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CONSTRAINED-INPUT CONSTRAINED-STATE STABILITY IN BANACH SPACE*

C.S. Kubrusly[1,2] and P.C.M. Vieira[1]

[1] Department of Research and Development, National Lab. for Scientific Comp. - LNCC, Rio de Janeiro, 22290, Brazil.

[2] Department of Electrical Engineering, Catholic University - PUC/RJ, Rio de Janeiro, 22453, Brazil.

Abstract: This paper deals with the class of all discrete time-varying bounded linear systems in a Banach space, for which the state sequence remains in a bounded region whenever the input sequence is constrained to (another) bounded region. A necessary and sufficient condition for membership to such a class is given, as well as a full description of the subclasses of it obtained by fixing "a priori" those bounded regions.

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1. INTRODUCTION

Consider a discrete time-varying linear system modelled as follows.

$$\text{(M)} \qquad x(n+1) = F(n)x(n) + u(n+1), \quad x(0) = u(0),$$

where, for each integer $n \geq 0$, $x(n)$ and $u(n)$ lie in a Banach space X, and $F(n)$ is a bounded linear operator from X into itself. Let D and E be bounded sets in X containing the origin as an interior point. Now consider the following (stability) property.

$$\text{(S)} \qquad u(n) \in D \quad \forall n \geq 0 \qquad \Longrightarrow \qquad x(n) \in E \quad \forall n \geq 0.$$

In this paper we shall be addressing to the following problems.

(I) Give a necessary and sufficient condition, on the model (M), for the existence of a pair (D, E) for which property (S) holds.

(II) Given a pair (D, E), describe the class of all models (M) such that property (S) holds.

The motivation for considering these problems is twofold. First, from the point of view of stability theory, it is shown in section 3 that the necessary and sufficient condition in problem (I) is actually uniform stability (here we are using "uniform stability" as a generic term meaning any of the equivalent concepts of stability in the uniform topology – e.g. see [3]). Secondly, from the point of view of optimal control theory, the solution of problems (I) and (II) is, in fact, fundamental for the existence of optimal strategies under state and control constraints (e.g. see [4,5]).

The paper is organized as follows. Notation and terminology are posed in section 2. Problem (I) is solved in section 3. The description required in problem (II) is given in section 4, whenever the sets D and E are (closed or open) balls in X. However, such a description comes out in terms of the norm of an operator $\mathcal{F}$ on $\ell_\infty(X)$ which, in general, is not straightforward computable. Section 5 deals with such a computational task, and an illustrative example is given in section 6.

2. NOTATION AND TERMINOLOGY

Given a linear space Y, $L[Y]$ will stand for the algebra of all linear transformations from Y into itself. Throughout this paper X will stand for a Banach space, and $B[X]$ will denote the Banach algebra of all bounded linear operators from X into itself. We shall use $\| \; \|$ to denote both the norm in X and the induced uniform norm in $B[X]$. By an o-neighbourhood in X we mean a subset of X for which the origin is an interior point, and by an o-ball in X we mean a local o-neighbourhood in X (i.e. B_ρ is an o-ball in X iff it is a nontrivial closed or open ball of radius $\rho > 0$ centered at the origin in X).

Let $s(X)$ be the linear space of all X-valued sequences $\mathbf{x} = \{x(n) \in X; n \geq 0\}$. For any $A \subset X$, set $s(A) = \{\mathbf{x} \in s(X) : x(n) \in A \;\; \forall n \geq 0\}$: the subset of $s(X)$ made up of all A-valued sequences. As usual, set $\ell_\infty(X) = \{\mathbf{x} \in s(X) : \sup_{n \geq 0} \| x(n) \| < \infty\}$ which, with its standard norm $\| \mathbf{x} \|_\infty = \sup_{n \geq 0} \| x(n) \|$, is the Banach space of all bounded X-valued sequences. Note that A is bounded in X iff $s(A) \subset \ell_\infty(X)$, and A is a bounded o-neighbourhood in X iff $s(A)$ is a bounded o-neighbourhood in $\ell_\infty(X)$. In particular, B_ρ is an o-ball (open or closed) in X iff $s(B_\rho)$ is an o-ball (open or closed, respectively) in $\ell_\infty(X)$, and these balls have the same radius.

Given a sequence of operators in $B[X]$, $\mathbf{F} = \{F(k) \in B[X]; k \geq 0\} \in s(B[X])$, set $\Phi(k,k) = I$ (throughout this paper I will denote the identity in $B[X]$) for every integer $k \geq 0$, and

$$\Phi(n,j) = \prod_{k=j}^{n-1} F(k) = F(n-1) \ldots F(j)$$

for every integers $0 \leq j < n$, so that

$$\Phi(n,j) = \Phi(n,k)\;\Phi(k,j)$$

for every integers j,k,n such that $0 \leq j \leq k \leq n$. The double sequence of operators $\{\Phi(n,j) \in B[X]; 0 \leq j \leq n\}$ is usually referred to as the *evolution operator process* associated with $\mathbf{F} \in s(B[X])$. Now consider a sequence of (linear) maps $\{\mathcal{F}_n : s(X) \to X; n \geq 0\}$ recursively defined as follows.

$$\mathcal{F}_{n+1} = F(n)\mathcal{F}_n + \mathcal{E}_{n+1}, \qquad \mathcal{F}_0 = \mathcal{E}_0,$$

where, for each $n \geq 0$, the (linear) map $\mathcal{E}_n : s(X) \to X$ is such that $\mathcal{E}_n \mathbf{u} = u(n)$ for all $\mathbf{u} = (u(0), u(1), \ldots) \in s(X)$. It is readily verified by induction that

$$\mathcal{F}_n = \sum_{j=0}^{n} \Phi(n,j)\mathcal{E}_j \qquad \forall n \geq 0.$$

Such a sequence defines a transformation $\mathcal{F} : s(X) \to s(X)$, given by

$$\mathcal{F}\mathbf{u} = \big(\mathcal{F}_0\mathbf{u}, \mathcal{F}_1\mathbf{u}, \ldots\big) \qquad \forall \mathbf{u} \in s(X),$$

which is clearly linear. $\mathcal{F} \in L[s(X)]$ will be referred to as the *input-state operator matrix* associated with $\mathbf{F} \in s(B[X])$. Note that $\mathcal{F}$ is invertible. Actually, $\mathcal{F}^{-1} \in L[s(X)]$ is given by

$$\mathcal{F}^{-1}\mathbf{x} = \big(\mathcal{E}_0\mathbf{x}, \big(\mathcal{E}_1 - F(0)\mathcal{E}_0\big)\mathbf{x}, \big(\mathcal{E}_2 - F(1)\mathcal{E}_1\big)\mathbf{x}, \ldots\big) \qquad \forall \mathbf{x} \in s(X).$$

Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ will be identified with the following infinite matrices of operators.

$$\mathcal{F} = \begin{pmatrix} I & & & & \\ F(0) & I & & & \\ \Phi(2,0) & F(1) & I & & \\ \Phi(3,0) & \Phi(3,1) & F(2) & I & \\ & & & & \ddots \end{pmatrix}, \quad \mathcal{F}^{-1} = \begin{pmatrix} I & & & & \\ -F(0) & I & & & \\ & -F(1) & I & & \\ & & -F(2) & I & \\ & & & & \ddots \end{pmatrix}$$

We shall use the same notation $\mathcal{F}$ $(\mathcal{F}^{-1})$ for the restriction of $\mathcal{F}$ $(\mathcal{F}^{-1})$ on $\ell_\infty(X) \subset s(X)$. If $\mathcal{F} \in B[\ell_\infty(X)]$ $(\mathcal{F}^{-1} \in B[\ell_\infty(X)])$ then, according to our previous convention, $\| \mathcal{F} \|_\infty$ $(\|\mathcal{F}^{-1}\|_\infty)$ will stand for the induced uniform norm of $\mathcal{F}$ $(\mathcal{F}^{-1})$ in $B[\ell_\infty(X)]$. Finally note that $\mathbf{F} \in \ell_\infty(B[X])$ iff $\sup_{k\geq 0} \| F(k) \| < \infty$, and (as it is readily verified)

$$\mathcal{F}^{-1} \in B[\ell_\infty(X)] \quad \Longrightarrow \quad \|\mathcal{F}^{-1}\|_\infty = 1 + \sup_{k\geq 0} \| F(k) \|,$$

$$\mathcal{F} \in B[\ell_\infty(X)] \quad \Longrightarrow \quad \|\mathcal{F}^{-1}\|_\infty \leq \| \mathcal{F} \|_\infty.$$

Hence

$$\mathcal{F} \in B[\ell_\infty(X)] \quad \Longrightarrow \quad \mathbf{F} \in \ell_\infty(B[X]) \quad \Longleftrightarrow \quad \mathcal{F}^{-1} \in B[\ell_\infty(X)],$$

and the converse of the above unilateral implication fails (e.g. take $\mathbf{F}$ constantly equal to I, so that $\mathcal{F}_n \mathbf{u} = (n+1)u$ for any sequence $\mathbf{u} \in \ell_\infty(X)$ constantly equal to an arbitrary $u \in X$).

3. D-INPUT E-STATE STABILITY

Consider an operator sequence $\mathbf{F} = \{F(k) : k \geq 0\} \in s(B[X])$. Given $\mathbf{u} = \{u(n) : n \geq 0\} \in s(X)$, let $\mathbf{x} = \{x(n) : n \geq 0\} \in s(X)$ be recursively defined by the following nonautonomous inhomogeneous difference equation in X.

$$(1-\mathrm{a}) \qquad x(n+1) = F(n)x(n) + u(n+1), \quad x(0) = u(0),$$

whose solution is

$$(1-\mathrm{b}) \qquad x(n) = \sum_{j=0}^{n} \Phi(n,j)u(j) \qquad \forall n \geq 0,$$

so that

$$(1-\mathrm{c}) \qquad \mathbf{x} = \mathcal{F}\mathbf{u}.$$

Here, $\{\Phi(n,j) \in B[X];\ 0 \leq j \leq n\}$ and $\mathcal{F} \in L[s(X)]$ are the evolution operator process and the input-state operator matrix, associated with $\mathbf{F} \in s(B[X])$, respectively. Now take any integer $k \geq 0$ and an arbitrary $x \in X$. Set $\mathbf{v} \in s(X)$ such that $v(k) = x$ and $v(j) = 0$ for every $0 \leq j \neq k$, so that $\mathbf{y} = \mathcal{F}\mathbf{v}$ is given by $y(n) = \Phi(n,k)x$ for every $n \geq k$ and $y(n) = 0$ if $0 \leq n < k$, according to (1-b). Set $x_k(n) = y(k+n)$ for every $n \geq 0$, and $\mathbf{x}_k = \{x_k(n) : n \geq 0\} \in s(X)$. Thus, for each $k \geq 0$,

$$(2-\mathrm{b}) \qquad x_k(n) = \Phi(k+n,k)x \qquad \forall n \geq 0,$$

which is the solution of the following nonautonomous homogeneous difference equation in X:

$$(2-\mathrm{a}) \qquad x_k(n+1) = F(k+n)x_k(n), \qquad x_k(0) = x.$$

Definitions. The (free) model (2), or equivalently the sequence $\mathbf{F} \in s(B[X])$, is *uniformly asymptotically equistable* if the family of sequences $\{\mathbf{x}_k \in s(X) : k \geq 0\}$ is uniformly equiconvergent to zero, or equivalently if

$$\lim_{n\to\infty} \sup_{k\geq 0} \|\Phi(k+n,k)\| = 0.$$

The (forced) model (1), or equivalently the sequence $\mathbf{F} \in s(B[X])$, is *bounded-input bounded-state stable* if $\mathcal{F} \in L[s(X)]$ is $\ell_\infty(X)$-invariant. That is, if $\mathcal{F}(\ell_\infty(X)) \subseteq \ell_\infty(X)$, or equivalently if $\mathcal{F} \in L[\ell_\infty(X)]$, which means that

$$\mathbf{u} \in \ell_\infty(X) \quad \Longrightarrow \quad \mathcal{F}\mathbf{u} \in \ell_\infty(X).$$

It is *D-input E-state stable* if there exist bounded o-neighbourhoods D and E in X such that $\mathcal{F}(s(D)) \subseteq s(E)$, which means that

$$\mathbf{u} \in s(D) \quad \Longrightarrow \quad \mathcal{F}\mathbf{u} \in s(E)$$

(i.e. $u(n) \in D \ \ \forall n \geq 0 \Longrightarrow x(n) \in E \ \ \forall n \geq 0$). Note that D-input E-state stability does not require nor ensures that the above implication does hold for every pair of bounded o-neighbourhoods. In fact, as we shall see later in this section, it never does; and it holds for $D = E$ if and only if $\mathbf{F} = \mathbf{0}$.

There is in current literature a fairly complete collection of necessary and sufficient conditions for uniform asymptotic equistability, which turns out to be equivalent to bounded-input bounded-state stability. We display below just a few of those well-known results that will be required later in the sequel (for proof see e.g. [3]).

Proposition 1. *The following assertions are equivalent.*

(a) $\mathbf{F}$ *is uniformly asymptotically equistable.*

(b) $\lim_{n\to\infty}\sup_{k\geq 0}\|\Phi(k+n,k)\|^{1/n}<1$.

(c) $\sup_{n\geq 0}\sum_{j=0}^{n}\|\Phi(n,j)\|^{p}<\infty$ *for an arbitrary* $p>0$.

(d) $\mathbf{F}$ *is bounded-input bounded-state stable.*

(e) $\|\mathcal{F}\mathbf{u}\|_\infty \leq \mu\|\mathbf{u}\|_\infty \quad \forall \mathbf{u}\in\ell_\infty(X),$ *for some* $\mu>0$.

It is worth noticing that, for time-invariant models, which means that $\mathbf{F}\in s(B[X])$ is a constant sequence, say $F(k)=F\in B[X]$ for every $k\geq 0$, the definition of uniform asymptotic equistability is naturally reduced to *uniform asymptotic stability*,

$$\lim_{n\to\infty}\parallel F^n\parallel = 0,$$

which is the time-invariant version of (a); and assertions (b) and (c) become, respectively,

$$r_\sigma(F)<1,$$

$$\sum_{j=0}^{\infty}\parallel F^j\parallel^p<\infty \quad \textit{for an arbitrary} \quad p>0,$$

where $r_\sigma(F):=\lim_{n\to\infty}\parallel F^n\parallel^{1/n}$ is the spectral radius of $F\in B[X]$. Let us also remark that it is implicitly assumed in (a) and (b) that the nonnegative sequence $\{\sup_{k\geq 0}\|\Phi(k+n,k)\|; n\geq 0\}$ is well-defined (i.e. $\sup_{k\geq 0}\|\Phi(k+n,k)\|<\infty$ for every $n\geq 0$), so that (for $n=1$) $\sup_{k\geq 0}\|F(k)\|<\infty$. That is, $\mathbf{F}\in\ell_\infty(B[X])$. Moreover, also note that $\mathcal{F}\in B[\ell_\infty(X)]$ iff (e) holds (actually, $\|\mathcal{F}\|_\infty=\min\{\mu>0 : (e)\ \textit{holds}\}$, by definition). Thus we may add the following further auxiliary results.

Proposition 2. *Each of the assertions below is also equivalent to* (a)-(e).

(f) $\mathcal{F}\in B[\ell_\infty(X)]$.

(g) $\mathcal{F},\mathcal{F}^{-1}\in B[\ell_\infty(X)]$ (i.e. $\mathcal{F}$ *is a topological isomorphism on* $\ell_\infty(X)$).

Now let us draw our attention to D-input E-state stability, which lies somewhere between two extremes. One of them is obtained by relaxing the boundedness assumption on D and the o-neighbourhoodness assumption on E, and the other by assuming that D and E are both o-balls in X. We shall show that these (aparently weaker at one hand and stronger on the other hand) two extremes are actually equivalent to D-input E-state stability. This is important because we can supply a complete characterization of all pairs $(B_\delta, B_\varepsilon)$ of o-balls in X for which $\mathcal{F}(s(B_\delta))\subseteq s(B_\varepsilon)$.

Proposition 3. *Consider an operator sequence* $\mathbf{F}\in s(B[X])$. *Each of the assertions below is also equivalent to* (a)-(g).

(h) *There exist o-balls* B_δ *and* B_ε *in* X *such that*

$$\mathbf{u} \in s(B_\delta) \quad \Longrightarrow \quad \mathcal{F}\mathbf{u} \in s(B_\varepsilon).$$

(i) *There exist an o-neighbourhood* $D \subset X$ *and a bounded set* $E \subset X$ *such that*

$$\mathbf{u} \in s(D) \quad \Longrightarrow \quad \mathcal{F}\mathbf{u} \in s(E).$$

Moreover, if (i) *holds, then* $D \subseteq E$, *so that* (i) *is equivalent to*

(j) $\mathbf{F}$ *is D-input E-state stable*;

and $D = E$ *if and only if* $\mathbf{F} = \mathbf{0}$.

Proof. First note that (e) $\Rightarrow$ (h) $\Rightarrow$ (i) trivially. From now on suppose (i) holds. Take a closed o-ball B_ρ contained in D (whose existence is ensured because the origin is an interior point of D). Take an arbitrary $\mathbf{0} \neq \mathbf{u} \in \ell_\infty(X)$, and set $\mathbf{v} = (\rho/\|\,\mathbf{u}\,\|_\infty)\mathbf{u} \in \ell_\infty(X)$. Since $\|\mathbf{v}\|_\infty = \rho$, it follows that $\mathbf{v} \in s(B_\rho) \subseteq s(D)$. Hence $(\rho/\|\,\mathbf{u}\,\|_\infty)\mathcal{F}\mathbf{u} = \mathcal{F}\mathbf{v} \in s(E)$, whenever (i) holds. Therefore, $\mathcal{F}\mathbf{u} = (\|\,\mathbf{u}\,\|_\infty/\rho)\mathcal{F}\mathbf{v} \in (\|\,\mathbf{u}\,\|_\infty/\rho)s(E) = s((\|\,\mathbf{u}\,\|_\infty/\rho)E) \subset \ell_\infty(X)$ (because E is bounded). Thus (i) $\Rightarrow$ (d). Now take $u, v \in D$ and $k \geq 0$, arbitrary. Set $\mathbf{u}, \mathbf{v} \in s(D)$ as follows: $u(k) = u$ and $u(j) = 0$ for every $0 \leq j \neq k$; and $v(k) = u$, $v(k+1) = v$ and $v(j) = 0$ if either $0 \leq j < k$ or $j > k+1$. Thus $\mathbf{x} = \mathcal{F}\mathbf{u}$ and $\mathbf{y} = \mathcal{F}\mathbf{v}$ are such that $x(k) = u$, $x(k+1) = F(k)u$, and $y(k+1) = F(k)u + v$, according to (1-b). Since (i) holds, $\mathbf{x} = \mathcal{F}\mathbf{u} \in s(E)$ and $\mathbf{y} = \mathcal{F}\mathbf{v} \in s(E)$. Thus $x(k) \in E$, so that $u \in E$. Hence $D \subseteq E$. Moreover, $F(k)u = x(k+1) \in E$ and $F(k)u + v = y(k+1) \in E$. Therefore, if $D = E$, then $F(k)u \in D$ and $F(k)u + v \in D$ for all $u, v \in D$ and every $k \geq 0$, which implies (by induction on n) that $nF(k)u \in D$ for all $u \in D$ and every $k, n \geq 0$. Hence $F(k)u = 0$ for all $u \in D$ and every $k \geq 0$ (because $D = E$ is bounded). In particular, $\rho\|\,F(k)\,\| = \sup_{\|u\| \leq \rho} \|\,F(k)u\,\| = 0$ for every $k \geq 0$, so that $\mathbf{F} = \mathbf{0}$. On the other hand, if $\mathbf{F} = \mathbf{0}$, then $\mathcal{F}\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \ell_\infty(X)$, so that $\mathcal{F}(s(A)) = s(A)$ for every bounded $A \subset X$. ■

4. INPUT AND STATE CONSTRAINTS

Since all the stability concepts discussed above are equivalent, we shall use the generic term *uniform stability* to refer to any of them. Thus, consider the class

$$\Sigma = \{\mathbf{F} \in s(B[X]) : \mathbf{F} \text{ } is \text{ } uniformly \text{ } stable \}$$

of all uniformly stable models as in (1) or (2). Recall that uniform stability clearly implies that $\mathbf{F} \in \ell_\infty(B[X])$. Actually (cf. Proposition 2),

$$\Sigma = \{\mathbf{F} \in \ell_\infty(B[X]) : \mathcal{F} \in B[\ell_\infty(X)]\}.$$

Now, given a pair (δ, ε) of positive numbers, let B_δ and B_ε be either both closed or both open o-balls, and set

$$\Sigma_{\delta,\varepsilon} = \{\mathbf{F} \in s(B[X]) : \mathcal{F}(s(B_\delta)) \subseteq s(B_\varepsilon)\},$$

the class of all models as in (1) for which the state sequence remains in a closed (open) o-ball of radius ε whenever the input sequence is constrained to a closed (open) o-ball of radius δ (i.e. $\mathbf{u} \in s(B_\delta) \Rightarrow \mathbf{x} = \mathcal{F}\mathbf{u} \in s(B_\varepsilon)$); and

$$\Sigma^{-1}_{\varepsilon,\delta} = \{\mathbf{F} \in s(B[X]) : \mathcal{F}^{-1}(s(B_\varepsilon)) \subseteq s(B_\delta)\},$$

the class of all models as in (1) with the following property: the set of all input sequences for which the state sequence remains in a closed (open) o-ball of radius ε is itself constrained to a closed (open) o-ball of radius δ (i.e. $\mathbf{x} \in s(B_\varepsilon) \Rightarrow \mathbf{u} = \mathcal{F}^{-1}\mathbf{x} \in s(B_\delta)$). Our aim in this section is to characterize each of these classes.

Proposition 4. *For any pair (δ, ε) of positive numbers,*

$$\Sigma_{\delta,\varepsilon} = \{\mathbf{F} \in \Sigma : \| \mathcal{F} \|_\infty \leq \varepsilon/\delta\},$$

$$\Sigma^{-1}_{\varepsilon,\delta} = \{\mathbf{F} \in \ell_\infty(B[X]) : \|\mathcal{F}^{-1}\|_\infty \leq \delta/\varepsilon\}.$$

Proof. Let (δ, ε) be an arbitrary pair of positive numbers. If $\mathbf{F} \in \Sigma_{\delta,\varepsilon}$, then $\mathbf{F} \in \Sigma$, because (f) $\iff$ (h) in Propositions 2 and 3. Hence $\delta\| \mathcal{F} \|_\infty = \sup_{\mathbf{u}\in s(B_\delta)} \|\mathcal{F}\mathbf{u}\|_\infty \leq \varepsilon$. On the other hand, if $\mathbf{F} \in \Sigma$ is such that $\delta\|\mathcal{F}\|_\infty \leq \varepsilon$, then $\|\mathcal{F}\mathbf{u}\|_\infty \leq (\varepsilon/\delta)\| \mathbf{u} \|_\infty$ for all $\mathbf{u} \in \ell_\infty(X)$, so that $\mathcal{F}\mathbf{u} \in s(B_\varepsilon)$ whenever $\mathbf{u} \in s(B_\delta)$ (i.e. $\mathbf{F} \in \Sigma_{\delta,\varepsilon}$). Similarly, if $\mathbf{F} \in \Sigma^{-1}_{\varepsilon,\delta}$, then $\mathbf{F} \in \ell_\infty(B[X])$, because $\mathcal{F}^{-1} \in B[\ell_\infty(X)] \iff \mathbf{F} \in \ell_\infty(B[X])$. Hence $\varepsilon\|\mathcal{F}^{-1}\|_\infty = \sup_{\mathbf{x}\in s(B_\varepsilon)} \|\mathcal{F}^{-1}\mathbf{x}\| \leq \delta$. On the other hand, if $\mathbf{F} \in \ell_\infty(B[X])$ is such that $\varepsilon\|\mathcal{F}^{-1}\|_\infty \leq \delta$, then $\|\mathcal{F}^{-1}\mathbf{x}\|_\infty \leq (\delta/\varepsilon)\|\mathbf{x}\|_\infty$ for all $\mathbf{x} \in \ell_\infty(X)$, so that $\mathcal{F}^{-1}\mathbf{x} \in s(B_\delta)$ (i.e. $\mathbf{F} \in \Sigma^{-1}_{\varepsilon,\delta}$). ■

It is worth remarking that, whereas uniform stability is a necessary and sufficient condition for a given $\mathbf{F}$ to belong to $\Sigma_{\delta,\varepsilon}$ for some pair (δ, ε), such a condition is not necessary for $\mathbf{F}$ to belong to $\Sigma^{-1}_{\varepsilon,\delta}$ for any pair (δ, ε). Any $\mathbf{F} \in \ell_\infty(B[X])$ belongs to $\Sigma^{-1}_{\varepsilon,\delta}$ for some pair (δ, ε). This happens because $\mathcal{F}^{-1}(s(B_\varepsilon))$ may not be an o-neighbourhood if $\mathbf{F} \notin \Sigma$. Actually, $\mathcal{F}^{-1}(s(B_\varepsilon))$ is an o-neighbourhood for every o-ball B_ε if and only if $\mathcal{F} \in B[\ell_\infty(X)]$ (i.e. if and only if $\mathbf{F} \in \Sigma$). However, in such a case (i.e. for $\mathbf{F} \in \Sigma$),

$$s\big(B_{\varepsilon/\|\mathcal{F}\|_\infty}\big) \subseteq \mathcal{F}^{-1}(s(B_\varepsilon)) \subseteq s(B_{\varepsilon\|\mathcal{F}^{-1}\|_\infty}) \qquad \forall \varepsilon > 0,$$

$$s\big(B_{\delta/\|\mathcal{F}^{-1}\|_\infty}\big) \subseteq \mathcal{F}(s(B_\delta)) \subseteq s(B_{\delta\|\mathcal{F}\|_\infty}) \qquad \forall \delta > 0,$$

since, according to Proposition 4,

$$\min\big\{\varepsilon/\delta : \mathbf{F} \in \Sigma_{\delta,\varepsilon}\big\} = \| \mathcal{F} \|_\infty,$$

$$\min\left\{\delta/\varepsilon : \mathbf{F}\in\Sigma^{-1}_{\varepsilon,\delta}\right\} = \|\mathcal{F}^{-1}\|_\infty.$$

5. ESTIMATING $\|\mathcal{F}\|_\infty$

In order to determine exactly which operator sequences belong to $\Sigma_{\delta,\varepsilon}$ or to $\Sigma^{-1}_{\varepsilon,\delta}$, for a given pair (δ,ε), we need (according to Proposition 4) to supply an expression for $\|\mathcal{F}\|_\infty$ or for $\|\mathcal{F}^{-1}\|_\infty$ in terms of $\{\|\Phi(n,j)\|; 0\le j\le n\}$. As far as the class $\Sigma^{-1}_{\varepsilon,\delta}$ is concerned, this becomes a trivial task. Actually $\|\mathcal{F}^{-1}\|_\infty = 1+\sup_{k\ge 0}\|F(k)\|$ for every $\mathbf{F}\in\ell_\infty(B[X])$ (so that, $1\le\|\mathcal{F}^{-1}\|_\infty$ and $\|\mathcal{F}^{-1}\|_\infty=1\Leftrightarrow\mathbf{F}=\mathbf{0}$), as we have already seen in section 1. Moreover,

$$\Sigma_{\delta,\varepsilon}\subseteq\Sigma^{-1}_{\delta,\varepsilon},\qquad \Sigma^{-1}_{\delta,\varepsilon}=\emptyset\quad if\quad \varepsilon<\delta,\qquad \Sigma_{\delta,\delta}=\Sigma^{-1}_{\delta,\delta}=\{\mathbf{0}\},$$

for any pair $(\delta>0,\varepsilon>0)$, according to Proposition 4, since (cf. section 1) $\|\mathcal{F}^{-1}\|_\infty\le\|\mathcal{F}\|_\infty$ whenever $\mathcal{F}\in B[\ell_\infty(X)]$. As matter of fact, we can supply tighter lower bounds for

$$\|\mathcal{F}\|_\infty = \sup_{\|\mathbf{u}\|_\infty\le 1}\sup_{n\ge 0}\Big\|\sum_{j=0}^{n}\Phi(n,j)u(j)\Big\| = \sup_{n\ge 0}\sup_{\|\mathbf{u}\|_\infty\le 1}\Big\|\sum_{j=0}^{n}\Phi(n,j)u(j)\Big\|$$

($\mathcal{F}\in B[\ell_\infty(X)]$); and the search of such bounds is the central theme of this section. Let us first recall the following trivial bounds for the norm of $\mathcal{F}\in B[\ell_\infty(X)]$ (finiteness being ensured according to Propositions 1-c and 2-f).

$$\sup_{n\ge 0}\Big\|\sum_{j=0}^{n}\Phi(n,j)\Big\|\le\|\mathcal{F}\|_\infty\le\sup_{n\ge 0}\sum_{j=0}^{n}\|\Phi(n,j)\|<\infty.$$

Proposition 5. *If* $\mathcal{F}\in B[\ell_\infty(X)]$, *then*

(a)
$$\|\mathcal{F}^{-1}\|_\infty\le\sup_{\substack{0\le j\le n\\ \Phi(n,j)\ne 0}}\|\Phi(n,j)\|\sum_{k=j}^{n}\|\Phi(k,j)\|^{-1}\le\|\mathcal{F}\|_\infty.$$

If H *is a Hilbert space and* $\mathcal{F}\in B[\ell_\infty(H)]$, *then*

(b)
$$\sup_{n\ge 0}\sum_{j=0}^{n}\|\Phi(n,j)\|^2\le\|\mathcal{F}\|^2_\infty.$$

Proof. Take an arbitrary integers $0\le j\le n$ such that $\Phi(n,j)\ne 0$ (recall that $\Phi(k,k)\ne 0\quad\forall k\ge 0$). Thus there exists $u\in X$ such that $\Phi(n,j)u\ne 0$. Since

$\Phi(n,j) = \Phi(n,k)\Phi(k,j)$ for every $j \le k \le n$, it follows that $\Phi(k,j)u \ne 0$ for every $k \in [j,n]$ whenever $u \in X$ is such that $\Phi(n,j)u \ne 0$. Hence we get the lower bound for $\| \mathcal{F} \|_\infty$ in (a).

$$\sup_{m\ge 0}\sup_{\|\mathbf{u}\|_\infty\le 1}\|\sum_{k=0}^{m}\Phi(m,k)u(k)\| \ge \sup_{\substack{\|u\|\le 1\\ \Phi(n,j)u\ne 0}}\|\sum_{k=0}^{n}\Phi(n,k)\frac{\Phi(k,j)u}{\|\Phi(k,j)u\|}\|$$

$$= \sup_{\substack{\|u\|\le 1\\ \Phi(n,j)u\ne 0}}\|\Phi(n,j)u\|\sum_{k=j}^{n}\frac{1}{\|\Phi(k,j)u\|} \ge \|\Phi(n,j)\|\sum_{k=j}^{n}\|\Phi(k,j)\|^{-1}$$

for every integers $0 \le j \le n$ such that $\Phi(n,j) \ne 0$. The remaining result in (a) is readily verified by noting that $\{j \ge 0 : F(j) = \Phi(j+1,j) \ne 0\} \subset \{0 \le j \le n : \Phi(n,j) \ne 0\}$, and recalling that $\|\mathcal{F}^{-1}\|_\infty = \sup_{j\ge 0}(1 + \|F(j)\|)$. Now, in a Hilbert space setting, we claim that

$$\text{(b}'\text{)}\qquad \sum_{k=0}^{m}\|\Phi(n,n-k)\|^2 \le \sup_{\|\mathbf{u}\|_\infty\le 1}\|\sum_{k=0}^{m}\Phi(n,n-k)u(n-k)\|^2$$

for every integers $0 \le m \le n$. In particular, for $m = n$, we have (by setting $j = n-k$)

$$\sum_{j=0}^{n}\|\phi(n,j)\|^2 \le \sup_{\|\mathbf{u}\|_\infty\le 1}\|\sum_{j=0}^{n}\Phi(n,j)u(j)\|^2$$

for every $n \ge 0$, which ensures the lower bound for $\| \mathcal{F} \|_\infty^2$ in (b). Next we shall verify the claimed inequality in (b$'$). Take an arbitrary $n > 0$ (otherwise, i.e. if $n = 0$, (b$'$) holds trivially), and note that the inequality in (b$'$) holds for $m = 0$. Suppose it holds for some $m \in [0, n-1]$ then, according to the parallelogram law,

$$\sum_{k=0}^{m+1}\|\Phi(n,n-k)\|^2 = \sum_{k=0}^{m}\|\Phi(n,n-k)\|^2 + \|\Phi(n,n-m-1)\|^2$$

$$\le \sup_{\|\mathbf{u}\|_\infty\le 1}\|\sum_{k=0}^{m}\Phi(n,n-k)u(n-k)\|^2 + \sup_{\|v\|\le 1}\|\Phi(n,n-m-1)v\|^2$$

$$= \sup_{\|\mathbf{u}\|_\infty\le 1}\Big(\|\sum_{k=0}^{m}\Phi(n,n-k)u(n-k)\|^2 + \|\Phi(n,n-m-1)u(n-m-1)\|^2\Big)$$

$$\le \frac{1}{2}\Big[\sup_{\|\mathbf{u}\|_\infty\le 1}\|\sum_{k=0}^{m}\Phi(n,n-k)u(n-k) + \Phi(n,n-m-1)u(n-m-1)\|^2$$

$$+\sup_{\|\mathbf{u}\|_\infty\leq 1}\Big\|\sum_{k=0}^{m}\Phi(n,n-k)u(n-k)-\Phi(n,n-m-1)u(n-m-1)\Big\|^2\Big]$$

Since the above quantities between brackets are identical, the inequality in (b′) holds for $m+1\in[1,n]$, which concludes the proof by induction. ∎

Now take arbitrary integers $0\leq j<n$, and consider the following Properties.

$$(\mathrm{P}_1)\qquad \|\Phi(n,j)\|=\|\Phi(n,k)\|\,\|\Phi(k,j)\|\qquad \forall k\in(j,n),$$

$$(\mathrm{P}_2)\qquad \|\Phi(n,j)\|=\prod_{k=j}^{n-1}\|F(k)\|.$$

It is a simple matter to verify that (P_1) and (P_2) are equivalent. Therefore, we shall refer to them as "Property (P)". A sequence $\mathbf{F}\in s(B[X])$ will be called *collectively normaloid* if Property (P) holds for every integers $0\leq j<n$. It will be called *collectively paranormaloid* if Property (P) holds for every integers $0\leq j<n$ such that $\Phi(n,j)\neq 0$. Recall that an operator $F\in B[X]$ is normaloid (i.e. $r_\sigma(T)=\|F\|$, e.g. see [2: p.267]) if and only if $\|F^n\|=\|F\|^n$ for every $n\geq 0$. We shall say that $F\in B[X]$ is paranormaloid iff $\|F^n\|=\|F\|^n$ for every $n\geq 0$ such that $\|F^n\|\neq 0$. Thus, a constant sequence $\mathbf{F}=\{F(k)=F;k\geq 0\}\in s(B[X])$ is collectively normaloid (collectively paranormaloid) if and only if $F\in B[X]$ is normaloid (paranormaloid). Note that, if $\mathbf{F}\in s(B[X])$ is collectively normaloid, then it is obviously collectively paranormaloid. However, the converse fails. For instance, take a sequence $\mathbf{F}\in s(B[X])$ constantly equal to a nilpotent operator $F\in B[X]$, say of order 3 (i.e. $F,F^2\neq 0$ and $F^3=0$, so that $r_\sigma(F)=0$). Thus, since F is not normaloid, $\mathbf{F}$ is not collectively normaloid. However, if F is such that $\|F^2\|=\|F\|^2$ (e.g. take $F=\begin{pmatrix}0&\alpha&0\\0&0&\alpha\\0&0&0\end{pmatrix}\in B[\mathbb{C}^3]$ for any $0\neq\alpha\in\mathbb{C}$, so that $\|F^2\|=\|F\|^2=|\alpha|^2$ and $F^3=0$), then it is paranormaloid, and so $\mathbf{F}$ is collectively paranormaloid.

Proposition 6. *If* $\mathbf{F}\in\Sigma$ *is collectively paranormaloid, then*

$$\|\mathcal{F}\|_\infty=\sup_{n\geq 0}\sum_{j=0}^{n}\|\Phi(n,j)\|=1+\sup_{n\geq 1}\sum_{\substack{j=0\\ \Phi(n,j)\neq 0}}^{n-1}\prod_{k=j}^{n-1}\|F(k)\|.$$

Proof. Take arbitrary integers $0\leq j\leq n$ for which $\Phi(n,j)\neq 0$, so that $\Phi(k,j)\neq 0$ for every $k\in[j,n]$. Since $\mathbf{F}\in\Sigma$ is collectively paranormaloid, $\mathcal{F}\in B[\ell_\infty(X)]$ and Property (P_1) holds (note that for $k=j$ or $k=n$ the identity in (P_1) holds trivially). Thus

$$\|\Phi(n,j)\| \sum_{k=j}^{n} \|\phi(k,j)\|^{-1} = \sum_{k=j}^{n} \|\Phi(n,k)\|,$$

so that the lower bound for $\|\mathcal{F}\|_\infty$ in Proposition 5-a coincides with the upper bound $\sup_{n\geq 0} \sum_{j=0}^{n} \|\Phi(n,j)\|$. Moreover, since Property (P_2) holds whenever $\Phi(n,j) \neq 0$,

$$\|\mathcal{F}\|_\infty = \sup_{n\geq 0} \sum_{j=0}^{n} \|\Phi(n,j)\| = 1 + \sup_{n\geq 1} \sum_{\substack{j=0 \\ \Phi(n,j)\neq 0}}^{n-1} \|\Phi(n,j)\|$$

$$= 1 + \sup_{n\geq 1} \sum_{\substack{j=0 \\ \Phi(n,j)\neq 0}}^{n-1} \prod_{k=j}^{n-1} \|F(k)\|.$$

■

6. AN ILLUSTRATIVE EXAMPLE

To illustrate the estimates for $\|\mathcal{F}\|_\infty$ given in the preceding section, we shall consider here a simple example regarding time-invariant models operating in a Hilbert space H. Thus, let $\mathbf{F} \in \Sigma$ be a nontrivial uniformly stable sequence constantly equal to $F \in B[H]$ $(F \neq 0)$, so that $r_\sigma(F) < 1$. Let

$$\lambda_1 = \sup_{\substack{k\geq 0 \\ \|F^k\|\neq 0}} \|F^k\| \sum_{j=0}^{k} \|F^j\|^{-1},$$

$$\lambda_2 = \Big(\sum_{j=0}^{\infty} \|F^j\|^2\Big)^{1/2},$$

be the lower bounds for $\|\mathcal{F}\|_\infty$ in Proposition 5-a and 5-b, respectively. Let μ be the upper bound $\sup_{n\geq 0} \sum_{j=0}^{n} \|\Phi(n,j)\| < \infty$ for $\|\mathcal{F}\|_\infty$, so that

$$\mu = \sum_{j=0}^{\infty} \|F^j\|.$$

First note that, if F is paranormaloid, then Proposition 6 ensures that

$$\lambda_1 = \|\mathcal{F}\|_\infty = \mu = \begin{cases} (1-\|F\|)^{-1}, & \text{if } F \text{ is normaloid,} \\ m_F + 1, & \text{if } F \text{ is not normaloid and } \|F\| = 1, \\ \frac{1-\|F\|^{m_F+1}}{1-\|F\|}, & \text{if } F \text{ is not normaloid and } \|F\| \neq 1, \end{cases}$$

where $m_F := \max\{k \geq 0 : \|F^k\| \neq 0\}$ whenever F is paranormaloid but not normaloid. Moreover, since $\|F\| \neq 0$,

$$\lambda_2 = \Big(\sum_{\substack{j=0 \\ \|F^j\| \neq 0}}^{\infty} \|F\|^{2j} \Big)^{1/2} < \sum_{\substack{j=0 \\ \|F^j\| \neq 0}}^{\infty} \|F\|^{j} = \lambda_1.$$

Next, let us consider a concrete example. Set $H = \ell_2$, the Hilbert space made up of all one-sided square-summable complex sequences. Let

$$F = shift(\alpha, \beta, \beta, \ldots) \in B[\ell_2]$$

be a weighted shift on ℓ_2, with nonnegative weighting sequence $\{\alpha, \beta, \beta, \ldots\}$. Suppose $0 \leq \beta \leq \alpha$, so that (e.g. see [1: p.234]) $\|F\| = \alpha$ and $\|F^k\| = \alpha\beta^{k-1}$ for every $k > 1$. Hence $r_\sigma(F) = \beta$. Since $F \neq 0$ and $r_\sigma(F) < 1$, it follows that $0 < \alpha$ and $\beta < 1$. That is, suppose the pair (α, β) belongs to

$$\Omega := \{(\alpha, \beta) \in I\!R^2 : \quad 0 \leq \beta \leq \alpha, \quad 0 < \alpha, \quad \beta < 1\},$$

and note that, for any $(\alpha, \beta) \in \Omega$,

$$\lambda_2^2 = 1 + \alpha^2/(1-\beta^2), \qquad \mu = 1 + \alpha/(1-\beta),$$

$$\lambda_1 = \begin{cases} 1+\alpha & \text{if } \alpha \geq \beta/(1-\beta), \\ 1/(1-\beta) \geq 1+\alpha & \text{if } \alpha \leq \beta/(1-\beta). \end{cases}$$

Let β_0 ($\simeq 0.69$) be the root of $\beta^3 - 2\beta^2 - 2\beta + 2 = 0$ in (0,1). Since the functions $\beta/(1-\beta)$, $2(1-\beta^2)/\beta^2$, and $[\beta(1+\beta)(2-\beta)/(1-\beta)]^{1/2}$ meet at $\beta = \beta_0$, let α_0 ($\simeq 2.2$) be their common value at $\beta = \beta_0$. Now consider the following partition of Ω

$$\Gamma_0 := \{(\alpha, \beta) \in \Omega : \ \alpha = 2(1-\beta^2)/\beta^2 \geq \alpha_0 \ \textit{or} \ \alpha = [\beta(1+\beta)(2-\beta)/(1-\beta)]^{1/2} \geq \alpha_0\},$$

$$\Omega_1 := \{(\alpha, \beta) \in \Omega : \quad \alpha > 2(1-\beta^2)/\beta^2 \quad \textit{and} \quad \alpha > [\beta(1+\beta)(2-\beta)/(1-\beta)]^{1/2}\},$$

$$\Gamma := \{(\alpha, \beta) \in \Omega : \quad \beta = 0 \quad \textit{or} \quad \beta = \alpha\},$$

$$\Omega_2 := \Omega \setminus (\Gamma \cup \Gamma_0 \cup \Omega_1).$$

It is readily verified that, for any $(\alpha, \beta) \in \Omega$,

$$\lambda_2 < \mu, \qquad \textit{and} \qquad \lambda_1 = \mu \iff (\alpha, \beta) \in \Gamma.$$

Actually, if $\beta = 0$ then F is nilpotent of order 2 (i.e. $F^2 = 0$), and F is normaloid whenever $\beta = \alpha$. Hence, if $(\alpha, \beta) \in \Gamma$ then F is paranormaloid, so that

$$\|\mathcal{F}\|_\infty = \mu \qquad if \qquad (\alpha, \beta) \in \Gamma.$$

Moreover, it is also readily verified that, for any $(\alpha, \beta) \in \Omega$,

$$\lambda_2 < \lambda_1 \qquad \Longleftrightarrow \qquad (\alpha, \beta) \in \Omega_2 \cup \Gamma,$$

$$\lambda_2 = \lambda_1 \qquad \Longleftrightarrow \qquad (\alpha, \beta) \in \Gamma_0,$$

$$\lambda_1 < \lambda_2 \qquad \Longleftrightarrow \qquad (\alpha, \beta) \in \Omega_1.$$

Therefore, we get the following estimates for $\|\mathcal{F}\|_\infty$:

$$\lambda_1 \leq \|\mathcal{F}\|_\infty \leq \mu \quad if \quad (\alpha, \beta) \in \Omega_2 \cup \Gamma_0,$$

$$\lambda_2 \leq \|\mathcal{F}\|_\infty \leq \mu \quad if \quad (\alpha, \beta) \in \Omega_1 \cup \Gamma_0.$$

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