i \circ v_i)D^*$ - see e.g. [5]), where $\mathcal{F}_{BK} \in B[B[H]]$ is given by

$$\mathcal{F}_{BK}(P) = F_{BK}\ P\ F_{BK}^* + \mathcal{T}(P)$$

for all $P \in B[H]$, with

$$F_{BK} := (A_0 - BK) + \sum_{k=1}^{\infty} \langle s; e_k \rangle A_k$$

in $B[H]$, and $\mathcal{T} \in B[B[H]]$ defined by

$$\mathcal{T}(P) = \sum_{k,\ell=1}^{\infty} \langle Ce_k; e_\ell \rangle A_k\ P\ A_\ell^*$$

for all $P \in B[H]$. Here $\{e_k;\ k \geq 1\}$ is supposed to be an orthonormal basis for H that ensures the above convergences in the uniform topologies of $B[H]$ and $B[B[H]]$, respectively. The existence of such an orthonormal basis (e.g. the one made up of eigenvectors of $S \in B_1^+[H]$), which may depend on the correlation operator $S \in B_1^+[H]$ but not on the bounded sequence $\{(A_0 - BK) \in B[H], A_k \in B[H];\ k \geq 1\}$, has been established in [6] (also see [1]).

Associated with $\mathcal{T}$ and $\mathcal{F}_{BK}$ set $\mathcal{T}^\# \in B[B[H]]$ and $\mathcal{F}_{BK}^\# \in B[B[H]]$ as follows: for all $P \in B[H]$,

$$\mathcal{T}^\#(P) = \sum_{k,\ell=1}^{\infty} \langle Ce_\ell; e_k \rangle A_\ell^*\ P\ A_k,$$

$$\mathcal{F}_{BK}^\#(P) = F_{BK}^*\, P\, F_{BK} + \mathcal{T}^\#(P).$$

Note that the feedback loop in model (1) is characterized by the product $BK \in B[H]$. An absence of it (i.e. the case of $BK = O$) leads to operators F_O, $\mathcal{F}_O$ and $\mathcal{F}_O^{\#}$. Set $F = F_O$, $\mathcal{F} = \mathcal{F}_O$ and $\mathcal{F}^{\#} = \mathcal{F}_O^{\#}$ for short, so that $F_{BK} = F - BK$ and, for all $P \in B[H]$,

$$\mathcal{F}(P) = FPF^* + \mathcal{T}(P), \qquad \mathcal{F}_{BK}(P) = (F-BK)P(F-BK)^* + \mathcal{T}(P),$$
$$\mathcal{F}^{\#}(P) = F^*PF + \mathcal{T}^{\#}(P), \qquad \mathcal{F}_{BK}^{\#}(P) = (F-BK)^*P(F-BK) + \mathcal{T}^{\#}(P).$$

4. THE CLASS Γ_B

Given $B \in B[H', H]$ set

$$\Gamma_B = \{K \in B[H, H'] : \quad r(\mathcal{F}_{BK}^{\#}) < 1\}.$$

Under the assumptions made in the previous section $r(\mathcal{F}_{BK}^{\#}) < 1$ is a necessary condition for mean-square stability of model (1) which, together with its dual $r(\mathcal{F}_{BK}) < 1$, is sufficient as well (cf. [7]). Thus a stabilizing state feedback controller K necessarily lies in Γ_B. Now consider the following functional $J_{\cdot}(\cdot) : \ell_1(B_1^+[H'']) \times \Gamma_B \to I\!R^+$.

$$J_{\mathbf{T}}(K) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \operatorname{tr}\left(DT_jD^*\mathcal{F}_{BK}^{\#^{i-j}}(M + K^*NK)\right) \tag{4.a}$$

for all $\mathbf{T} = \{T_i \in B_1^+[H'']; \ i \geq 0\} \in \ell_1(B_1^+[H''])$ and $K \in \Gamma_B$. We shall restate the optimization problem (2.a) in terms of such a functional, but first let us verify that in fact it is well defined. Note that $\sum_{j=0}^{i} \operatorname{tr}(DT_jD^*\mathcal{F}_{BK}^{\#^{i-j}}(M + K^*NK)) \leq ||D||^2||M + K^*NK|| \sum_{j=0}^{i} ||T_j||_1||\mathcal{F}_{BK}^{\#^{i-j}}||$ for every $i \geq 0$. Set $a = \{||\mathcal{F}_{BK}^{\#^i}||; \ i \geq 0\}$ and $b = \{||T_i||_1; \ i \geq 0\}$ which are in ℓ_1 (for $r(\mathcal{F}_{BK}^{\#}) < 1$ - see e.g. [4] - and $\mathbf{T} \in \ell_1(B_1[H''])$). Thus the convolution (or the Cauchy product) $c = \{\sum_{j=0}^{i} ||\mathcal{F}_{BK}^{\#^{i-j}}|| \ ||T_j||_1; \ i \geq 0\} = a * b$ lies itself in ℓ_1 as well with $||c||_1 \leq ||a||_1||b||_1$ (see e.g. [2, p.529]). Hence $\sum_{i=0}^{\infty}\sum_{j=0}^{i} ||\mathcal{F}_{BK}^{\#^{i-j}}|| \ ||T_j||_1 \leq (\sum_{i=0}^{\infty} ||\mathcal{F}_{BK}^{\#^i}||)||\mathbf{T}||_1$. Moreover, since $B^+[H]$ is invariant under $\mathcal{F}_{BK}^{\#} : B[H] \to B[H]$ (cf. [7]), $M + K^*NK \in B^+[H]$ and $T_i \in B^+[H'']$ for every $i \geq 0$ (for $\mathbf{T} \in \ell_1(B_1^+[H''])$), it follows that $\operatorname{tr}(DT_jD^*\mathcal{F}_{BK}^{\#^{i-j}}(M + K^*NK)) = \operatorname{tr}(T_j^{\frac{1}{2}}D^*\mathcal{F}_{BK}^{\#^{i-j}}(M + K^*NK)DT_j^{\frac{1}{2}}) \geq 0$ for every $0 \leq j \leq i$. Hence

$$0 \leq J_{\mathbf{T}}(K) \leq \left(||D||^2||M + K^*NK|| \sum_{i=0}^{\infty} \left\|\mathcal{F}_{BK}^{\#^i}\right\|\right) ||\mathbf{T}||_1 < \infty$$

for all $\mathbf{T} \in \ell_1(B_1^+[H''])$ and $K \in \Gamma_B$. Therefore $J_{\cdot}(K) : \ell_1(B_1^+[H'']) \to I\!R^+$ is a (uniformly) bounded functional for each $K \in \Gamma_B$. Thus, for any set $\Lambda \subseteq \ell_1(B_1^+[H''])$ such that $\Lambda \setminus \{\mathbf{O}\} \neq \emptyset$, the functional $J_\Lambda : \Gamma_B \to I\!R^+$, given by the formula

$$J_\Lambda(K) = \sup_{\mathbf{O} \neq \mathbf{T} \in \Lambda} \frac{J_{\mathbf{T}}(K)}{||\mathbf{T}||_1} \tag{4.b}$$

for every $K \in \Gamma_B$, is well defined.

Remark 1. Note that $r(\mathcal{F}^{\#}_{BK}) < 1$ is not equivalent to $r(\mathcal{F}_{BK}) < 1$ for the general bilinear case where $\mathcal{T}$ is not null; and this is the reason why the upper symbol $^{\#}$ can not be suppressed from the definition of Γ_B. To check this consider the following particular case with $BK = O$. Set $H = \ell_2$, take an arbitrary real $\alpha \neq 0$, and set A_0, $A_1 \in B[\ell_2]$ as follows.

$$A_0 h = \alpha(\eta_2, \eta_4, \eta_6, \cdots), \qquad A_1 h = \alpha(\eta_1, \eta_3, \eta_5, \cdots),$$

for all $h = (\eta_1, \eta_2, \eta_3, \cdots) \in \ell_2$; so that $A_0 A_0^* = A_1 A_1^* = \alpha^2 I$, $A_0^* A_0 = \alpha^2 \mathrm{diag}(0, 1, 0, 1, \cdots)$ and $A_1^* A_1 = \alpha^2 \mathrm{diag}(1, 0, 1, 0, \cdots)$. Suppose $s = 0$ and $S = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, \cdots) \in B_1^+[\ell_2]$ with $\sigma_1 = 1$, and set $A_k = O$ for every $k \geq 2$. Thus

$$\mathcal{F}(I) = A_0 A_0^* + A_1 A_1^* = 2\alpha^2 I, \qquad \mathcal{F}^{\#}(I) = A_0^* A_0 + A_1^* A_1 = \alpha^2 I.$$

Hence $\mathcal{F}^i(I) = (2\alpha^2)^i I$ and $\mathcal{F}^{\#^i}(I) = (\alpha^2)^i I$, so that (cf. Property P$_2$ in [7]) $||\mathcal{F}^i||^{\frac{1}{i}} = 2\alpha^2$ and $||\mathcal{F}^{\#^i}||^{\frac{1}{i}} = \alpha^2$, for every $i \geq 0$. Therefore, by the Beurling-Gelfand formula for the spectral radius (see e.g. [2, p.567]), $r(\mathcal{F}) = 2r(\mathcal{F}^{\#}) = 2\alpha^2$.

5. PROBLEM RE-STATEMENT

For any $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$ the state correlation sequence $\{Q_i;\ i \geq 0\}$ evolves as in (3), and for any $K \in \Gamma_B$ the functionals in (4) are well defined. This will be enough to establish that $\mathbf{z} \in \ell_2(\mathcal{H} \oplus \mathcal{H}')$ whenever $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$ and $K \in \Gamma_B$, so that the optimization problem posed in Section 2 makes sense. Indeed, since

$$z_i = M^{\frac{1}{2}} x_i \oplus N^{\frac{1}{2}} K x_i \in \mathcal{H} \oplus \mathcal{H}',$$

it follows by linearity of the trace that

$$\begin{aligned}
||z_i||^2_{\mathcal{H}\oplus\mathcal{H}'} &= ||M^{\frac{1}{2}} x_i||^2_{\mathcal{H}} + ||N^{\frac{1}{2}} K x_i||^2_{\mathcal{H}'} \\
&= \varepsilon\langle M x_i; x_i\rangle + \varepsilon\langle N K x_i; K x_i\rangle \\
&= \mathrm{tr}(Q_i M) + \mathrm{tr}(Q_i K^* N K) \\
&= \sum_{0 \leq j \leq i} \mathrm{tr}(\mathcal{F}^{i-j}_{BK}(D R_j D^*)(M + K^* N K))
\end{aligned}$$

for every $i \geq 0$ whenever $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$, according to (3-b) (recall: $||L^{\frac{1}{2}} x||^2_{\mathcal{H}} = \varepsilon\langle Lx; x\rangle = \mathrm{tr}(\mathcal{E}(x \circ x)L)$ for every $x \in \mathcal{H}$ and $L \in B^+[H]$ - see e.g. [5]). However $\mathrm{tr}(\mathcal{F}^i_{BK}(T)L) = \mathrm{tr}(T\mathcal{F}^{\#^i}_{BK}(L))$ for every $i \geq 0$, $T \in B_1[H]$ and $L \in B[H]$ (cf. [7]). Hence

$$||z_i||^2_{\mathcal{H}\oplus\mathcal{H}'} = \sum_{j=0}^{i} \mathrm{tr}\left(D R_j D^* \mathcal{F}^{\#^{i-j}}_{BK}(M + K^* N K)\right)$$

for every $i \geq 0$ whenever $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$. Associated with any $\mathbf{v} = \{v_i \in \mathcal{H}'';\ i \geq 0\} \in \ell_2(\mathcal{H}'')$ set $\mathbf{R} = \{R_i = \mathcal{E}(v_i \circ v_i) \in B_1^+[H''];\ i \geq 0\} \in \ell_1(B_1^+[H''])$, so that

$$||\mathbf{v}||^2_{\ell_2(\mathcal{H}'')} = ||\mathbf{R}||_1 = \sum_{i=0}^{\infty} ||R_i||_1 = \sum_{i=0}^{\infty} \operatorname{tr}(R_i);$$

and set $\Lambda(\mathcal{V}_{\mathbf{w}}) = \{\mathbf{T} \in \ell_1(B_1^+[H'']) : T_i = \mathcal{E}(u_i \circ u_i) \ \ \forall i \geq 0 \ \text{ for some } \ \mathbf{u} \in \mathcal{V}_{\mathbf{w}}\}$, so that $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$ if and only if $\mathbf{R} \in \Lambda(\mathcal{V}_{\mathbf{w}})$. Therefore

$$(5) \qquad ||\mathbf{z}||^2_{\ell_2(\mathcal{H}\oplus\mathcal{H}')} = \sum_{i=0}^{\infty} ||z_i||^2_{\mathcal{H}\oplus\mathcal{H}'} = \sum_{i=0}^{\infty} (\varepsilon\langle Mx_i; x_i\rangle + \varepsilon\langle NKx_i; Kx_i\rangle)$$
$$= \sum_{i=0}^{\infty} \operatorname{tr}(Q_i(M + K^*NK)) = J_{\mathbf{R}}(K)$$

for every $\mathbf{R} \in \Lambda(\mathcal{V}_{\mathbf{w}})$ and $K \in \Gamma_B$ (cf. (4-a)), and hence $\mathbf{z} = \{z_i \in \mathcal{H}\oplus\mathcal{H}';\ i \geq 0\} \in \ell_2(\mathcal{H}\oplus\mathcal{H}')$ whenever $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$ and $K \in \Gamma_B$. Conclusion: the optimization problem in (2.a) can be equivalently stated as follows. Find $K \in \Gamma_B$ that minimizes

$$(2.b) \qquad \sup_{\mathbf{O}\neq\mathbf{R}\in\Lambda(\mathcal{V}_{\mathbf{w}})} \frac{J_{\mathbf{R}}(K)}{||\mathbf{R}||_1}.$$

Note that $\Lambda(\mathcal{V}_{\mathbf{w}}) \setminus \{\mathbf{O}\} \neq \emptyset$ since $\mathcal{V}_{\mathbf{w}} \setminus \{\mathbf{0}\} \neq \emptyset$. In other words (cf. (4-b)), find $K_P \in \Gamma_B$ such that

$$(2.c) \qquad J_{\Lambda(\mathcal{V}_{\mathbf{w}})}(K_P) \leq J_{\Lambda(\mathcal{V}_{\mathbf{w}})}(K)$$

for all $K \in \Gamma_B$.

6. AUXILIARY RESULTS

Propositions 1 to 4 below will be required for proving the main results of the next section. Throughout the remaining of the paper we shall refer to the setup of the previous sections.

Proposition 1. *Take $Q \in B[H]$ arbitrary. For any $P \in B^+[H]$ set*

$$K_P = (B^*PB + N)^{-1}B^*PF$$

in $B[H, H']$. The following assertions are equivalent.

(a) $\quad Q + K_P^*NK_P = P - \mathcal{F}^{\#}_{BK_P}(P).$

(b) $\quad Q + K^*NK = P - \mathcal{F}^{\#}_{BK}(P) + (K - K_P)^*(B^*PB + N)(K - K_P)$ *for an arbitrary* $K \in B[H, H']$.

(c) $\qquad Q = P - \mathcal{F}^{\#}(P) + K_P^*(B^*PB + N)K_P.$

Proof. First note that $(B^*PB+N) \in G^+[H']$ for any $P \in B^+[H]$ since $N \in G^+[H']$, so that $K_P \in B[H,H']$ is well defined. A trivial but somewhat lengthy algebraic manipulation leads to

$$\mathcal{F}^{\#}_{BK}(P) + K^*NK = \mathcal{F}^{\#}_{BK_P}(P) + K_P^*NK_P + (K-K_P)^*(B^*PB+N)(K-K_P)$$

for any $K \in B[H,H']$. Thus (a) implies (b). Moreover, recalling that $\mathcal{F}^{\#} = \mathcal{F}^{\#}_{BO}$, we get: (b) implies (c) trivially (set $K = O$ in (b)) and, according to the above identity with $K = O$, (c) implies (a). $\square$

Proposition 2. *Take H, H', $B \in B[H',H]$, and $K \in B[H,H']$ arbitrary. Suppose $r(\mathcal{F}^{\#}_{BK}) < 1$. If*

(6) $$K^*K = P - \mathcal{F}^{\#}(P)$$

holds for some $P \in B^+[H]$, then $r(\mathcal{F}^{\#}) < 1$.

Proof. First note that (recall: $F_{BK} = F - BK$)

$$\mathcal{F}_{BK}(P) = \mathcal{F}(P) - F_{BK}P(BK)^* - (BK)PF_{BK}^* - (BK)P(BK)^*$$

for any $P \in B[H]$ and $K \in B[H,H']$. Now take $P_0 \in B_1^+[H]$ and $\alpha > 0$ arbitrary, and consider the sequences $\{P_i \in B[H];\ i \geq 0\}$ and $\{\widehat{P}_i \in B[H];\ i \geq 0\}$ recursively defined as follows.

$$P_{i+1} = \mathcal{F}(P_i) = \mathcal{F}_{BK}(P_i) + F_{BK}P_i(BK)^* + (BK)P_iF_{BK}^* + (BK)P_i(BK)^*,$$

$$\widehat{P}_{i+1} = (1+\alpha)\mathcal{F}_{BK}(\widehat{P}_i) + (1+\alpha^{-1})(BK)P_i(BK)^*, \qquad \widehat{P}_0 = P_0.$$

Recall that $B_1[H]$ and $B^+[H]$ are invariant under $\mathcal{F} \in B[B[H]]$, and so are they under $\mathcal{T} \in B[B[H]]$ and $\mathcal{F}_{BK} \in B[B[H]]$ (cf. [7]). Thus $P_i \in B_1^+[H]$ and $\widehat{P}_i \in B_1^+[H]$ for every $i \geq 0$ by induction. Moreover, for each $i \geq 0$,

$$O \leq P_i \leq \widehat{P}_i.$$

Indeed (recall: $\mathcal{F}_{BK}(P) = F_{BK}PF_{BK}^* + \mathcal{T}(P)$ for all $P \in B[H]$)

$$\begin{aligned} O &\leq \left(\alpha^{\frac{1}{2}}F_{BK} - \alpha^{-\frac{1}{2}}BK\right)P_i\left(\alpha^{\frac{1}{2}}F_{BK} - \alpha^{-\frac{1}{2}}BK\right)^* + \alpha\mathcal{T}(P_i) \\ &= (1+\alpha)\mathcal{F}_{BK}(P_i - \widehat{P}_i) + \widehat{P}_{i+1} - P_{i+1}, \end{aligned}$$

so that $(1+\alpha)\mathcal{F}_{BK}(\widehat{P}_i - P_i) \leq \widehat{P}_{i+1} - P_{i+1}$, and hence $O \leq \widehat{P}_i - P_i$, for every $i \geq 0$ by induction. Thus

$$||P_i||_1 \leq ||\widehat{P}_i||_1$$

for each $i \geq 0$, since $O \leq \mathrm{tr}(T) = ||T||_1$ for any $T \in B_1^+[H]$ and $\mathrm{tr} : B_1[H] \to \mathbb{C}$ is linear. However, by setting

$$U_0 = P_0 \qquad \text{and} \qquad U_{i+1} = (1 + \alpha^{-1})(BK)P_i(BK)^*$$

in $B_1^+[H]$, it follows that

$$\widehat{P}_i = \sum_{j=0}^{i} (1+\alpha)^j \mathcal{F}_{BK}^j (U_{i-j}).$$

Therefore

$$||\widehat{P}_i||_1 \leq \sum_{j=0}^{i} \left\| \left[(1+\alpha)\mathcal{F}_{BK}^{\#}\right]^j \right\| \, ||U_{i-j}||_1$$

for every $i \geq 0$, since $||\mathcal{F}_{BK}^j(T)||_1 \leq ||\mathcal{F}_{BK}^{\#^j}|| \; ||T||_1$ for each $j \geq 0$ and every $T \in B_1[H]$ (cf. Property P_5 in [7]). If $r(\mathcal{F}_{BK}^{\#}) < 1$ then we can take $\alpha > 0$ small enough so that $r((1+\alpha)\mathcal{F}_{BK}^{\#}) = (1+\alpha)r(\mathcal{F}_{BK}^{\#}) < 1$ and hence (see e.g. [4])

$$\sum_{j=0}^{\infty} \left\| \left[(1+\alpha)\mathcal{F}_{BK}^{\#}\right]^j \right\| < \infty.$$

Moreover, if (6) holds for some $P \in B^+[H]$, then

$$\begin{aligned}
&||KP_jK^*||_1 = \mathrm{tr}(K\mathcal{F}^j(P_0)K^*) = \mathrm{tr}(K^*K\mathcal{F}^j(P_0)) \\
&= \mathrm{tr}(P\mathcal{F}^j(P_0)) - \mathrm{tr}(\mathcal{F}^{\#}(P)\mathcal{F}^j(P_0)) = \mathrm{tr}(P\mathcal{F}^j(P_0)) - \mathrm{tr}(P\mathcal{F}^{j+1}(P_0)) \\
&= \mathrm{tr}(P^{\frac{1}{2}}\mathcal{F}^j(P_0)P^{\frac{1}{2}}) - \mathrm{tr}(P^{\frac{1}{2}}\mathcal{F}^{j+1}(P_0)P^{\frac{1}{2}}) = ||P^{\frac{1}{2}}\mathcal{F}^j(P_0)P^{\frac{1}{2}}||_1 - ||P^{\frac{1}{2}}\mathcal{F}^{j+1}(P_0)P^{\frac{1}{2}}||_1
\end{aligned}$$

for each $j \geq 0$ (cf. Proposition P_4 in [7]); so that

$$\begin{aligned}
\sum_{j=0}^{n} ||U_{j+1}||_1 &\leq (1+\alpha^{-1})||B||^2 \sum_{j=0}^{n} ||KP_jK^*||_1 \\
&= (1+\alpha^{-1})||B||^2 \left(||P^{\frac{1}{2}}P_0P^{\frac{1}{2}}||_1 - ||P^{\frac{1}{2}}\mathcal{F}^{n+1}(P_0)P^{\frac{1}{2}}||_1 \right) \\
&\leq (1+\alpha^{-1})||B||^2 ||P^{\frac{1}{2}}P_0P^{\frac{1}{2}}||_1
\end{aligned}$$

for every $n \geq 0$. Since $\{||[(1+\alpha)\mathcal{F}_{BK}^{\#}]^j||;\ j \geq 0\} \in \ell_1$ and $\{||U_j||_1;\ j \geq 0\} \in \ell_1$, their convolution (or their Cauchy product) $\{\sum_{j=0}^{i} ||[(1+\alpha)\mathcal{F}_{BK}^{\#}]^j||\; ||U_{i-j}||_1;\ i \geq 0\}$ lies in ℓ_1 as well (cf. [2, p.77]). Therefore

$$\sum_{i=0}^{\infty} ||\widehat{P}_i||_1 < \infty,$$

so that

$$\sum_{i=0}^{\infty} ||\mathcal{F}^i(P_0)||_1 = \sum_{i=0}^{\infty} ||P_i||_1 \leq \sum_{i=0}^{\infty} ||\widehat{P}_i||_1 < \infty$$

for all $P_0 \in B_1^+[H]$. Thus $r(\mathcal{F}^\#) < 1$ (cf. Proof of Lemma 2 in [7]). $\square$

Proposition 3. *Take H, H', $B \in B[H', H]$, and $K \in B[H, H']$ arbitrary. Suppose* $r(\mathcal{F}^\#_{BK}) < 1$.

(a) *For each $Q \in B[H]$ there exists a unique solution $P \in B[H]$ to the Lyapunov-like equation*

$$Q = P - \mathcal{F}^\#_{BK}(P). \tag{7}$$

(b) *Take $Q \in B^+[H]$ and $P \in B[H]$ arbitrary. If* (7) *holds then $P \in B^+[H]$.*

Proof. (a) Take $Q \in B[H]$ arbitrary, and note that

$$\begin{aligned} Q - \mathcal{F}^{\#^{n+1}}_{BK}(Q) &= \sum_{i=0}^{n} \left(\mathcal{F}^{\#^i}_{BK}(Q) - \mathcal{F}^{\#^{i+1}}_{BK}(Q) \right) \\ &= \sum_{i=0}^{n} \mathcal{F}^{\#^i}_{BK}(Q) - \mathcal{F}^\#_{BK}\left(\sum_{i=0}^{n} \mathcal{F}^{\#^i}_{BK}(Q) \right) \end{aligned}$$

for every $n \geq 0$, since $\mathcal{F}^\#_{BK} : B[H] \to B[H]$ is linear. Suppose $r(\mathcal{F}^\#_{BK}) < 1$. Then, as is well known, $\{\sum_{i=0}^n \mathcal{F}^{\#^i}_{BK} \in B[B[H]];\ n \geq 0\}$ converges in $B[B[H]]$ (see e.g. [1], [7]). Thus, given $Q \in B[H]$, $\{\sum_{i=0}^n \mathcal{F}^{\#^i}_{BK}(Q) \in B[H];\ n \geq 0\}$ converges in $B[H]$ to, say, $P_Q = \sum_{i=0}^\infty \mathcal{F}^{\#^i}_{BK}(Q) \in B[H]$, and hence $\mathcal{F}^{\#^n}_{BK}(Q) \to O$ in $B[H]$ as $n \to \infty$. Therefore, according to the above equation,

$$Q = P_Q - \mathcal{F}^\#_{BK}(P_Q),$$

since $\mathcal{F}^\#_{BK} : B[H] \to B[H]$ is continuous. That is, $P_Q \in B[H]$ is a solution to (7). On the other hand, if $P \in B[H]$ is a solution to (7) for a given $Q \in B[H]$, then

$$\sum_{i=0}^{n} \mathcal{F}^{\#^i}_{BK}(Q) = \sum_{i=0}^{n} \left(\mathcal{F}^{\#^i}_{BK}(P) - \mathcal{F}^{\#^{i+1}}_{BK}(P) \right) = P - \mathcal{F}^{\#^{n+1}}_{BK}(P)$$

for every $n \geq 0$. Therefore, since $r(\mathcal{F}^\#_{BK}) < 1$,

$$0 \leq \left\| \sum_{i=0}^{n} \mathcal{F}^{\#^i}_{BK}(Q) - P \right\| = \left\| \mathcal{F}^{\#^{n+1}}_{BK}(P) \right\| \leq \left\| \mathcal{F}^{\#^{n+1}}_{BK} \right\| \, ||P|| \to 0 \quad \text{as} \quad n \to \infty,$$

and hence $P = P_Q$ by uniqueness of the limit.

(b) If (7) holds for some $Q \in B^+[H]$ and $P \in B[H]$, then

$$O \leq \sum_{i=0}^{n} \mathcal{F}^{\#^i}_{BK}(Q) = P - \mathcal{F}^{\#^{n+1}}_{BK}(P)$$

for each $n \geq 0$, since $B^+[H]$ is invariant under $\mathcal{F}^{\#}_{BK} : B[H] \to B[H]$ (cf. [7]). Thus

$$\mathcal{F}^{\#^{n+1}}_{BK}(P) \leq P$$

for every $n \geq 0$. Since $r(\mathcal{F}^{\#}_{BK}) < 1$ it follows that $\mathcal{F}^{\#^{n}}_{BK} \to \mathcal{O}$ in $B[B[H]]$ as $n \to \infty$, and hence $\mathcal{F}^{\#^{n}}_{BK}(P) \to O$ in $B[H]$ as $n \to \infty$. Therefore $O \leq P$. $\square$

Proposition 4. *Suppose there exists* $\widehat{B} \in B[H]$ *such that* $r(\mathcal{F}^{\#}_{\widehat{B}M}) < 1$ *and take* $Q \in B^+[H]$ *arbitrary. If*

$$M + Q + K^*NK = P - \mathcal{F}^{\#}_{BK}(P) \tag{8}$$

holds for some $P \in B^+[H]$, *then* $r(\mathcal{F}^{\#}_{BK}) < 1$.

Proof. Consider the separable Hilbert space $\widetilde{H}$ obtained by the following direct (orthogonal) sum: $\widetilde{H} = H \oplus H \oplus H'$. Now set

$$\widetilde{K} = \begin{pmatrix} M^{\frac{1}{2}} \\ Q^{\frac{1}{2}} \\ N^{\frac{1}{2}}K \end{pmatrix} \in B[H, \widetilde{H}],$$

$$\widetilde{B} = \begin{pmatrix} \widehat{B}M^{\frac{1}{2}} & O & -BN^{-\frac{1}{2}} \end{pmatrix} \in B[\widetilde{H}, H].$$

Since $\widetilde{B}\widetilde{K} = \widehat{B}M - BK$ we get $F_{BK} - \widetilde{B}\widetilde{K} = F - \widehat{B}M$, and hence

$$(\mathcal{F}^{\#}_{BK})_{\widetilde{B}\widetilde{K}} = \mathcal{F}^{\#}_{\widehat{B}M}.$$

Here $(\mathcal{F}^{\#}_{BK})_{\widetilde{B}\widetilde{K}} \in B[B[H]]$ is obtained from $\mathcal{F}^{\#}_{BK}$ as $\mathcal{F}^{\#}_{BK}$ is itself obtained from $\mathcal{F}^{\#}$. In other words,

$$(\mathcal{F}^{\#}_{BK})_{\widetilde{B}\widetilde{K}}(P) = (F_{BK} - \widetilde{B}\widetilde{K})^* P (F_{BK} - \widetilde{B}\widetilde{K}) + \mathcal{T}^{\#}(P)$$

for all $P \in B[H]$. Therefore, since $r(\mathcal{F}^{\#}_{\widehat{B}M}) < 1$,

$$r((\mathcal{F}^{\#}_{BK})_{\widetilde{B}\widetilde{K}}) < 1.$$

Moreover, if (8) holds for some $P \in B^+[H]$, then

$$\widetilde{K}^*\widetilde{K} = P - \mathcal{F}^{\#}_{BK}(P),$$

for $\widetilde{K}^*\widetilde{K} = M + Q + K^*NK$. Thus $r(\mathcal{F}^{\#}_{BK}) < 1$ according to Proposition 2. $\square$

7. MAIN RESULTS

Under appropriate conditions a solution to the quadratic stochastic optimal control problem posed in (2) is given in terms of the unique nonnegative solution to the associated Riccati-like operator equation. This will be properly stated in the theorem below whose proof relies on the following lemma.

Lemma. *Suppose there exists $\widehat{B} \in B[H]$ such that $r(\mathcal{F}^{\#}_{\widehat{B}M}) < 1$. If $\Gamma_B \neq \emptyset$ then there exists a unique nonnegative solution $P \in B^+[H]$ to the Riccati-like equation*

$$M = P - \mathcal{F}^{\#}(P) + F^* PB(B^* PB + N)^{-1} B^* PF. \tag{9}$$

Moreover $(B^* PB + N)^{-1} B^* PF \in \Gamma_B$.

Proof. Consider the above hypothesis. Claim: for each $i \geq 0$ there exists $P_i \in B^+[H]$ and $K_i \in \Gamma_B$ such that

$$M + K_i^* N K_i = P_i - \mathcal{F}^{\#}_{BK_i}(P_i). \tag{10}$$

Indeed, take $K_0 \in \Gamma_B$ arbitrary so that $r(\mathcal{F}^{\#}_{BK_0}) < 1$. Thus, according to Proposition 3, there exists a unique $P_0 \in B[H]$ which in fact lies in $B^+[H]$ such that

$$M + K_0^* N K_0 = P_0 - \mathcal{F}^{\#}_{BK_0}(P_0).$$

Hence the claimed result holds for $i = 0$. Suppose it holds for some $i \geq 0$. Set

$$K_{i+1} = (B^* P_i B + N)^{-1} B^* P_i F$$

in $B[H, H']$. By adding $(K_i - K_{i+1})^*(B^* P_i B + N)(K_i - K_{i+1})$ at both sides of equation (10), and recalling that (b) $\Longrightarrow$ (a) in Proposition 1, we get

$$M + (K_i - K_{i+1})^*(B^* P_i B + N)(K_i - K_{i+1}) + K_{i+1}^* N K_{i+1} = P_i - \mathcal{F}^{\#}_{BK_{i+1}}(P_i).$$

Thus, according to Proposition 4, $r(\mathcal{F}^{\#}_{BK_{i+1}}) < 1$ (i.e. $K_{i+1} \in \Gamma_B$). Therefore, by Proposition 3, there exists a unique $P_{i+1} \in B[H]$ which in fact lies in $B^+[H]$ such that

$$M + K_{i+1}^* N K_{i+1} = P_{i+1} - \mathcal{F}^{\#}_{BK_{i+1}}(P_{i+1});$$

which concludes the proof of the claimed statement by induction. Note that from the above two equations we get

$$O \leq (K_i - K_{i+1})^*(B^* P_i B + N)(K_i - K_{i+1}) = (P_i - P_{i+1}) - \mathcal{F}^{\#}_{BK_{i+1}}(P_i - P_{i+1})$$

for every $i \geq 0$. Hence $(P_i - P_{i+1}) \in B^+[H]$, since $r(\mathcal{F}^{\#}_{BK_{i+1}}) < 1$, for each $i \geq 0$ (cf. Proposition 3-b). Thus $\{P_i;\ i \geq 0\}$ is a monotone bounded sequence of self-adjoint operators (actually a nonincreasing sequence of nonnegative operators), so that it converges strongly (see e.g. [9, p.79]). Let $P_\infty \in B[H]$ be its strong limit which in fact lies in $B^+[H]$ because $B^+[H]$ is weakly closed in $B[H]$. That is

$$P_i \xrightarrow{\ s\ } P_\infty, \tag{11}$$

so that $B^*P_iF \xrightarrow{\ s\ } B^*P_\infty F$, $F^*P_iB \xrightarrow{\ s\ } F^*P_\infty B$ and $(B^*P_iB+N) \xrightarrow{\ s\ } (B^*P_\infty B+N)$. Since $N \in G^+[H']$ there exists $\gamma > 0$ such that $\gamma I \leq N \leq B^*P_iB + N$ and hence $||(B^*P_iB+N)^{-1}|| < \gamma^{-1}$, for every $i \geq 0$. Therefore the strongly convergent sequence of invertible operators $\{(B^*P_iB+N);\ i \geq 0\}$ has a bounded inverse sequence $\{(B^*P_iB+N)^{-1};\ i \geq 0\}$. Thus $(B^*P_iB+N)^{-1} \xrightarrow{\ s\ } (B^*P_\infty B+N)^{-1}$ (see e.g. [9, p.105]). Hence

$$\begin{aligned}
K_{i+1} &= (B^*P_iB+N)^{-1}B^*P_iF \xrightarrow{\ s\ } K_\infty := (B^*P_\infty B+N)^{-1}B^*P_\infty F,\\
K_{i+1}^* &= F^*P_iB(B^*P_iB+N)^{-1} \xrightarrow{\ s\ } K_\infty^* = F^*P_\infty B(B^*P_\infty B+N)^{-1},
\end{aligned}$$

so that

$$K_i^*NK_i \xrightarrow{\ s\ } K_\infty^*NK_\infty, \tag{12}$$

$$(F-BK_i)^*P_i(F-BK_i) \xrightarrow{\ s\ } (F-BK_\infty)^*P_\infty(F-BK_\infty)$$

(see e.g. [9, p.80]). The above convergence leads to

$$\mathcal{F}^{\#}_{BK_i}(P_i) \xrightarrow{\ s\ } \mathcal{F}^{\#}_{BK_\infty}(P_\infty) \tag{13}$$

because $\mathcal{T}^{\#}(P_i) \xrightarrow{\ s\ } \mathcal{T}^{\#}(P_\infty)$. Indeed $\mathcal{T}^{\#} \in B[B[H]]$ is the limit in $B[B[H]]$ of $\{\mathcal{T}^{\#}_n \in B[B[H]];\ n \geq 1\}$ which is defined as follows: for each $n \geq 1$

$$\mathcal{T}^{\#}_n(P) = \sum_{k,\ell=1}^{n} \langle Ce_\ell; e_k\rangle A_\ell^* P A_k$$

for all $P \in B[H]$ (cf. [6]). Thus $\mathcal{T}^{\#}_n(P_i) \xrightarrow{\ s\ } \mathcal{T}^{\#}_n(P_\infty)$ for every $n \geq 1$ (i.e. $||\mathcal{T}^{\#}_n(P_i - P_\infty)h|| \to 0$ as $i \to \infty$ for every $n \geq 1$ and all $h \in H$). Hence, by the triangle inequality in H,

$$\begin{aligned}
0 \leq \limsup_{i\to\infty} ||\mathcal{T}^{\#}(P_i-P_\infty)h|| &\leq \limsup_{i\to\infty} ||(\mathcal{T}^{\#}_n - \mathcal{T}^{\#})(P_i-P_\infty)h||\\
+ \limsup_{i\to\infty} ||\mathcal{T}^{\#}_n(P_i-P_\infty)h|| &\leq ||\mathcal{T}^{\#}_n - \mathcal{T}^{\#}|| \sup_{i\geq 0} ||P_i - P_\infty||\,||h||
\end{aligned}$$

for every $n \geq 1$ an all $h \in H$. Therefore, since $||\mathcal{T}^{\#}_n - \mathcal{T}^{\#}|| \to 0$ as $n \to \infty$ and $\sup_{i\geq 0} ||P_i - P_\infty|| < \infty$ (for $P_i \xrightarrow{\ s\ } P_\infty$), it follows that $||\mathcal{T}^{\#}(P_i-P_\infty)h|| \to 0$ as $i \to \infty$ for all $h \in H$ (i.e. $\mathcal{T}^{\#}(P_i) \xrightarrow{\ s\ } \mathcal{T}^{\#}(P_\infty)$ as claimed above). Then, according to (10)–(13), $P_\infty \in B^+[H]$ is a solution to

$$M + K_\infty^*NK_\infty = P_\infty - \mathcal{F}^{\#}_{BK_\infty}(P_\infty) \tag{14}$$

so that $r(\mathcal{F}^{\#}_{BK_\infty}) < 1$ (i.e. $K_\infty \in \Gamma_B$) according to Proposition 4. Moreover $P_\infty \in B^+[H]$ is a solution to (9), for (a) $\Longrightarrow$ (c) in Proposition 1. On the other hand, if $P \in B^+[H]$ is a solution to (9), then it is a solution to

$$M + K_P^* N K_P = P - \mathcal{F}^{\#}_{BK_P}(P) \tag{15}$$

(cf. Proposition 1: (c) $\Longrightarrow$ (a)) with

$$K_P := (B^*PB + N)^{-1}B^*PF$$

in $B[H, H']$. Hence $r(\mathcal{F}^{\#}_{BK_P}) < 1$ (i.e. $K_P \in \Gamma_B$) by Proposition 4. From (15) and (14) it follows that

$$M + K_\infty^* N K_\infty = P - \mathcal{F}^{\#}_{BK_\infty}(P) + (K_\infty - K_P)^*(B^*PB + N)(K_\infty - K_P),$$
$$M + K_P^* N K_P = P_\infty - \mathcal{F}^{\#}_{BK_P}(P_\infty) + (K_P - K_\infty)^*(B^*P_\infty B + N)(K_P - K_\infty),$$

respectively, for (a) $\Longrightarrow$ (b) in Proposition 1. By subtracting the above two equations from (14) and (15), respectively, we get

$$O \le (K_\infty - K_P)(B^*PB + N)(K_\infty - K_P) = (P_\infty - P) - \mathcal{F}^{\#}_{BK_\infty}(P_\infty - P),$$
$$O \le (K_P - K_\infty)(B^*P_\infty B + N)(K_P - K_\infty) = (P - P_\infty) - \mathcal{F}^{\#}_{BK_P}(P - P_\infty).$$

Therefore $O \le (P_\infty - P) \le O$ according to Proposition 3-b. Thus $P = P_\infty$. $\square$

Definition. Take a separable Hilbert space H_0 and $L \in B[H, H_0]$ arbitrary. The pair $(L, \mathcal{F})$ is *detectable* if there exists $\widehat{B} \in B[H_0, H]$ such that $r(\mathcal{F}^{\#}_{\widehat{B}L}) < 1$.

This is the natural extension to discrete bilinear models as in (1) of the usual definition of detectability for discrete linear models: a pair (L, F) is detectable, with $L \in B[H, H_0]$ and $F \in B[H]$ for some separable Hilbert space H_0, if there exists $\widehat{B} \in B[H_0, H]$ such that $r(F - \widehat{B}L) < 1$. In fact, if the discrete bilinear model in (1) is particularized to a linear one (e.g. by setting $A_k = O$ for every $k \ge 1$), then $\mathcal{F}^{\#}_{\widehat{B}L}(P) = (F - \widehat{B}L)^* P (F - \widehat{B}L)$ for all $P \in B[H]$ so that (see e.g. [1]) $r(\mathcal{F}^{\#}_{\widehat{B}L}) = r((F - \widehat{B}L)^*)^2 = r(F - \widehat{B}L)^2 = r(\mathcal{F}_{\widehat{B}L})$. It is worth noting that the identity $r(\mathcal{F}^{\#}_{BK}) = r(\mathcal{F}_{BK})$, which holds for the linear case, does not necessarily hold for the bilinear case (cf. Remark 1).

Using the above definition and applying the preceding lemma we get the following full statement for the solution to the quadratic optimal control problem posed in (2), which mirror its linear counterpart (see e.g. [3]).

Theorem. *If $(M, \mathcal{F})$ is detectable and Γ_B is not empty, then there exists* $K_P \in \Gamma_B$ such that

$$J_{\Lambda(\mathcal{V}_{\mathrm{w}})}(K_P) = \min_{K \in \Gamma_B} J_{\Lambda(\mathcal{V}_{\mathrm{w}})}(K).$$

Moreover $K_P = (B^*PB + N)^{-1}B^*PF$ *where* $P \in B^+[H]$ *is the unique nonnegative solution to the algebraic Riccati-like operator equation*

$$M = P - \mathcal{F}^{\#}(P) + F^*PB(B^*PB + N)^{-1}B^*PF,$$

and $J_{\Lambda(\mathcal{V}_{\mathbf{w}})}(K_P) = ||D^*PD||$.

Proof. The previous lemma ensures the existence of a unique nonnegative solution $P \in B^+[H]$ to the above Riccati-like operator equation, and that $K_P := (B^*PB+N)^{-1}B^*PF \in \Gamma_B$. Thus

$$M + K^*NK = P - \mathcal{F}^{\#}_{BK}(P) + (K - K_P)^*(B^*PB + N)(K - K_P)$$

for any $K \in B[H, H']$ (since (c) $\Rightarrow$ (b) in Proposition 1), so that

$$\begin{aligned}\text{tr}(Q_i(M + K^*NK)) &= \text{tr}(Q_iP) - \text{tr}(Q_i\mathcal{F}^{\#}_{BK}(P)) \\ &\quad + \text{tr}(Q_i(K - K_P)^*(B^*PB + N)(K - K_P))\end{aligned} \tag{16}$$

for every $i \geq 0$ and any $K \in B[H, H']$. However (cf. Property P_4 in [7])

$$\begin{aligned}\text{tr}(Q_iP) - \text{tr}(Q_i\mathcal{F}^{\#}_{BK}(P)) &= \text{tr}(Q_iP) - \text{tr}(\mathcal{F}_{BK}(Q_i)P) \\ &= \text{tr}(Q_iP) - \text{tr}(Q_{i+1}P) + \text{tr}(DR_{i+1}D^*P)\end{aligned}$$

for every $i \geq 0$ and any $K \in B[H, H']$ whenever $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$, according to (3-a). Hence

$$\begin{aligned}0 \leq \sum_{i=0}^{\infty}(\text{tr}(Q_iP) - \text{tr}(Q_i\mathcal{F}^{\#}_{BK}(P)) &= \text{tr}(Q_0P) - \lim_{n\to\infty}\text{tr}(Q_nP) + \sum_{i=0}^{\infty}\text{tr}(DR_{i+1}D^*P) \\ = \text{tr}(DR_0D^*P) + \sum_{i=1}^{\infty}\text{tr}(DR_iD^*P) &= \sum_{i=0}^{\infty}\text{tr}(D^*PDR_i) \leq ||D^*PD|| \;\; ||\mathbf{R}||_1\end{aligned} \tag{17}$$

for any $K \in \Gamma_B$ and $\mathbf{R} \in \Lambda(\mathcal{V}_{\mathbf{w}})$, since $\lim_{n\to\infty}\text{tr}(Q_nP) = 0$ whenever $K \in \Gamma_B$ and $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$. Actually $0 \leq \text{tr}(Q_nP) \leq ||Q_n||_1||P||$ for every $n \geq 0$ and $||Q_n||_1 \to 0$ as $n \to \infty$ because $\mathbf{Q} := \{Q_i \in B_1^+[H];\ i \geq 0\} \in \ell_1(B_1^+[H])$ whenever $K \in \Gamma_B$ and $\mathbf{v} \in \mathcal{V}_{\mathbf{w}}$. Indeed (cf. (3-b) $||Q_i||_1 = \text{tr}(Q_i) = \sum_{j=0}^{i}\text{tr}(\mathcal{F}^{i-j}_{BK}(DR_jD^*)) = \sum_{j=0}^{i}\text{tr}(DR_jD^*\mathcal{F}^{\#^{i-j}}_{BK}(I))$ for every $i \geq 0$, so that $||\mathbf{Q}||_1 = \sum_{i=0}^{\infty}||Q_i||_1 \leq (||D||^2\sum_{i=0}^{\infty}||\mathcal{F}^{\#^i}_{BK}||)||\mathbf{R}||_1$ for $K \in \Gamma_B$ and $\mathbf{R} \in \Lambda(\mathcal{V}_{\mathbf{w}})$, by using the very same arguments put forward in Section 4. Thus $\mathbf{Q} \in \ell_1(B_1^+[H])$ and we can also ensure convergence for the series

$$\begin{aligned}0 &\leq \sum_{i=0}^{\infty}\text{tr}(Q_i(K - K_P)^*(B^*PB + N)(K - K_P)) \\ &\leq ||(K - K_P)^*(B^*PB + N)(K - K_P)|| \;\; ||\mathbf{Q}||_1 \\ &\leq \left(||(K - K_P)^*(B^*PB + N)(K - K_P)|| \;\; ||D||^2\sum_{i=0}^{\infty}\left\|\mathcal{F}^{\#^i}_{BK}\right\|\right)||\mathbf{R}||_1\end{aligned} \tag{18}$$

whenever $K \in \Gamma_B$ and $\mathbf{R} \in \Lambda(\mathcal{V}_\mathbf{w})$. Therefore, according to (5) and (16)–(18),

$$\begin{aligned} J_\mathbf{R}(K) &= \sum_{i=0}^{\infty} \operatorname{tr}(Q_i(M + K^*NK)) \\ &= \sum_{i=0}^{\infty} \operatorname{tr}(DR_iD^*P) + \sum_{i=0}^{\infty} \operatorname{tr}(Q_i(K - K_P)^*(B^*PB + N)(K - K_P)) \\ &\le \left(||D^*PD|| + ||(K - K_P)^*(B^*PB + N)(K - K_P)|| \;\; ||D||^2 \sum_{i=0}^{\infty} \left\| \mathcal{F}_{BK}^{\#^i} \right\| \right) ||\mathbf{R}||_1 \end{aligned}$$

for every $K \in \Gamma_B$ and $\mathbf{R} \in \Lambda(\mathcal{V}_\mathbf{w})$. Hence

$$\sum_{i=0}^{\infty} \operatorname{tr}(DR_iD^*P) = J_\mathbf{R}(K_P) \le J_\mathbf{R}(K)$$

for any $K \in \Gamma_B$ and $\mathbf{R} \in \Lambda(\mathcal{V}_\mathbf{w})$, and (cf. (4-b))

$$\begin{aligned} J_{\Lambda(\mathcal{V}_\mathbf{w})}(K) &= \sup_{\mathbf{O} \neq \mathbf{R} \in \Lambda(\mathcal{V}_\mathbf{w})} \frac{J_\mathbf{R}(K)}{||\mathbf{R}||_1} \\ &\le ||D^*PD|| + ||(K - K_P)^*(B^*PB + N)(K - K_P)| \;\; ||D||^2 \sum_{i=0}^{\infty} \left\| \mathcal{F}_{BK}^{\#^i} \right\| \end{aligned}$$

for every $K \in \Gamma_B$. Thus

$$\sup_{\mathbf{O} \neq \mathbf{R} \in \Lambda(\mathcal{V}_\mathbf{w})} \frac{\sum_{i=0}^{\infty} \operatorname{tr}(DR_iD^*P)}{||\mathbf{R}||_1} = J_{\Lambda(\mathcal{V}_\mathbf{w})}(K_P) \le J_{\Lambda(\mathcal{V}_\mathbf{w})}(K)$$

for all $K \in \Gamma_B$, and

$$J_{\Lambda(\mathcal{V}_\mathbf{w})}(K_P) \le ||D^*PD||.$$

To complete the proof we still need to verify that $||D^*PD|| \le J_{\Lambda(\mathcal{V}_\mathbf{w})}(K_P)$. For that we proceed as follows. Take $h \in H''$ with $||h|| = 1$ arbitrary, and consider an orthonormal basis $\{h_k;\ k \ge 0\}$ for H'' such that $h_0 = h$. Now take $\mathbf{R}(h) = (R_0(h), O, O, \cdots)$ from $\Lambda(\mathcal{V}_\mathbf{w})$ with $R_0(h) = \operatorname{diag}(1, 0, 0, \cdots)$ (i.e. take $\mathbf{R}(h) = \{R_i(h) \in B_1^+[H''];\ i \ge 0\} \in \Lambda(\mathcal{V}_\mathbf{w})$ where $R_i(h) = O$ for every $i \ge 1$ and $R_0(h) = (h \circ h)$). Note that

$$||\mathbf{R}(h)||_1 = \sum_{i=0}^{\infty} ||R_i(h)||_1 = ||R_0(h)||_1 = \operatorname{tr}(R_0(h)) = \sum_{k=0}^{\infty} \langle R_0(h)h_k; h_k \rangle = 1,$$

and

$$\begin{aligned} \sum_{i=0}^{\infty} \operatorname{tr}(DR_i(h)D^*P) &= \operatorname{tr}(R_0(h)D^*PD) = \sum_{k=0}^{\infty} \langle R_0(h)D^*PDh_k; h_k \rangle \\ &= \sum_{k=0}^{\infty} \langle D^*PDh_k; R_0(h)h_k \rangle = \langle D^*PDh_0; h_0 \rangle = ||P^{\frac{1}{2}}Dh||^2. \end{aligned}$$

Hence

$$J_{\Lambda(\mathcal{V}_{\mathbf{w}})}(K_P) = \sup_{\mathbf{O}\neq\mathbf{R}\in\Lambda(\mathcal{V}_{\mathbf{w}})} \frac{\sum_{i=0}^{\infty}\mathrm{tr}(DR_iD^*P)}{||\mathbf{R}||_1} \geq \sup_{||h||=1} ||P^{\frac{1}{2}}Dh||^2 = ||P^{\frac{1}{2}}D||^2 = ||D^*PD||.$$

□

Remark 2. We actually have proved a bit more than the theorem statement, viz.

$$J_{\mathbf{R}}(K_P) = \min_{K\in\Gamma_B} J_{\mathbf{R}}(K) = \sum_{i=0}^{\infty}\mathrm{tr}(DR_iD^*P)$$

for any $\mathbf{R}\in\Lambda(\mathcal{V}_{\mathbf{w}})$. Therefore, for any $\mathbf{v}\in\mathcal{V}_{\mathbf{w}}$, K_P minimizes $||\mathbf{z}||_{\ell_2(\mathcal{H}\oplus\mathcal{H}')}$ over all $K\in\Gamma_B$.

8. CONCLUDING REMARKS

In this final section we shall consider a recursive approximation scheme for the optimization problem as in Remark 2. Precisely, we shall show how the previous time-invariant infinite-time optimal control problem can be strongly approximated by a sequence of time-varying finite-time ones. This, which will be stated below as a corollary to the preceding theorem, again mirror the linear case.

Take an arbitrary integer $n\geq 1$. Consider a time-varying version of (1) with $K\in B[H,H']$ replaced by $K_i\in B[H,H']$,

$$\widehat{x}_{i+1} = \left[A_0 + \sum_{k=1}^{\infty}\langle w_i; e_k\rangle A_k\right]\widehat{x}_i - BK_i\widehat{x}_i + Dv_{i+1}, \qquad \widehat{x}_0 = x_0 = Dv_0,$$

for each $i=0,\cdots,n-1$; whose finite sequence $\{\widehat{Q}_i = \mathcal{E}(\widehat{x}_i\circ\widehat{x}_i)\in B_1^+[H];\ i=0,\cdots,n\}$ of state correlations is such that

$$\widehat{Q}_{i+1} = \mathcal{F}_{BK_i}(\widehat{Q}_i) + DR_{i+1}D^*, \qquad \widehat{Q}_0 = Q_0 = DR_0D^*$$

(see Section 3). Now, given $\mathbf{v}\in\mathcal{V}_{\mathbf{w}}$ and $W\in B^+[H]$ arbitrary, set

$$(19)\qquad J_{\mathbf{R}}^{W}(n) = \inf_{\{K_i;\ i=0,\cdots,n-1\}}\left(\sum_{i=0}^{n-1}(\varepsilon\langle M\widehat{x}_i;\widehat{x}_i\rangle + \varepsilon\langle NK_i\widehat{x}_i;K_i\widehat{x}_i\rangle) + \varepsilon\langle W\widehat{x}_n;\widehat{x}_n\rangle\right)$$
$$= \inf_{\{K_i;\ i=0,\cdots,n-1\}}\left(\sum_{i=0}^{n-1}\mathrm{tr}(\widehat{Q}_i(M+K_i^*NK_i) + \mathrm{tr}(\widehat{Q}_nW)\right)$$

(cf. eq. (5)). Next consider a finite sequence in $B[H]$ backward recursively defined as follows: $P_W(n,n)=W$,

$$(20)\qquad P_W(i,n) = \mathcal{F}^{\#}_{BK_i(n)}(P_W(i+1,n)) + K_i^*(n)NK_i(n) + M,$$

with $K_i(n) \in B[H, H']$ given by

$$K_i(n) = (B^* P_W(i+1,n)B + N)^{-1} B^* P_W(i+1,n)F,$$

for each $i = 0, \cdots, n-1$. It is readily verified by induction on i that $P_W(i,n) \in B^+[H]$ (since $B^+[H]$ is invariant under $\mathcal{F}^{\#}_{BK} : B[H] \to B[H]$ for any $K \in B[H,H']$ - cf. [7]) and that $K_i(n)$ is well defined (i.e. $(B^* P_W(i+1,n)B + N) \in G^+[H']$), for every $i = 0, \cdots, n-1$.

Corollary. *Take* $\mathbf{R} \in \Lambda(\mathcal{V}_{\mathbf{w}})$ *and* $W \in B^+[H]$ *arbitrary. For each* $n \geq 1$ *set* $J^W_{\mathbf{R}}(n)$ *as in* (19) *and consider the finite sequence* $\{P_W(i,n) \in B^+[H];\ i = 0, \cdots, n\}$ *defined in* (20). *Then*

$$J^W_{\mathbf{R}}(n) = \sum_{i=0}^{n} \operatorname{tr}(DR_i D^* P_W(i,n)).$$

Moreover, under the assumptions of Lemma,

$$P_W(i,n) \xrightarrow{s} P \qquad \text{as} \qquad n \to \infty$$

for each $i \geq 0$, *where* $P \in B^+[H]$ *is the unique nonnegative solution to* (9), *and*

$$J^W_{\mathbf{R}}(n) \longrightarrow \inf_{K \in \Gamma_B} J_{\mathbf{R}}(K) = J_{\mathbf{R}}(K_P) = \sum_{i=0}^{\infty} \operatorname{tr}(DR_i D^* P) \qquad \text{as} \qquad n \to \infty.$$

Proof. Take $n \geq 1$ arbitrary. Recalling that (a) $\Longrightarrow$ (b) in Proposition 1 we get from (20) that

$$\begin{aligned} M + K_i^* N K_i = {} & P_W(i,n) - \mathcal{F}^{\#}_{BK_i}(P_W(i+1,n)) \\ & + (K_i - K_i(n))^*(B^* P_W(i+1,n)B + N)(K_i - K_i(n)) \end{aligned}$$

for each $i = 0, \cdots, n-1$. Therefore

$$\begin{aligned} \sum_{i=0}^{n-1} \operatorname{tr}(\widehat{Q}_i(M + K_i^* N K_i)) = {} & \operatorname{tr}(\widehat{Q}_0 P_W(0,n)) - \operatorname{tr}(\widehat{Q}_n P_W(n,n)) \\ & + \sum_{i=0}^{n-1} \operatorname{tr}(DR_{i+1} D^* P_W(i+1,n)) \\ & + \sum_{i=0}^{n-1} \operatorname{tr}(\widehat{Q}_i (K_i - K_i(n))^*(B^* P_W(i+1,n)B^* + N)(K_i - K_i(n)) \end{aligned}$$

(reason: recall that (cf. Property P_4 in [7]) $\operatorname{tr}(\widehat{Q}_i \mathcal{F}^{\#}_{BK_i}(P_W(i+1,n))) = \operatorname{tr}(\mathcal{F}_{BK_i}(\widehat{Q}_i) P_W(i+1,n)) = \operatorname{tr}(\widehat{Q}_{i+1} P_W(i+1,n)) - \operatorname{tr}(DR_{i+1} D^* P_W(i+1,n))$). Since $P_W(n,n) = W \in B^+[H]$ and $\widehat{Q}_0 = DR_0 D^*$, it follows that

$$J^W_{\mathbf{R}}(n) = \sum_{i=0}^{n} \operatorname{tr}(DR_i D^* P_W(i,n))$$

according to (19). Now note from (5) and (19) that, for $W = O$,

$$0 \leq J^O_{\mathbf{R}}(n) \leq J^O_{\mathbf{R}}(n+1) \leq \inf_{K \in \Gamma_B} J_{\mathbf{R}}(K) = J_{\mathbf{R}}(K_P), \tag{21}$$

so that

$$0 \leq \sum_{i=0}^{n} \operatorname{tr}(DR_iD^*P_O(i,n)) \leq \sum_{i=0}^{n+1} \operatorname{tr}(DR_iD^*P_O(i,n+1)) \leq \sum_{i=0}^{\infty} \operatorname{tr}(DR_iD^*P),$$

where $P \in B^+[H]$ is the unique nonnegative solution to (9), according to Remark 2. From now on take $i \in [0,n)$ arbitrary. Recall that both P and $P_W(i,n)$ do not depend on H'', $D \in B[H'',H]$ and $\mathbf{R} \in \Lambda(\mathcal{V}_{\mathbf{w}}) \subset \ell_1(B_1^+[H''])$. In particular, for $H'' = H$, $D = I$, and $\mathbf{R} = (O,\cdots,O,(r \circ r),O,\cdots) \in \Lambda(\mathcal{V}_{\mathbf{w}})$ with null operators except at the ith position and $r \in \mathcal{H}''$ arbitrary, the above inequalities lead to

$$O \leq P_O(i,n) \leq P_O(i,n+1) \leq P$$

(reason: $\operatorname{tr}((r \circ r)L) = \operatorname{tr}(r \circ L^*r) = \langle r; L^*r \rangle = \langle Lr; r \rangle$ for all $r \in H''$ and $L \in B[H'']$, so that $O \leq L$ whenever $0 \leq \operatorname{tr}((r \circ r)L)$ for all $r \in H''$ - see e.g. [5]). Hence

$$P_O(i,n) \xrightarrow{s} P_O(i,\infty) \qquad \text{as} \qquad n \to \infty$$

for some $P_O(i,\infty) \in B^+[H]$, since $\{P_O(i,n) \in B^+[H];\ n \geq 1\}$ is a monotone bounded sequence of self-adjoint operators and $B^+[H]$ is weakly closed in $B[H]$ (cf. proof of Lemma). Also note that $P_W(n-j,n) = P_W(n+1-j,n+1)$ for every $j = 0,\cdots,n$, which is readily verified by induction on j. In particular, for $j = n-i$, $P_W(i,n) = P_W(i+1,n+1)$. Thus

$$M + K_i^*(n)NK_i(n) = P_W(i,n) - \mathcal{F}^{\#}_{BK_i(n)}(P_W(i,n-1)),$$

$$K_i(n) = (B^*P_W(i,n-1)B + N)^{-1}B^*P_W(i,n-1)F,$$

for each $n \geq 2$, according to (20). Therefore (cf. proof of Lemma)

$$M + K_i(\infty)^*NK_i(\infty) = P_O(i,\infty) - \mathcal{F}^{\#}_{BK_i(\infty)}(P_O(i,\infty)),$$

where $K_i(\infty) \in B[H,H']$ is given by

$$K_i(\infty) = (B^*P_O(i,\infty)B + N)^{-1}B^*P_O(i,\infty)F.$$

Hence $P_O(i,\infty) \in B^+[H]$ is a solution to (9), for (a) $\Longrightarrow$ (c) in Proposition 1, so that

$$P_O(i,\infty) = P$$

by the preceding lemma. Next note that

$$J^O_{\mathbf{R}}(n) \leq J^W_{\mathbf{R}}(n) \leq J_{\mathbf{R}}(K) + \operatorname{tr}(Q_nW) \tag{22}$$

since $J^W_{\mathbf{R}}(n) \leq \sum_{i=0}^{n-1} \mathrm{tr}(Q_i(M + K^*NK)) + \mathrm{tr}(Q_nW) \leq J_{\mathbf{R}}(K) + \mathrm{tr}(Q_nW)$ for any $K \in \Gamma_B$, according to (5) and (19). In particular, for $K = K_P$ it follows from Remark 2 that

$$\begin{aligned}\sum_{i=0}^{n} \mathrm{tr}(DR_iD^*P_O(i,n)) &\leq \sum_{i=0}^{n} \mathrm{tr}(DR_iD^*P_W(i,n)) \\ &\leq \sum_{i=0}^{\infty} \mathrm{tr}(DR_iD^*P) + \sum_{i=0}^{n} \mathrm{tr}\left(DR_iD^*\mathcal{F}^{\#^{n-i}}_{BK_p}(W)\right).\end{aligned}$$

To verify the second of the above inequalities recall that (cf. (3-b)) $\mathrm{tr}(Q_nW) = \sum_{i=0}^{n} \mathrm{tr}\left(\mathcal{F}^{n-i}_{BK}(DR_iD^*)W\right) = \sum_{i=0}^{n} \mathrm{tr}(DR_iD^*\mathcal{F}^{\#^{n-i}}_{BK}(W))$. Therefore, as we have already verified in this proof,

$$P_O(i,n) \leq P_W(i,n) \leq P + \mathcal{F}^{\#^{n-i}}_{BK_P}(W).$$

Since $P_O(i,n) \xrightarrow{s} P$ and $\mathcal{F}^{\#^{n-i}}_{BK_P} \to \mathcal{O}$ in $B[B[H]]$, as $n \to \infty$ (for $K_P \in \Gamma_B$), it follows that

$$P_W(i,n) \xrightarrow{s} P \qquad \text{as} \qquad n \to \infty.$$

Thus $\sup_{n\geq 0} ||P_W(i,n)|| < \infty$, $||P_W^{\frac{1}{2}}(i,n)x(\omega)||^2 \longrightarrow ||P^{\frac{1}{2}}x(\omega)||^2$ as $n \to \infty$ and $||P_W^{\frac{1}{2}}(i,n)x(\omega)||^2 \leq \sup_{n\geq 0} ||P_W(i,n)|| \, ||x(\omega)||^2$ almost everywhere on Ω for any $x \in \mathcal{H}$. Hence the Lebesgue dominated convergence theorem (see e.g. [2, p.151]) ensures that $||P_W^{\frac{1}{2}}(i,n)x||^2_{\mathcal{H}} \longrightarrow ||P^{\frac{1}{2}}x||^2_{\mathcal{H}}$ as $n \to \infty$ for every $x \in \mathcal{H}$. Then (recall: $||L^{\frac{1}{2}}x||^2_{\mathcal{H}} = \mathrm{tr}(\mathcal{E}(x \circ x)L)$ for every $x \in \mathcal{H}$ and $L \in B^+[H]$) $\mathrm{tr}(DR_iD^*P_W(i,n)) \longrightarrow \mathrm{tr}(DR_iD^*P)$ as $n \to \infty$, so that (cf. Remark 2)

$$\lim_{m\to\infty} \ \lim_{n\to\infty} \ \sum_{i=0}^{m} \mathrm{tr}(DR_iD^*P_W(i,n)) = J_{\mathbf{R}}(K_P).$$

From (21) $\{J^O_{\mathbf{R}}(n); \ n \geq 1\}$ converges and

$$\sum_{i=0}^{m} \mathrm{tr}(DR_iD^*P_O(i,n)) \leq J^O_{\mathbf{R}}(n) \leq J_{\mathbf{R}}(K_P)$$

for every $m \leq n$. Therefore

$$J^O_{\mathbf{R}}(n) \longrightarrow J_{\mathbf{R}}(K_P) \quad \text{as} \quad n \to \infty.$$

Since $\mathrm{tr}(Q_nW) \longrightarrow 0$ as $n \to \infty$ for any $K \in \Gamma_B$ (cf. proof of Theorem) it follows, from (22) with $K = K_P$, that

$$J^W_{\mathbf{R}}(n) \longrightarrow J_{\mathbf{R}}(K_P) \quad \text{as} \quad n \to \infty. \qquad \square$$

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