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MEAN-SQUARE STABILITY FOR DISCRETE BILINEAR SYSTEMS IN HILBERT SPACE*

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Abstract. A discrete system is uniformly mean-square stable if uniform convergence is preserved between input and state correlation sequences, and if nuclearity is preserved between the input and state correlation limits. This paper supplies a contraction-free condition ensuring mean-square stability for infinite-dimensional bilinear systems evolving in a separable Hilbert space.

Key words. Discrete bilinear systems; infinite-dimensional systems; operator theory; stability.

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1. NOTATION

Let $B[X]$ be the Banach algebra of all bounded linear operators from a Banach space X into itself. Both the norm in X and the induced uniform norm in $B[X]$ will be denoted by $||\cdot||$, and $r(\cdot)$ will stand for the spectral radius in $B[X]$. Throughout the paper we assume that H is a separable nontrivial complex Hilbert space, with inner product $\langle\cdot;\cdot\rangle$. Let $T^* \in B[H]$ be the adjoint of $T \in B[H]$, and set $|T| = (T^*T)^{1/2} \in B^+[H]$, where $B^+[H] := \{T \in B[H] : \ 0 \leq T\}$ is the weakly closed convex cone of all self-adjoint nonnegative operators in $B[H]$. Let $B_\infty[H]$ be the class of all compact operators from $B[H]$. If $T \in B_\infty[H]$ (or equivalenlty, $|T| \in B_\infty[H]$), let $\{\lambda_k;\ k \geq 0\}$ be the nonincreasing nonnegative null sequence made up of all eigenvalues of $|T|$, each nonzero one counted according to its multiplicity, and set $||T||_1 = \sum_{k=0}^\infty \lambda_k$. Let $B_1[H] := \{T \in B_\infty :\ ||T||_1 < \infty\}$ denote the class of all nuclear operators from $B[H]$, and set $B_1^+[H] = B_1[H] \cap B^+[H]$. $||\cdot||_1$ is a norm in $B_1[H]$ and $(B_1[H], ||\ ||_1)$ is a Banach space. Indeed, $B_1[H]$ is a two-sided ideal of $B[H]$, so that $\max\{||LT||_1, ||TL||_1\} \leq ||L||\ ||T||_1$ for every $T \in B_1[H]$. The trace of $T \in B_1[H]$ is defined as $\mathrm{tr}(T) = \sum_{k=0}^\infty \langle Te_k; e_k\rangle$, which does not depend on the choice of the orthonormal basis $\{e_k;\ k \geq 0\}$ for H, and $|\mathrm{tr}(T)| \leq \mathrm{tr}(|T|) = ||T||_1$ for every $T \in B_1[H]$. Finally, for any $f, g \in H$, let $(f \circ g) \in B_1[H]$ be defined as $(f \circ g)h = \langle h; g\rangle f$ for all $h \in H$, so that $(f \circ f) \in B_1^+[H]$. (For a systematic presentation on nuclear operators, the reader is referred to [5, 13].)

2. INTRODUCTION

Consider a discrete bilinear system operating in a stochastic environment, whose model is given by the following infinite-dimensional difference equation.

$$x_{i+1} = \Big[A_0 + \sum_{k=1}^{\infty} \langle w_i; e_k\rangle A_k\Big]\, x_k + u_{i+1}, \qquad x_0 = u_0, \tag{1}$$

where $\{A_k \in B[H];\ k \geq 0\}$ is a bounded sequence, $\{e_k;\ k \geq 1\}$ is an orthonormal basis for H, and $\{x_i;\ i \geq 0\}$, $\{w_i;\ i \geq 0\}$ and $\{u_i;\ i \geq 0\}$ are H-valued second-order random sequences. Let us make the following simplifying assumptions on the stochastic environment (a more general setup can be found in [12] but the one we set below will suffice our present needs): $\{w_i;\ i \geq 0\}$ is an independent sequence which is stationary in expectation and correlation, and $\{u_i;\ i \geq 0\}$ is a zero-mean independent sequence which is independent of $\{w_i;\ i \geq 0\}$. Let $s \in H$, $S \in B_1^+[H]$ and $C := S - s \circ s \in B_1^+[H]$ denote, respectively, expectation, correlation and covariance associated with the random sequence $\{w_i;\ i \geq 0\}$. Let $\{R_i \in B_1^+[H];\ i \geq 0\}$ and $\{Q_i \in B_1^+[H];\ i \geq 0\}$ stand for the correlation sequences associated with $\{u_i;\ i \geq 0\}$ and $\{x_i;\ i \geq 0\}$, respectively (for a formal definition of expectation, correlation and covariance of H-valued second-order random variables, see e.g. [2, 9]). It has been shown in [10] that the state correlation sequence $\{Q_i;\ i \geq 0\}$ envolves as follows.

$$Q_{i+1} = \mathcal{F}(Q_i) + R_{i+1}, \qquad Q_0 = R_0. \tag{2}$$

Here, $\mathcal{F} \in B[B[H]]$ is given by $\mathcal{F}(Q) = FQF^* + \mathcal{T}(Q)$ for all $Q \in B[H]$, with $F := A_0 + \sum_{k=1}^{\infty} \langle s; e_k \rangle A_k$ in $B[H]$, and $\mathcal{T} \in B[B[H]]$ defined by

$$\mathcal{T}(Q) = \sum_{k,\ell=1}^{\infty} \langle Ce_\ell; e_k \rangle A_k \; Q \; A_\ell^*$$

for all $Q \in B[H]$; where the above convergences are in the uniform topologies of $B[H]$ and $B[B[H]]$, respectively. By setting $F_0 := F$ and $F_k := A_k$ for every $k \geq 1$, the operator $\mathcal{F} \in B[B[H]]$ can be concisely written as

$$\mathcal{F}(Q) = \sum_{k,\ell=0}^{\infty} \langle \mathbf{C}\mathbf{e}_\ell; \mathbf{e}_k \rangle F_k \; Q \; F_\ell^*$$

for all $Q \in B[H]$. Here, $\{\mathbf{e}_0 = 1 \oplus 0,\ \mathbf{e}_k = 0 \oplus e_k;\ k \geq 1\}$ is an orthonormal basis for the Hilbert space $\mathbf{H} = \mathbb{C} \oplus H$, and $\mathbf{C} := (1 \oplus C) \in B_1^+[\mathbf{H}]$; with $\oplus$ denoting direct orthogonal sum (see e.g. [3]). In the sequel, we shall also need to define operators associated with $\mathcal{T}$ and $\mathcal{F}$, say $\mathcal{T}^\# \in B[B[H]]$ and $\mathcal{F}^\# \in B[B[H]]$, according to the following rule: for all $Q \in B[H]$,

$$\mathcal{T}^\#(Q) = \sum_{k,\ell=1}^{\infty} \langle Ce_\ell; e_k \rangle A_\ell^* \; Q \; A_k, \qquad \mathcal{F}^\#(Q) = \sum_{k,\ell=0}^{\infty} \langle \mathbf{C}\mathbf{e}_\ell; \mathbf{e}_k \rangle F_\ell^* \; Q \; F_k.$$

The mean-square stability problem for the infinite-dimensional bilinear model (1) is concerned with the asymptotic behaviour of the state correlation sequence whose evolution was described in (2) (see e.g. [10] and, for continous-time versions, [4, 15]; where the role played by mean-square stability in stochastic system theory is addressed as well). Precisely, it is the problem of finding conditions on the operator $\mathcal{F}$ which ensure that the following definition is fulfilled.

Definition. Consider the preceding setup. The model (1) is (uniformly) *mean-square stable* if the state correlation sequence $\{Q_i;\ i \geq 0\}$ converges in $B[H]$ to a correlation operator (i.e. to an operator in $B_1^+[H]$) whenever the input correlation sequence $\{R_i;\ i \geq 0\}$ converges in $B[H]$ to a correlation operator.

A sufficient condition for mean-square stability was given in [10]: *If there exist real constants* $\sigma \geq 1$ *and* $0 < \alpha < 1$ *such that* (i) $||F^i|| \leq \sigma\, \alpha^i$ *for all* $i \geq 0$ *and* (ii) $\max\{||\mathcal{T}||, ||\mathcal{T}^\#||\} < (1-\alpha^2)/\sigma^2$, *then the model* (1) *is mean-square stable.* Actually, the condition (ii) occuring in [10] was defined differently but should have been defined as above. It imposes that $\mathcal{T}$ and $\mathcal{T}^\#$ are both strict contractions, which may be thought of as too stringent a requirement for stability purposes. The aim of the present paper is to provide a condition for mean-square stability without requiring that $\mathcal{T}$ or $\mathcal{T}^{\#}$ are contractions. This will be established in Section 4, by using the results of Section 3. Precisely, we shall show that $\max\{r(\mathcal{F}), r(\mathcal{F}^{\#})\} < 1$ implies mean-square stability as defined above.

3. INTERMEDIATE RESULTS

Given an arbitrary bounded sequence of operators $\{F_k \in B[H];\; k \geq 1\}$, and a nonnegative nuclear operator $C \in B_1^+[H]$, set $\mathcal{F}_n \in B[B[H]]$ as follows. For each $n \geq 0$,

$$\mathcal{F}_n(Q) = \sum_{k,\ell=0}^{n} \langle Ce_\ell; e_k\rangle F_k Q F_\ell^*$$

for all $Q \in B[H]$, where $\{e_k;\; k \geq 0\}$ is an orthonormal basis for H which ensures that the sequence $\{\mathcal{F}_n;\; n \geq 0\}$ converges in $B[B[H]]$. The existence of such an orthonormal basis, which may depend on the operator C but not on the bounded sequence $\{F_k;\; k \geq 0\}$, has been established in [10]. Let $\mathcal{F} \in B[B[H]]$ be the limit of $\{\mathcal{F}_n;\; n \geq 0\}$, and write

$$\mathcal{F}(Q) = \sum_{k,\ell=0}^{\infty} \langle Ce_\ell; e_k\rangle F_k Q F_\ell^*$$

for all $Q \in B[H]$. Now set $\mathcal{F}_n^{\#} \in B[B[H]]$ as follows. For each $n \geq 0$,

$$\mathcal{F}_n^{\#}(Q) = \sum_{k,\ell=0}^{n} \langle Ce_\ell; e_k\rangle F_\ell^* Q F_k$$

for all $Q \in B[H]$. Since $\{\mathcal{F}_n;\; n \geq 0\}$ converges uniformly for an arbitrary bounded sequence $\{F_k;\; k \geq 0\}$, and since boundedness for $\{F_k;\; k \geq 0\}$ and for $\{F_k^*;\; k \geq 0\}$ are equivalent, it follows that $\{\mathcal{F}_n^{\#};\; n \geq 0\}$ also converges in $B[B[H]]$. Let $\mathcal{F}^{\#} \in B[B[H]]$ be its limit, so that

$$\mathcal{F}^{\#}(Q) = \sum_{k,\ell=0}^{\infty} \langle Ce_\ell; e_k\rangle F_\ell^* Q F_k$$

for all $Q \in B[H]$. Propositions (P_1) to (P_5) below comprise the properties of the operators $\mathcal{F}$ and $\mathcal{F}^{\#}$ that will be needed in the sequel. Note that, if any of them holds true, then it does hold for $\mathcal{F}$ interchanged with $\mathcal{F}^{\#}$.

Propositions. *For every integer* $i \geq 1$,

(P_1) $\quad \mathcal{F}^i(B^+[H]) \subseteq B^+[H]$, (P$_2$) $\quad ||\mathcal{F}^i|| = ||\mathcal{F}^i(I)||,$

(P$_3$) $\quad \mathcal{F}^i(B_1[H]) \subseteq B_1[H],$

(P$_4$) $\quad \mathrm{tr}(D\mathcal{F}^i(Q)) = \mathrm{tr}(Q\mathcal{F}^{\#i}(D)), \qquad D \in B[H], Q \in B_1[H],$

(P$_5$) $\quad \sup_{||Q||_1=1} ||\mathcal{F}^i(Q)||_1 = ||\mathcal{F}^{\#^i}||.$

Proof of (P$_1$). Let $\{f_j;\ j \geq 0\}$ be an orthonormal basis for H made up of eigenvectors of $C \in B_1^+[H]$ (whose existence is ensured by the spectral theorem for compact normal operators - see e.g. [1 p.438]). For each $j \geq 0$ let $\gamma_j \geq 0$ be the eigenvalue of C associated with the eigenvector f_j. Take an arbitrary integer $n \geq 1$, and set $F_j(n) = \sum_{k=0}^n \langle f_j; e_k\rangle F_k$ in $B[H]$ for each $j \geq 0$. Take $g \in H$ and $h \in H$ arbitrary. It has been shown in [3] that

$$\langle \mathcal{F}_n(Q)g; h\rangle = \sum_{j\geq 0} \gamma_j \langle QF_j^*(n)g; F_j^*(n)h\rangle$$

for all $Q \in B[H]$. Therefore, by induction on i we get

$$\langle \mathcal{F}_n^i(Q)g; h\rangle = \sum_{j_1,\cdots,j_i\geq 0} \gamma_{j_1}\cdots\gamma_{j_i} \langle QF_{j_i}^*(n)\cdots F_{j_1}^*(n)g; F_{j_i}^*(n)\cdots F_{j_1}^*(n)h\rangle \tag{3}$$

for every integer $i \geq 1$ and all $Q \in B[H]$. Thus, if $Q \in B^+[H]$, then

$$\langle \mathcal{F}_n^i(Q)h; h\rangle = \sum_{j_1,\cdots,j_i\geq 0} \gamma_{j_1}\cdots\gamma_{j_i} ||Q^{1/2}F_{j_i}^*(n)\cdots F_{j_1}^*(n)h||^2, \tag{4}$$

and hence $\mathcal{F}_n^i(Q) \in B^+[H]$ for every $i \geq 1$. Since $\mathcal{F}_n \to \mathcal{F}$ as $n \to \infty$ in $B[B[H]]$, it follows that $\mathcal{F}_n^i \to \mathcal{F}^i$ as $n \to \infty$ in $B[B[H]]$, and so $\mathcal{F}_n^i(Q) \to \mathcal{F}^i(Q)$ as $n \to \infty$ in $B[H]$ for every $Q \in B[H]$ and each $i \geq 1$. Therefore, $\mathcal{F}^i(Q) \in B^+[H]$ whenever $Q \in B^+[H]$, because $B^+[H]$ is closed in $B[H]$. □

Proof of (P$_2$). Take $g \in H$, $h \in H$, $n \geq 0$, and $i \geq 1$ arbitrary. By using the Schwarz inequality three times, it follows from (3) and (4) that

$$\begin{aligned}|\langle \mathcal{F}_n^i(Q)g; h\rangle| &\leq ||Q|| \sum_{j_1,\cdots,j_i\geq 0} \gamma_{j_1}\cdots\gamma_{j_i} ||F_{j_i}^*(n)\cdots F_{j_1}^*(n)g||\ ||F_{j_i}^*(n)\cdots F_{j_1}^*(n)h|| \\ &\leq ||Q||\langle \mathcal{F}_n^i(I)g; g\rangle^{1/2}\ \langle \mathcal{F}_n^i(I)h; h\rangle^{1/2} \leq ||Q||\ ||\mathcal{F}_n^i(I)||\ ||g||\ ||h||\end{aligned}$$

for every $Q \in B[H]$. Thus, $||\mathcal{F}_n^i|| = ||\mathcal{F}_n^i(I)||$, since

$$||\mathcal{F}_n^i(I)|| \leq ||\mathcal{F}_n^i|| = \sup_{||Q||=1} ||\mathcal{F}_n^i(Q)|| = \sup_{||Q||=1}\ \sup_{||g||=||h||=1} |\langle \mathcal{F}_n^i(Q)g; h\rangle| \leq ||\mathcal{F}_n^i(I)||.$$

Therefore, $||\mathcal{F}^i|| = ||\mathcal{F}^i(I)||$. Indeed, $||\mathcal{F}_n^i|| \to ||\mathcal{F}^i||$ and $||\mathcal{F}_n^i(I)|| \to ||\mathcal{F}^i(I)||$, because $\mathcal{F}_n^i \to \mathcal{F}^i$ in $B[B[H]]$, as $n \to \infty$ for each $i \geq 1$. □

Proof of (P$_3$). Take $Q \in B_1[H]$ arbitrary. First note that, since $B_1[H]$ is a two-sided ideal of $B[H]$, $D\mathcal{F}_n(Q) \in B_1[H]$ and $Q\mathcal{F}_n^{\#}(D) \in B_1[H]$ for any $D \in B[H]$ and every integer $n \geq 0$. Since $\mathrm{tr} : B_1[H] \to \mathbb{C}$ is a linear functional, and since

$$\mathrm{tr}(DE) \;=\; \mathrm{tr}(ED) \tag{5}$$

for every $D \in B[H]$ and $E \in B_1[H]$, we get by the very definition of $\mathcal{F}_n$ and ${\mathcal{F}_n}^{\#}$ that

$$\mathrm{tr}(D\mathcal{F}_n(Q)) \;=\; \mathrm{tr}(Q\,{\mathcal{F}_n}^{\#}(D)) \tag{6}$$

for each $n \geq 0$ and every $D \in B[H]$. Now, take arbitrary integres $n, \nu \geq 0$, and consider the polar decomposition $\mathcal{F}_{n+\nu}(Q) - \mathcal{F}_n(Q) = V|\mathcal{F}_{n+\nu}(Q) - \mathcal{F}_n(Q)|$, where $V = V_{n,\nu}(Q) \in B[H]$ is a partial isometry, so that $|\mathcal{F}_{n+\nu}(Q) - \mathcal{F}_n(Q)| = V^*(\mathcal{F}_{n+\nu}(Q) - \mathcal{F}_n(Q))$ (see e.g. [6 pp.74, 262]). From (6) we get

$$\begin{aligned} ||\mathcal{F}_{n+\nu}(Q) - \mathcal{F}_n(Q)||_1 &= \mathrm{tr}(V^*\mathcal{F}_{n+\nu}(Q)) - \mathrm{tr}(V^*\mathcal{F}_n(Q)) \\ &= \mathrm{tr}(Q\mathcal{F}_{n+\nu}^{\#}(V^*)) - \mathrm{tr}(Q\mathcal{F}_n^{\#}(V^*)) \leq ||Q||_1 ||\mathcal{F}_{n+\nu}^{\#} - \mathcal{F}_n^{\#}||, \end{aligned}$$

by linearity of the trace and recalling that a partial isometry is a contraction (so that $||V^*|| \leq 1$). Thus, since $\{\mathcal{F}_n^{\#};\ n \geq 0\}$ converges in $B[B[H]]$, $\lim_{n\to\infty} \sup_{\nu\geq 0}\ ||\mathcal{F}_{n+\nu}(Q) - \mathcal{F}_n(Q)||_1 = 0$. Hence, $\{\mathcal{F}_n(Q);\ n \geq 0\}$ converges in the Banach space $B_1[H]$. However, $\{\mathcal{F}_n(Q);\ n \geq 0\}$ converges to $\mathcal{F}(Q)$ in $B[H]$. Therefore,

$$\mathcal{F}_n(Q) \to \mathcal{F}(Q) \qquad \text{as } n \to \infty \qquad \text{in } B_1[H], \tag{7}$$

since convergence in $B_1[H]$ impleis convergence in $B[H]$, clearly to the same limit. Thus $\mathcal{F}(Q) \in B_1[H]$, so that $B_1[H]$ is invariant under $\mathcal{F}$, and the inclusion in (P$_3$) is trivially verified by induction. □

Proof of (P$_4$). Take $D \in B[H]$ and $Q \in B_1[H]$ arbitrary. Convergence in (7) implies that

$$D\mathcal{F}_n(Q) \to D\mathcal{F}(Q) \qquad \text{as } n \to \infty \qquad \text{in } B_1[H].$$

Moreover, since $\{\mathcal{F}_n^{\#}(D);\ n \geq 0\}$ converges to $\mathcal{F}^{\#}(D)$ in $B[H]$,

$$Q\mathcal{F}_n^{\#}(D) \to Q\mathcal{F}^{\#}(D) \qquad \text{as } n \to \infty \qquad \text{in } B_1[H].$$

Since $\mathrm{tr} : B_1[H] \to \mathbb{C}$ is continuous, the above two convergence results lead to

$$\mathrm{tr}(D\,\mathcal{F}(Q)) \;=\; \mathrm{tr}(Q\,\mathcal{F}^{\#}(D)), \tag{8}$$

according to (6). Thus, (P_4) holds for $i = 1$. Suppose it holds for some $i \geq 1$. Then, since $\mathcal{F}(Q) \in B_1[H]$, and according to (5) and (8),

$$\operatorname{tr}(D\mathcal{F}^{i+1}(Q)) = \operatorname{tr}(\mathcal{F}(Q)\mathcal{F}^{\#^i}(D)) = \operatorname{tr}(\mathcal{F}^{\#^i}(D)\mathcal{F}(Q)) = \operatorname{tr}(Q\,\mathcal{F}^{\#^{i+1}}(D)),$$

which completes the proof by induction. $\square$

Proof of (P_5). Take $Q \in B_1[H]$ and $i \geq 1$ arbitrary. Consider the polar decomposition $\mathcal{F}^i(Q) = V_i|\mathcal{F}^i(Q)|$, where $V_i = V_i(Q) \in B[H]$ is a partial isometry, so that $|\mathcal{F}^i(Q)| = V_i^*\mathcal{F}^i(Q)$. From ($P_4$) we get

$$||\mathcal{F}^i(Q)||_1 = \operatorname{tr}(V_i^*\mathcal{F}^i(Q)) = \operatorname{tr}(Q\,\mathcal{F}^{\#^i}(V_i^*)) \leq ||Q||_1||\mathcal{F}^{\#^i}||,$$

since $||V_i^*|| \leq 1$. On the other hand, from (P_4) and (5), by using the very definition of trace, and according to the Fourier series theorem (see e.g. [1 p.155]), we have

$$\begin{aligned}\operatorname{tr}(\mathcal{F}^i(f \circ g)) &= \operatorname{tr}((f \circ g)\mathcal{F}^{\#^i}(I)) = \operatorname{tr}(\mathcal{F}^{\#^i}(I)(f \circ g))\\ &= \sum_{k=0}^{\infty}\langle\mathcal{F}^{\#^i}(I)(f \circ g)h_k; h_k\rangle = \sum_{k=0}^{\infty}\langle\mathcal{F}^{\#^i}(I)f; h_k\rangle\ \langle h_k; g\rangle = \langle\mathcal{F}^{\#^i}(I)f; g\rangle\end{aligned}$$

for every $f, g, \in H$, and for any orthonormal basis $\{h_k;\ k \geq 0\}$ for H. Hence,

$$|\langle\mathcal{F}^{\#}(I)f; g\rangle| \ \leq\ \operatorname{tr}(|\mathcal{F}^i(f \circ g)|) = ||\mathcal{F}^i(f \circ g)||_1,$$

for all $f, g \in H$. Therefore, since $||f \circ g||_1 = ||f||\ ||g||$ for every $f, g \in H$,

$$||\mathcal{F}^{\#^i}(I)|| = \sup_{||f||=||g||=1} |\langle\mathcal{F}^{\#^i}(I)f; g\rangle| \leq \sup_{||Q||_1=1} ||\mathcal{F}^i(Q)||_1.$$

Finally, recall that $||\mathcal{F}^{\#^i}|| = ||\mathcal{F}^{\#^i}(I)||$ (cf. (P_2) with $\mathcal{F}$ replaced by $\mathcal{F}^{\#}$). $\square$

4. FINAL RESULTS

Given an operator sequence $\{R_i \in B[H];\ i \geq 0\}$ consider another operator sequence $\{Q_i \in B[H];\ i \geq 0\}$ recursively defined by a linear autonomous difference equation,

$$Q_{i+1} = \mathcal{F}(Q_i) + R_{i+1}, \qquad\qquad Q_0 = R_0,$$

with $\mathcal{F} \in B[B[H]]$ defined as in the previous section, whose solution is given by

$$Q_i = \sum_{j=0}^{i}\mathcal{F}^{i-j}(R_j)$$

for each $i \geq 0$. Hence, the operator sequence $\{Q_i;\ i \geq 0\}$ is $B^+[H]$-valued and/or $B_1[H]$-valued whenever the operator sequence $\{R_j;\ j \geq 0\}$ has the same properties, according to propositions (P_1) and (P_3). As we have seen in Section 2, the above linear model in $B[H]$ describes the state correlation evolution for an infinite-dimensional bilinear system if $\{R_i;\ i \geq 0\}$ and $\{Q_i;\ i \geq 0\}$ are identified with input and state correlation sequences, respectivelly (and hence restricted to be $B_1^+[H]$-valued). The theorem below shows that mean-square stability as defined in Section 2 is ensured whenever $r(\mathcal{F}) < 1$ and $r(\mathcal{F}^\#) < 1$. Before stating it we shall give equivalent conditions for $r(\mathcal{F}) < 1$ and for $r(\mathcal{F}^\#) < 1$ (further conditions for $r(\mathcal{F}) < 1$ were established in [3]).

Lemma 1. *The following assertions are equivalent.*

(a_1) $r(\mathcal{F}) < 1$.

(b_1) $\{Q_i \in B[H];\ i \geq 0\}$ *converges in* $B[H]$ *whenever* $\{R_i \in B[H];\ i \geq 0\}$ *converges in* $B[H]$.

(c_1) $\{Q_i \in B^+[H];\ i \geq 0\}$ *converges in* $B[H]$ *whenever* $\{R_i \in B^+[H];\ i \geq 0\}$ *converges in* $B[H]$.

Lemma 2. *The following assertions are equivalent.*

(a_2) $r(\mathcal{F}^\#) < 1$.

(b_2) $\{Q_i \in B_1[H];\ i \geq 0\}$ *converges in* $B_1[H]$ *whenever* $\{R_i \in B_1[H];\ i \geq 0\}$ *converges in* $B_1[H]$.

(c_2) $\{Q_i \in B_1^+[H]\ i \geq 0\}$ *converges in* $B_1[H]$ *whenever* $\{R_i \in B_1^+[H];\ i \geq 0\}$ *converges in* $B_1[H]$.

Theorem. *If* $r(\mathcal{F}) < 1$ *and* $r(\mathcal{F}^\#) < 1$, *then*

(mss) $\{Q_i \in B_1^+[H];\ i \geq 0\}$ *converges in* $B[H]$ *to a nuclear operator whenever* $\{R_i \in B_1^+[H];\ i \geq 0\}$ *converges in* $B[H]$ *to a nuclear operator.*

Proof of Lemma 1. Since $B[H]$ is a Banach space, and since $\mathcal{F} \in B[B[H]]$, we get (a_1) $\iff$ (b_1) (see e.g. [11]). Note that (b_1) $\implies$ (c_1) trivially. Now, suppose (c_1) holds, and set $R_i = I$ for every $i \geq 0$. Thus, $\{Q_i = \sum_{j=0}^{i} \mathcal{F}^j(I);\ i \geq 0\}$ converges in $B[H]$, so that $\mathcal{F}^i(I) \to 0$ as $i \to \infty$ in $B[H]$. Hence, $||\mathcal{F}^i|| \to 0$ as $i \to \infty$, according to (P_2). Equivalently (see e.g. [7]), $r(\mathcal{F}) < 1$. Then, (c_1) $\implies$ (a_1). $\square$

Proof of Lemma 2. Consider proposition (P_3) and set $\mathcal{F}_1 = \mathcal{F}|_{B_1[H]} : B_1[H] \to B_1[H]$, so that $\mathcal{F}_1^i(Q) = \mathcal{F}^i(Q)$ for all $Q \in B_1[H]$ and every $i \geq 0$. Thus,

$$||\mathcal{F}_1^i||_1 := \sup_{||Q||_1=1} ||\mathcal{F}_1^i(Q)||_1 = \sup_{||Q||_1=1} ||\mathcal{F}^i(Q)||_1 = ||\mathcal{F}^{\#^i}||$$

for each $i \geq 0$, according to (P_5). Hence, $\mathcal{F}_1 \in B[B_1[H]]$. Moreover, letting $r_1(\cdot)$ stand for the spectral radius in the Banach algebra $B[B_1[H]]$, we get

$$r_1(\mathcal{F}_1) = r(\mathcal{F}^{\#}) \tag{9}$$

by the Beurling-Gelfand formula for the spectral radius (see e.g. [1 p.326]). Now, if $R_i \in B_1[H]$ for every $i \geq 0$, then $Q_i = \sum_{j=0}^{i} \mathcal{F}_1^{i-j}(R_j)$ for each $i \geq 0$, so that

$$Q_{i+1} = \mathcal{F}_1(Q_i) + R_{i+1}, \qquad Q_0 = R_0.$$

Therefore, since $B_1[H]$ is a Banach space and since $\mathcal{F}_1 \in B[B_1[H]]$, it follows that $r_1(\mathcal{F}_1) < 1$ if and only if (b_2) holds (see e.g. [11]). Thus, (a_2) $\iff$ (b_2) by (9); and (b_2) $\Longrightarrow$ (c_2) trivially. To verify that (c_2) $\Longrightarrow$ (a_2) proceed as follows. Take an arbitrary $R \in B_1[H]$. Recalling that any operator in $B[H]$ has a Cartesian decomposition (whose real and imaginary parts are self-adjoint), and that any self-adjoint operator in $B[H]$ can be decomposed in negative and positive parts (see e.g. [1 p.472]), we get

$$R = (R_1 - R_2) + \sqrt{-1}\,(R_3 - R_4)$$

where $R_m \in B_1^+[H]$ for each $m = 1, 2, 3, 4$. Now suppose (c_2) holds, and set $R_{i,m} = R_m \in B_1^+[H]$ for each $m = 1, 2, 3, 4$ and every $i \geq 0$. Thus, $\{Q_{i,m} = \sum_{j=0}^{i} \mathcal{F}^j(R_m) \in B_1^+[H];\ i \geq 0\}$ converges in $B_1[H]$, so that $\sup_{i\geq 0} ||Q_{i,m}||_1 < \infty$, for each $m = 1, 2, 3, 4$. Hence, for every $i \geq 0$,

$$\begin{aligned}\sum_{j=0}^{i} ||\mathcal{F}_1^j(R)||_1 &\leq \sum_{j=0}^{i}\sum_{m=1}^{4} ||\mathcal{F}^j(R_m)||_1 = \sum_{j=0}^{i}\sum_{m=1}^{4} \mathrm{tr}(\mathcal{F}^j(R_m)) = \sum_{m=1}^{4} \mathrm{tr}\Big(\sum_{j=0}^{i} \mathcal{F}^j(R_m)\Big) \\ &= \sum_{m=1}^{4} \mathrm{tr}(Q_{i,m}) = \sum_{m=1}^{4} ||Q_{i,m}||_1 \leq \sum_{m=1}^{4} \sup_{i\geq 0} ||Q_{i,m}||_1 < \infty\end{aligned}$$

for all R in the Banach space $B_1[H]$. Equivalently (see e.g. [7]), the operator $\mathcal{F}_1 \in B[B_1[H]]$ is such that $r_1(\mathcal{F}_1) < 1$. Then, (c_2) $\Longrightarrow$ (a_2) by (9). □

Proof of Theorem. Suppose $r(\mathcal{F}) < 1$. It is well-known (see e.g. [11]) that $r(\mathcal{F}) < 1$ implies the existence of $(\mathcal{I} - \mathcal{F})^{-1} \in B[B[H]]$, the convergence of $\{\sum_{j=0}^{i} \mathcal{F}^j \in B[B[H]];\ i \geq 0\}$ in $B[B[H]]$ to $(\mathcal{I} - \mathcal{F})^{-1}$, and that the limits in (b_1) of Lemma 1 are such that $Q_\infty = (\mathcal{I} - \mathcal{F})^{-1}(R_\infty)$ (and so are the limits in (c_1)). Take an arbitrary sequence $\{R_i \in B_1^+[H];\ i \geq 0\}$ that converges in $B[H]$ to a nuclear operator $R_\infty \in B_1^+[H]$. Since (a_1) $\Longrightarrow$ (c_1) in Lemma 1, $\{Q_i \in B_1^+[H];\ i \geq 0\}$ converges in $B[H]$ to $Q_\infty = (\mathcal{I} - \mathcal{F})^{-1}(R_\infty)$. Set $P_i = \sum_{j=0}^{i} \mathcal{F}^j(R_\infty)$ in $B_1^+[H]$, so that $||P_i - Q_\infty|| \leq \|\sum_{j=0}^{i} \mathcal{F}^j - (\mathcal{I} - \mathcal{F})^{-1}\|\ ||R_\infty||$, for every $i \geq 0$. Hence,

$$\lim_{i\to\infty} ||P_i - Q_\infty|| = 0. \tag{10}$$

Now note that, according to (P_1), (P_3) and (P_5),

$$||P_i||_1 = \mathrm{tr}(P_i) = \sum_{j=0}^{i} \mathrm{tr}(\mathcal{F}^i(R_\infty)) = \sum_{j=0}^{i} ||\mathcal{F}^j(R_\infty)||_1 \leq \Big(\sum_{j=0}^{i} ||\mathcal{F}^{\#^i}||\Big)||R_\infty||_1$$

for every $i \geq 0$. If $r(\mathcal{F}^{\#}) < 1$, then

$$\sup_{i\geq 1} ||P_i||_1 < \infty, \tag{11}$$

because $\sum_{j=0}^{\infty} ||\mathcal{F}^{\#^i}|| < \infty$ whenever $r(\mathcal{F}^{\#}) < 1$, (see e.g. [7]). From (10) and (11) it follows that $Q_\infty \in B_1[H]$ (see e.g. [8, 14 p.179]), and hence $Q_\infty \in B_1^+[H]$ (because $B^+[H]$ is closed in $B[H]$). Thus, $r(\mathcal{F}) < 1$ and $r(\mathcal{F}^{\#}) < 1 \Longrightarrow$ (mss). $\square$

Remark. It is worth remarking that (mss) $\Longrightarrow$ $r(\mathcal{F}^{\#}) < 1$. This can be verified by applying the same technique used to show that (c_2) $\Longrightarrow$ (a_2) in Lemma 2. Here one has to notice that (mss) ensures that $\{Q_{i,m} \in B_1^+[H];\ i \geq 0\}$ converges in $B[H]$ to, say, $Q_{\infty,m} \in B_1^+[H]$ for each $m = 1, 2, 3, 4$. However, $0 \leq Q_{i,m} \leq Q_{\infty,m} \in B_1[H]$, since $0 \leq \mathcal{F}^{i+1}(R_m) = Q_{i+1,m} - Q_{i,m}$, for every $i \geq 0$ and each $m = 1, 2, 3, 4$. Hence, convergence in $B[H]$ actually leads to convergence in $B_1[H]$ (see e.g. [8]).

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