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LYAPUNOV EQUATION FOR INFINITE-DIMENSIONAL DISCRETE BILINEAR SYSTEMS*

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Abstract. Mean-square stability for discrete systems requires that uniform convergence is preserved between input and state correlation sequences. Such a convergence preserving property holds for an infinite-dimensional bilinear system if and only if the associate Lyapunov equation has a unique strictly positive solution.

Key words. Discrete bilinear systems; infinite-dimensional systems; operator theory; stability.

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1. NOTATION

Let $B[X]$ be the Banach algebra of all bounded linear operators from a Banach space X into itself, and let $G[X]$ be the group of all invertible operators from $B[X]$. Both the norm in X and the induced uniform norm in $B[X]$ will be denoted by $\|\cdot\|$, and $r_\sigma(\cdot)$ will stand for the spectral radius in $B[X]$. An operator $T \in B[X]$ is said to be similar to a strict contraction if $\|W^{-1}TW\| < 1$ for some $W \in G[X]$. Throughout this paper, H will stand for a separable nontrivial complex Hilbert space with inner product $\langle\cdot;\cdot\rangle$, and an upper star * will denote adjoint of a Hilbert-space operator, as usual. The Hilbert space $\oplus_{j=0}^{\infty}H$ obtained by the direct (orthogonal) sum of countably infinite copies of H will be denoted by $\ell_2(H)$, and its inner product by $\langle\cdot;\cdot\rangle_{\ell_2(H)}$. Let $B^+[H]$ be the weakly closed cone of $B[H]$ made up of all self-adjoint nonnegative operators, let $G^+[H] = B^+[H] \cap G[H]$ be the class of all strictly positive operators, and let $B^+_\infty[H]$ be the class of all compact operators from $B^+[H]$. The trace of $T \in B^+_\infty[H]$ is defined as $\mathrm{tr}(T) = \sum_{k=1}^{\infty}\langle Te_k; e_k\rangle = \sum_{k=1}^{\infty}\lambda_k$, where $\{e_k;\ k \geq 1\}$ is any orthonormal basis for H, and $\{\lambda_k \geq 0;\ k \geq 1\}$ is a sequence made up of all eigenvalues of $T \in B^+_\infty[H]$, each nonzero one counted according to its multiplicity. Set $B^+_1[H] = \{T \in B^+_\infty[H] : \mathrm{tr}(T) < \infty\}$, the class of all nonnegative nuclear operators. Finally, for any $h \in H$, let $(h \circ h) \in B^+_1[H]$ be defined as $(h \circ h)x = \langle x; h\rangle h$ for all $x \in H$.

2. INTRODUCTION

Given an operator sequence $\{R_i \in B[X];\ i \geq 0\}$, consider another operator sequence $\{Q_i \in B[X];\ i \geq 0\}$ recursively defined by a linear autonomous difference equation

$$Q_{i+1} = \mathcal{F}(Q_i) + R_{i+1}, \qquad Q_0 = R_0, \tag{1}$$

where $\mathcal{F}$ stands for a bounded linear transformation from the Banach space $B[H]$ into itself. Recall that $\{Q_i \in B[H];\ i \geq 0\}$ converges in $B[H]$ whenever $\{R_i \in B[H];\ i \geq 0\}$ converges in $B[H]$ if and only if [6]

$$r_\sigma(\mathcal{F}) < 1. \tag{2}$$

As a background for our further discussion let us consider first a well-known particular case, viz. $\mathcal{F} = \mathcal{F}_F$, where $\mathcal{F}_F \in B[B[H]]$ is defined by

$$\mathcal{F}_F(Q) = F\,Q\,F^* \qquad \forall Q \in B[H]$$

for some $F \in B[H]$. Since $\|\mathcal{F}_F^i\| = \|F^i\|^2$ for every $i \geq 0$, $r_\sigma(\mathcal{F}_F) = r_\sigma(F)^2$ according to the Gelfand formula for the spectral radius. Hence, $r_\sigma(\mathcal{F}_F) < 1$ if and only if $r_\sigma(F) < 1$. However, as is well known (see e.g. [1, 3, 7, 9, 10]), $r_\sigma(F) < 1$ if and only if

(3) for every $V \in G^+[H]$ there exists a unique solution $W \in G^+[H]$ for the Lyapunov equation $V = W - \mathcal{F}(W)$.

The purpose of this paper is to show that the equivalence between (2) and (3) still holds for a more general case where, instead of setting $\mathcal{F} = \mathcal{F}_F$, we take $\mathcal{F} := \mathcal{F}_F + \mathcal{T}$ with $\mathcal{T}$ in $B[B[H]]$ defined as follows.

$$\mathcal{T}(Q) = \sum_{k,\ell=1}^{\infty} \langle Ce_\ell; e_k \rangle A_k \, Q \, A_\ell^* \qquad \forall Q \in B[H],$$

where $\{A_k \in B[H];\ k \geq 0\}$ is an arbitrary bounded sequence of operators, $C \in B_1^+[H]$ is any nonnegative nuclear operator, and $\{e_k;\ k \geq 1\}$ is a suitable orthonormal basis for H ensuring convergence for the above infinite series.

Such a deterministic linear setup, concerning the convergence preserving property of the linear model (1), naturally arrises in the mean-square stability problem for infinite-dimensional discrete (either linear or bilinear) systems operating in a stochastic environment. Actually, mean-square stability for an infinite-dimensional discrete (either linear or bilinear) system requires that the state correlation sequence, say $\{Q_i \in B_1^+[H];\ i \geq 0\}$, converges in $B[H]$ whenever the input correlation sequence, say $\{R_i \in B_1^+[H];\ i \geq 0\}$, converges in $B[H]$; and such a state correlation sequence evolves according to the linear model (1), with $\mathcal{F} = \mathcal{F}_F$ for the case of a linear system (see e.g. [4, 11]), or with $\mathcal{F} = \mathcal{F}_F + \mathcal{T}$ for the case of a bilinear system (see e.g. [5] and, for the continuous-time case, [2, 12]).

Our main result is presented in section 4 by using the auxiliary results developed in section 3. The present approach was motivated by the earlier works on finite-dimensional stochastic bilinear systems in [8] and on infinite-dimensional deterministic linear systems in [7], extending the former to an infinite-dimensional setting and the latter to the underlying class of operators $\mathcal{F} = \mathcal{F}_F + \mathcal{T}$.

3. PRELIMINARIES

Consider a bounded sequence of operators $\{A_k \in B[H];\ k \geq 1\}$, and let $\{e_k;\ k \geq 1\}$ be an orthonormal basis for H. Given $C \in B_1^+[H]$, set $\mathcal{T}_n \in B[B[H]]$ as follows. For each integer $n \geq 1$,

$$\mathcal{T}_n(Q) = \sum_{k,\ell=1}^{n} \langle Ce_\ell; e_k \rangle A_k \, Q \, A_\ell^* \qquad \forall Q \in B[H].$$

Assumption. $\mathcal{T}_n \to \mathcal{T} \in B[B[H]]$ as $n \to \infty$ in $B[B[H]]$.

Under the above assumption, write

$$\mathcal{T}(Q) = \sum_{k,\ell=1}^{\infty} \langle Ce_\ell; e_k \rangle A_k \, Q \, A_\ell^* \qquad \forall Q \in B[H]$$

and, given $F \in B[H]$, let $\mathcal{F} \in B[B[H]]$ be defined as follows.

$$\mathcal{F}(Q) = F\,Q\,F^* + \mathcal{T}(Q) \qquad \forall Q \in B[H].$$

Remark. Note that the very definition of $\{\mathcal{T}_n \in B[B[H]];\ n \geq 1\}$ in terms of a bounded sequence $\{A_k \in B[H];\ k \geq 1\}$ and a nonnegative nuclear operator $C \in B_1^+[H]$ is not enough to ensure its convergence as assumed above. For instance, take $c \in H$ such that $|\sum_{k=1}^n \langle c; e_k\rangle| \to \infty$ as $n \to \infty$ (e.g. with $H = \ell_2$, take $c = (\,1,\ 1/2,\ 1/3,\ \cdots) \in \ell_2$ and let $\{e_k;\ k \geq 1\}$ be the standard orthonormal basis for ℓ_2), and set $C = (c \circ c) \in B_1^+[H]$. Thus, $\sum_{k,\ell=1}^n \langle Ce_\ell; e_k\rangle = \sum_{k,\ell=1}^n \langle e_\ell; c\rangle\,\langle c; e_k\rangle = |\sum_{k=1}^n \langle c; e_k\rangle|^2$ for each $n \geq 1$. Therefore, with $A_k = I$ for every $k \geq 1$, $\mathcal{T}_n(I) = |\sum_{k=1}^n \langle c; e_k\rangle|^2 I$ for each $n \geq 1$, so that $\|\mathcal{T}_n(I)\| \to \infty$ as $n \to \infty$. On the other hand, the assumed uniform convergence of $\{\mathcal{T}_n \in B[B[H]];\ n \geq 1\}$ can always be achieved by choosing a suitable orthonormal basis for H. For instance, take any $s \in H$ and set $S = C + (s \circ s)$. Since $S \in B_1^+[H]$, let $\{e_k;\ k \geq 1\}$ be an orthonormal basis for H made up of eigenvectors of S (whose existence is ensured by the spectral theorem for compact normal operators). Let $\lambda_k \geq 0$ be the eingevalue of S associated with the eigenvector e_k for each $k \geq 1$, and set $\mathcal{L}_n \in B[B[H]]$ for every $n \geq 1$ as follows.

$$\mathcal{L}_n(Q) = \sum_{k,\ell=1}^{n} \langle Se_\ell; e_k\rangle A_k\,Q\,A_\ell^* = \sum_{k=1}^{n} \lambda_k\,A_k\,QA_k^* \qquad \forall Q \in B[H].$$

Since $\sum_{k=1}^\infty \lambda_k = \sum_{k=1}^\infty \langle Se_k; e_k\rangle = \operatorname{tr}(S) < \infty$,

$$\sup_{\nu\geq 1} \|\mathcal{L}_{n+\nu} - \mathcal{L}_n\| \;\leq\; \Big(\sup_{k\geq 1} \|A_k\|^2\Big) \sum_{k=n+1}^{\infty} \lambda_k \to 0 \quad \text{as} \quad n \to \infty,$$

so that $\{\mathcal{L}_n \in B[B[H]];\ n \geq 1\}$ converges in the Banach space $B[B[H]]$. Now set $\mathcal{K}_n \in B[B[H]]$ such that

$$\mathcal{K}_n(Q) = \sum_{k,\ell=1}^{n} \langle (s\circ s)e_\ell; e_k\rangle A_k\,Q\,A_\ell^* = M_n\,Q\,M_n^* \qquad \forall Q \in B[H],$$

where $M_n := \sum_{k=1}^n \langle s; e_k\rangle A_k \in B[H]$, for every $n \geq 1$; and note that $(\sum_{k=m}^p |\langle s; e_k\rangle|)^2 = \sum_{k,\ell=m}^p |\langle (s\circ s)e_\ell; e_k\rangle| \leq \sum_{k,\ell=m}^p |\langle Se_\ell; e_k\rangle| = \sum_{k=m}^p \lambda_k$ for every $m, p \geq 1$, since $(s \circ s) \leq S$. Thus,

$$\sup_{\nu\geq 1} \|M_{n+\nu} - M_n\| \;\leq\; \Big(\sup_{k\geq 1} \|A_k\|\Big) \Big(\sum_{k=n+1}^{\infty} \lambda_k\Big)^{1/2} \to 0 \quad \text{as} \quad n \to \infty,$$

and hence $\{M_n \in B[H];\ n \geq 1\}$ converges in the Banach space $B[H]$. By letting $M \in B[H]$ be its limit, it is readily verified that

$$\sup_{\|Q\|=1} \|\mathcal{K}_n(Q) - MQM^*\| \;\leq\; \|M_n - M\|^2 + 2\|M\|\,\|M_n - M\| \to 0 \quad \text{as} \quad n \to \infty,$$

so that $\{\mathcal{K}_n \in B[B[H]];\ n \geq 1\}$ also converges in $B[B[H]]$. Therefore, $\{\mathcal{T}_n \in B[B[H]];\ n \geq 1\}$ converges in $B[B[H]]$, since $\mathcal{T}_n = \mathcal{L}_n - \mathcal{K}_n$ for every $n \geq 1$.

Finally, for any $W \in G^+[H]$ let $\mathcal{W} \in G[B[H]]$ be defined by

$$\mathcal{W}(Q) = W\,Q\,W \qquad \forall Q \in B[H],$$

so that $\mathcal{W}^{-1} \in G[B[H]]$ is such that

$$\mathcal{W}^{-1}(Q) = W^{-1}\,Q\,W^{-1} \qquad \forall Q \in B[H].$$

Propositions. *For every* $W \in G^+[H]$,

(P$_1$)
$$\mathcal{W}^{-1}\mathcal{F}\mathcal{W}(B^+[H]) \;\subseteq B^+[H],$$

(P$_2$)
$$\|\mathcal{W}^{-1}\mathcal{F}\mathcal{W}\| = \|\mathcal{W}^{-1}\mathcal{F}\mathcal{W}(I)\|.$$

Proof. Consider the direct (orthogonal) sum $\mathbf{H} = \mathbb{C} \oplus H$, which is a Hilbert space with inner product given by

$$\langle \mathbf{x}; \mathbf{y} \rangle = \xi\overline{v} + \langle x; y \rangle$$

for all $\mathbf{x} = \xi \oplus x$ and $\mathbf{y} = v \oplus y$ in $\mathbf{H}$ ($\xi, v \in \mathbb{C}$, the upper bar denoting complex conjugate, and $x, y \in H$). Given the orthonormal basis $\{e_k;\ k \geq 1\}$ for H set, for each $k \geq 0$,

$$\mathbf{e}_k = \begin{cases} 1 \oplus 0 & \text{if} \quad k = 0, \\ 0 \oplus e_k & \text{if} \quad k \geq 1; \end{cases}$$

so that $\{\mathbf{e}_k;\ k \geq 0\}$ is an orthonormal basis for $\mathbf{H}$. Let $\mathbf{C} = (1 \oplus C) \in B_1^+[\mathbf{H}]$ be the direct sum of the identity on $\mathbb{C}$ with $C \in B_1^+[H]$, so that

$$\langle \mathbf{C}\,\mathbf{x}; \mathbf{y} \rangle = \langle \xi \oplus Cx;\ v \oplus y \rangle = \xi\overline{v} + \langle Cx; y \rangle$$

for all $\mathbf{x} = \xi \oplus x \in \mathbf{H}$ and $\mathbf{y} = v \oplus y \in \mathbf{H}$. In particular,

$$\langle \mathbf{C}\mathbf{e}_\ell; \mathbf{e}_k \rangle = \begin{cases} 1 & \text{if} \quad k = \ell = 0, \\ \langle Ce_\ell; e_k \rangle & \text{if} \quad k, \ell \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for each $n \geq 1$, set $\mathcal{F}_n \in B[B[H]]$ as follows.

$$\mathcal{F}_n(Q) = FQF^* + \mathcal{T}_n(Q) = \sum_{k,\ell=0}^{n} \langle \mathbf{C}\mathbf{e}_\ell; \mathbf{e}_k \rangle F_k Q F_\ell^* \qquad \forall Q \in B[H],$$

with $F_0 := F \in B[H]$ and $F_k := A_k \in B[H]$ for every $k \geq 1$, so that $\mathcal{F}_n \to \mathcal{F} \in B[B[H]]$ as $n \to \infty$ in $B[B[H]]$, where

$$\mathcal{F}(Q) = FQF^* + \mathcal{T}(Q) = \sum_{k,\ell=0}^{\infty} \langle \mathbf{C}\mathbf{e}_\ell; \mathbf{e}_k \rangle F_k Q F_\ell^* \qquad \forall Q \in B[H],$$

according to the convergence assumption on $\{\mathcal{T}_n \in B[B[H]];\ n \geq 1\}$. Now, since $\mathbf{C} \in B_1^+[\mathbf{H}]$, the spectral theorem says that

$$\mathbf{C}\mathbf{x} = \sum_{j=0}^{\infty} \gamma_j \langle \mathbf{x}; \mathbf{f}_j \rangle \mathbf{f}_j \qquad \forall \mathbf{x} \in \mathbf{H}$$

for some orthonormal basis $\{\mathbf{f}_j;\ j \geq 0\}$ for $\mathbf{H}$, where $\gamma_j \geq 0$ for every $j \geq 0$ (actually, $\sum_{j=0}^{\infty} \gamma_j = \operatorname{tr}(\mathbf{C}) < \infty$). Then, by continuity of the inner product,

$$\langle \mathbf{C}\mathbf{e}_\ell; \mathbf{e}_k \rangle = \sum_{j=0}^{\infty} \gamma_j \langle \mathbf{e}_\ell; \mathbf{f}_j \rangle \ \langle \mathbf{f}_j; \mathbf{e}_k \rangle$$

for every $k, \ell \geq 0$. Hence, for each $n \geq 1$,

$$\begin{aligned}
\langle \mathcal{F}_n(Q)x; y \rangle &= \sum_{k,\ell=0}^{n} \sum_{j=0}^{\infty} \gamma_j \langle \mathbf{e}_\ell; \mathbf{f}_j \rangle \ \langle \mathbf{f}_j; \mathbf{e}_k \rangle \ \langle Q F_\ell^* x; F_k^* y \rangle \\
&= \sum_{j=0}^{\infty} \gamma_j \sum_{k,\ell=0}^{n} \langle Q \langle \mathbf{e}_\ell; \mathbf{f}_j \rangle F_\ell^* x;\ \langle \mathbf{e}_k; \mathbf{f}_j \rangle F_k^* y \rangle \\
&= \sum_{j=0}^{\infty} \gamma_j \Big\langle Q \sum_{\ell=0}^{n} \langle \mathbf{e}_\ell; \mathbf{f}_j \rangle F_\ell^* x;\ \sum_{k=0}^{n} \langle \mathbf{e}_k; \mathbf{f}_j \rangle F_k^* y \Big\rangle
\end{aligned}$$

for all $x, y \in H$, since addition is continuous. Thus, if $Q \in B^+[H]$, then

$$\langle \mathcal{F}_n(Q)x; x \rangle = \sum_{j=0}^{\infty} \gamma_j \Big\| Q^{1/2} \sum_{k=0}^{n} \langle \mathbf{e}_\ell; \mathbf{f}_j \rangle F_k^* x \Big\|^2 \qquad \forall x \in H,$$

which implies that $\mathcal{F}_n(Q) \in B^+[H]$, for every $n \geq 1$. Therefore,

$$(\mathrm{P}_1') \qquad \mathcal{F}(B^+[H]) \subseteq B^+[H],$$

because $B^+[H]$ is closed in $B[H]$. Now, by using the Schwarz inequality twice, and recalling that $\|\mathcal{F}_n(I)^{1/2}\| = \|\mathcal{F}_n(I)\|^{1/2}$ since $\mathcal{F}_n(I) \in B^+[H]$ for every $n \geq 1$, we get

$$\begin{aligned}
|\langle \mathcal{F}_n(Q)x; y \rangle| &\leq \|Q\| \sum_{j=0}^{\infty} \gamma_j \Big\| \sum_{\ell=0}^{n} \langle \mathbf{e}_\ell; \mathbf{f}_j \rangle F_\ell^* x \Big\| \ \Big\| \sum_{k=0}^{n} \langle \mathbf{e}_k; \mathbf{f}_j \rangle F_k^* y \Big\| \\
&\leq \|Q\| \Big(\sum_{j=0}^{\infty} \gamma_j \Big\| \sum_{\ell=0}^{n} \langle \mathbf{e}_\ell; \mathbf{f}_j \rangle F_\ell^* x \Big\|^2 \Big)^{1/2} \Big(\sum_{j=0}^{\infty} \gamma_j \Big\| \sum_{k=0}^{n} \langle \mathbf{e}_k; \mathbf{f}_j \rangle F_k^* y \Big\|^2 \Big)^{1/2} \\
&= \|Q\| \langle \mathcal{F}_n(I)x; x \rangle^{1/2} \langle \mathcal{F}_n(I)y; y \rangle^{1/2} \\
&= \|Q\| \ \|\mathcal{F}_n(I)^{1/2} x\| \ \|\mathcal{F}_n(I)^{1/2} y\| \ \leq \|Q\| \ \|\mathcal{F}_n(I)\| \ \|x\| \ \|y\|
\end{aligned}$$

for all $x, y \in H$ and every $n \geq 1$, so that

$$\|\mathcal{F}_n(Q)\| = \sup_{\|x\|=\|y\|=1} |\langle \mathcal{F}_n(Q)x; y\rangle| \quad \leq \quad \|Q\| \, \|\mathcal{F}_n(I)\|$$

for all $Q \in B[H]$ and every $n \geq 1$. Hence, $\|\mathcal{F}_n\| = \|\mathcal{F}_n(I)\|$, because

$$\|\mathcal{F}_n(I)\| \leq \|\mathcal{F}_n\| = \sup_{\|Q\|=1} \|\mathcal{F}_n(Q)\| \quad \leq \quad \|\mathcal{F}_n(I)\|,$$

for every $n \geq 1$. Therefore, since $\mathcal{F}_n \to \mathcal{F}$ as $n \to \infty$ in $B[B[H]]$,

$$(\mathrm{P}_2') \qquad \|\mathcal{F}\| \quad = \quad \|\mathcal{F}(I)\|.$$

Finally, take an arbitrary $W \in G^+[H]$ and set $\tilde{F}_k = W^{-1}F_kW \in B[H]$ for every $k \geq 0$. Thus,

$$\mathcal{W}^{-1}\mathcal{F}\mathcal{W}(Q) = W^{-1}\mathcal{F}(WQW)W^{-1} = \sum_{k,\ell=0}^{\infty} \langle \mathbf{C}\mathbf{e}_\ell; \mathbf{e}_k\rangle \tilde{F}_k Q \tilde{F}_\ell^* \qquad \forall Q \in B[H].$$

Hence, (P_1') implies (P_1), and (P_2') implies (P_2). □

Lemma. *For any $W \in G^+[H]$ and any $\alpha \in (0, \infty)$,*

$$\begin{aligned} \|\mathcal{W}^{-1}\mathcal{F}\mathcal{W}\| \leq \alpha \quad &\Longleftrightarrow \quad \alpha W^2 - \mathcal{F}(W^2) \in B^+[H], \\ \|\mathcal{W}^{-1}\mathcal{F}\mathcal{W}\| < \alpha \quad &\Longleftrightarrow \quad \alpha W^2 - \mathcal{F}(W^2) \in G^+[H]. \end{aligned}$$

Proof. Take an arbitrary $W \in G^+[H]$ and set $\tilde{\mathcal{F}} = \mathcal{W}^{-1}\mathcal{F}\mathcal{W} \in B[B[H]]$, so that $\tilde{\mathcal{F}}(I) = W^{-1}\mathcal{F}(W^2)W^{-1} \in B[B[H]]$. Thus, for any $\alpha \in (0, \infty)$,

$$\langle (\alpha W^2 - \mathcal{F}(W^2))x; x\rangle = \langle (\alpha I - \tilde{\mathcal{F}}(I))Wx; Wx\rangle \qquad \forall x \in H.$$

Hence, $\alpha W^2 - \mathcal{F}(W^2) \in B^+[H]$ $(\in G^+[H])$ if and only if $\alpha I - \tilde{\mathcal{F}}(I) \in B^+[H]$ $(\in G^+[H])$, which in turn is equivalent (cf. [7]) to $\|\tilde{\mathcal{F}}(I)^{1/2}\| \leq \alpha^{1/2}$ $(< \alpha^{1/2})$, since $\tilde{\mathcal{F}}(I) \in B^+[H]$ by Proposition (P_1). However, by Proposition (P_2), $\|\tilde{\mathcal{F}}\| = \|\tilde{\mathcal{F}}(I)\| = \|\tilde{\mathcal{F}}(I)^{1/2}\|^2$. Thus, the above inequality is equivalent to $\|\tilde{\mathcal{F}}\| \leq \alpha$ $(< \alpha)$, which completes the proof. □

4. CONCLUSION

In this final section we shall conclude the anounced proof for the equivalence between assertions (2) and (3) with $\mathcal{F} = \mathcal{F}_F$ replaced by $\mathcal{F} = \mathcal{F}_F + \mathcal{T}$. As commented on section 2, this supplies a necessary and sufficient condition for the convergence preserving property between input and state correlation sequences, as required in the mean-square stability problem, for infinite-dimensional discrete bilinear systems.

Theorem. *The following assertions are equivalent.*

(a) $r_\sigma(\mathcal{F}) < 1$.

(b) *$\mathcal{F}$ is similar to a strict contraction.*

(c) *There exists $W \in G^+[H]$ such that $W - \mathcal{F}(W) \in G^+[H]$.*

(d) *For every $V \in G^+[H]$ there exists $W \in G^+[H]$ such that*

$$V = W - \mathcal{F}(W).$$

Moreover, if the above holds, then the solution $W \in G^+[H]$ of the Lyapunov-like equation $V = W - \mathcal{F}(W)$, for any $V \in G^+[H]$, is unique and given by

$$W = \sum_{j=0}^{\infty} \mathcal{F}^j(V) \quad = \quad (\mathcal{I} - \mathcal{F})^{-1}(V)$$

(with $\mathcal{I}$ standing for the identity in $B[B[H]]$), where the above convergence is in the uniform topology of $B[B[H]]$.

Proof. It is trivially verified that (d) $\Longrightarrow$ (c), and (c) $\Longrightarrow$ (b) according to the previous Lemma. Since similarity preserves the spectrum, $r_\sigma(\mathcal{F}) = r_\sigma(\mathcal{W}^{-1}\mathcal{F}\mathcal{W}) \leq \|\mathcal{W}^{-1}\mathcal{F}\mathcal{W}\|$ for every $\mathcal{W} \in G[B[H]]$, so that (b) $\Longrightarrow$ (a). To verify that (a) $\Longrightarrow$ (d), proceed as follows. Suppose $r_\sigma(\mathcal{F}) < 1$ (so that $\sum_{j=0}^{\infty} \|\mathcal{F}^j\| < \infty$) and take an arbitrary $V \in G^+[H]$. According to Proposition (P_1) (with $W = I$) we get by induction on j that $\mathcal{F}^j(V) \in B^+[H]$ for every $j \geq 0$. Then, for every $n \geq 0$,

$$\|(V^{1/2})^{-1}\|^2\|x\|^2 \;\; \leq \;\; \|V^{1/2}x\|^2 \;\; \leq \;\; \sum_{j=0}^{n} \|\mathcal{F}^j(V)^{1/2}x\|^2 \;\; \leq \;\; \Bigg(\sum_{j=0}^{\infty} \|\mathcal{F}^j\|\Bigg)\|V\|\|x\|^2$$

for all $x \in H$. Hence, we may define a map $\Phi : H \to \ell_2(H)$, given by

$$\Phi x = \bigoplus_{j=0}^{\infty} \mathcal{F}^j(V)^{1/2}x \qquad \forall x \in H,$$

which is clearly linear and bounded, with

$$\|\Phi x\|^2_{\ell_2(H)} = \sum_{j=0}^{\infty} \|\mathcal{F}^j(V)^{1/2}x\|^2 \qquad \forall x \in H,$$

so that it has a bounded inverse on its (closed) range. Thus, $\Phi^*\Phi \in G^+[H]$. By the continuity of the inner product we get (cf. proof of Proposition (P_1)), for every $x \in H$,

$$\begin{aligned}\langle \mathcal{F}(\Phi^*\Phi)x; x\rangle &= \Big\langle \sum_{k,\ell=0}^{\infty} \langle \mathbf{C}\mathbf{e}_\ell; \mathbf{e}_k\rangle F_k \Phi^*\Phi F_\ell^* x; x\Big\rangle \\ &= \sum_{k,\ell=0}^{\infty} \langle \mathbf{C}\mathbf{e}_\ell; \mathbf{e}_k\rangle \; \langle \Phi F_\ell^* x; \Phi F_k^* x\rangle_{\ell_2(H)}.\end{aligned}$$

However, for each $k, \ell \geq 0$ and every $x \in H$,

$$\begin{aligned}\langle \Phi F_\ell^* x ; \Phi F_k^* x\rangle_{\ell_2(H)} &= \sum_{j=0}^{\infty} \langle \mathcal{F}^j(V)^{1/2} F_\ell^* x ; \mathcal{F}^j(V)^{1/2} F_k^* x\rangle \\ &= \sum_{j=0}^{\infty} \langle F_k \mathcal{F}^j(V) F_\ell^* x ; x\rangle \;\;=\;\; \Big\langle F_k \Big(\sum_{j=0}^{\infty} \mathcal{F}^j(V)\Big) F_\ell^* x ; x\Big\rangle,\end{aligned}$$

since $\{\sum_{j=0}^{n} \mathcal{F}^j ;\; n \geq 0\}$ converges in $B[B[H]]$ whenever $r_\sigma(\mathcal{F}) < 1$. Therefore,

$$\begin{aligned}\langle \mathcal{F}(\Phi^*\Phi) x ; x\rangle &= \Big\langle \sum_{k,\ell=0}^{\infty} \langle \mathbf{C}\mathbf{e}_\ell ; \mathbf{e}_k\rangle F_k \Big(\sum_{j=0}^{\infty} \mathcal{F}^j(V)\Big) F_\ell^* x ; x\Big\rangle \\ &= \Big\langle \mathcal{F}\Big(\sum_{j=0}^{\infty} \mathcal{F}^j(V)\Big) x ; x\Big\rangle = \sum_{j=0}^{\infty} \langle \mathcal{F}^{j+1}(V) x ; x\rangle \\ &= \sum_{j=1}^{\infty} \|\mathcal{F}^j(V)^{1/2} x\|^2 = \|\Phi x\|^2_{\ell_2(H)} - \|V^{1/2} x\|^2\end{aligned}$$

for every $x \in H$, so that

$$\langle (V - \Phi^*\Phi + \mathcal{F}(\Phi^*\Phi)) x ; x\rangle = \|V^{1/2} x\|^2 - \|\Phi x\|^2_{\ell_2(H)} + \langle \mathcal{F}(\Phi^*\Phi) x ; x\rangle = 0$$

for all $x \in H$. Hence, $V = \Phi^*\Phi - \mathcal{F}(\Phi^*\Phi)$. Thus, (a) $\Longrightarrow$ (d) with $W = \Phi^*\Phi \in G^+[H]$. Moreover, such an operator is unique. Indeed, if $V = W - \mathcal{F}(W) \in G^+[H]$ for some $W \in G^+[H]$, then

$$\|\mathcal{F}^j(V)^{1/2} x\|^2 = \langle \mathcal{F}^j(V) x ; x\rangle = \|\mathcal{F}^j(W)^{1/2} x\|^2 - \|\mathcal{F}^{j+1}(W)^{1/2} x\|^2$$

for all $x \in H$ and every $j \geq 0$. Hence, for every $x \in H$,

$$\|\Phi x\|^2_{\ell_2(H)} = \lim_{n\to\infty} \sum_{j=0}^{n} \|\mathcal{F}^j(V)^{1/2} x\|^2 = \lim_{n\to\infty} \Big(\|W^{1/2} x\|^2 - \|\mathcal{F}^{n+1}(W)^{1/2} x\|^2\Big) = \|W^{1/2} x\|^2,$$

since $\|\mathcal{F}^n(W)^{1/2} x\|^2 \leq \|\mathcal{F}^n\|\, \|W\|\, \|x\|^2 \to 0$ as $n \to \infty$, because $r_\sigma(\mathcal{F}) < 1$. Therefore,

$$\langle (\Phi^*\Phi - W)\, x ; x\rangle = \|\Phi x\|^2_{\ell_2(H)} - \|W^{1/2} x\|^2 = 0 \qquad \forall x \in H,$$

so that $W = \Phi^*\Phi$, which proves uniqueness. Finally, if $W \in G^+[H]$ is the solution of $V = W - \mathcal{F}(W) \in G^+[H]$, then $\sum_{j=0}^{n} \mathcal{F}^j(V) = \sum_{j=0}^{n} \big(\mathcal{F}^j(W) - \mathcal{F}^{j+1}(W)\big) = W - \mathcal{F}^{n+1}(W) \in B^+[H]$ for each $n \geq 0$. Thus,

$$\Big\|\sum_{j=0}^{n} \mathcal{F}^j(V) - W\Big\| = \|\mathcal{F}^{n+1}(W)\| \leq \|\mathcal{F}^{n+1}\|\, \|W\| \to 0 \quad as \quad n \to \infty,$$

because $r_\sigma(\mathcal{F}) < 1$. Hence, $\sum_{j=0}^{n} \mathcal{F}^j(V) \to W$ as $n \to \infty$ in $B[H]$. However, since $r_\sigma(\mathcal{F}) < 1$ actually implies that $(\mathcal{I} - \mathcal{F}) \in G[B[H]]$ and that $\{\sum_{j=0}^{n} \mathcal{F}^j \in B[B[H]];\ n \geq 0\}$ converges in $B[B[H]]$ to $(\mathcal{I} - \mathcal{F})^{-1} \in G[B[H]]$, we get the final claim. $\square$

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