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STRONG STABILITY DOES NOT IMPLY SIMILARITY TO A CONTRACTION

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Abstract. We shall exhibit a strongly asymptotically stable discrete linear system whose model operator is not similar to a contraction.

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1. NOTATION AND TERMINOLOGY

Let H be a Hilbert space. Set $\ell_2(H) = \Sigma_{k=1}^{\infty} \oplus H$, the Hilbert space obtained by the direct sum of countably infinite copies of H, which is made up of all one-sided square-summable H-valued sequences. An element in $\ell_2(H)$ (i.e. a sequence $\{x_k \in H; k \geq 1\}$ such that $\Sigma_{k=1}^{\infty}\|x_k\|^2 < \infty$) will be denoted by $\Sigma_{k=1}^{\infty} \oplus x_k$. Let $B[H]$ be the Banach algebra of all bounded linear operators of H into itself. If $\{L_k \in B[H]; k \geq 1\}$ is a bounded sequence of operators, then its direct sum (i.e. the operator $L \in B[\ell_2(H)]$ such that $L\Sigma_{k=1}^{\infty} \oplus x_k = \Sigma_{k=1}^{\infty} \oplus L_k x_k$ for all $\Sigma_{k=1}^{\infty} \oplus x_k \in \ell_2(H)$) will be denoted by $L = \Sigma_{k=1}^{\infty} \oplus L_k$. Set $B^+[H] = \{A \in B[H] : A = A^* \textit{ and } 0 \leq \langle Ax; x\rangle \quad \forall x \in H\}$, the weakly closed cone of $B[H]$ made up of all self-adjoint nonnegative operators. If $A \in B^+[H]$ we shall write, as usual, $0 \leq A$. Let $G[H]$ be the group of all invertible operators from $B[H]$ (i.e. $G[H] = \{W \in B[H] : \exists W^{-1} \in B[H]\}$). Set $G^+[H] = B^+[H] \cap G[H] = \{Q \in B^+[H] : \alpha\|x\|^2 \leq \langle Qx; x\rangle \quad \forall x \in H, \textit{ for some } \alpha > 0\}$, the class of all strictly positive operators from $B[H]$. By a contraction (a strict contraction) we mean an operator $C \in B[H]$ such that $\|C\| \leq 1 \quad (\|C\| < 1)$. Thus, an operator $T \in B[H]$ is similar to a contraction (s.c.), or similar to a strict contraction (s.s.c.), if there exists $W \in G[H]$ such that $\|WTW^{-1}\| \leq 1$, or $\|WTW^{-1}\| < 1$, respectively. Recall that [6] $T \in B[H]$ is similar to a contraction if and only if there exists $Q \in G^+[H]$ such that

$$0 \leq Q - T^*QT.$$

2. INTRODUCTION

Consider a discrete time-invariant infinite-dimensional free bounded linear system modelled by the following autonomous homogeneous difference equation.

$$\text{(M)} \qquad x_{n+1} = T\,x_n, \qquad x_0 = x,$$

for every integer $n \geq 0$, where the initial condition x lies in a Hilbert space H and $T \in B[H]$. The model (M) (or equivalently, the operator T) is strongly asymptotically stable if the state sequence $\{x_n = T^n x \in H; n \geq 0\}$ converges to zero for all initial conditions $x \in H$. Moreover, if the state sequence converges to zero uniformly for all initial conditions $x \in H$, or equivalently if $\sup_{\|x\|\leq 1}\|T^n x\| \to 0 \textit{ as } n \to \infty$ (i.e. if $\|T^n\| \to 0 \textit{ as } n \to \infty$), then the model (M) (or the operator T) is said to be uniformly asymptotically stable. On the other hand, if the state sequence converges weakly to zero for all initial conditions $x \in H$ (i.e. if $\langle T^n x; y\rangle \to 0 \textit{ as } n \to \infty, \forall x, y \in H$), then the model (M) (or the operator T) is said to be weakly asymptotically stable. Therefore, asymptotic stability for a discrete linear system modelled as in (M) is equivalent to convergence to zero of the model operator power sequence $\{T^n \in B[H]; n \geq 0\}$. According to the topology for which the above sequence may converge to zero, it is naturally associated the concepts of weak, strong and uniform asymptotic stability. These will be denoted by $T^n \xrightarrow{w} 0$, $T^n \xrightarrow{s} 0$ and $T^n \xrightarrow{u} 0$, respectively; and are trivially related as follows: $T^n \xrightarrow{u} 0 \quad\Longrightarrow\quad T^n \xrightarrow{s} 0 \quad\Longrightarrow\quad T^n \xrightarrow{w} 0$.

Consider the Gelfand formula for the spectral radius of $T \in B[H]$ (i.e. $r_\sigma(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$), and recall that [8] $r_\sigma(T) = \inf_{W\in G[H]} \|WTW^{-1}\|$. Thus, T is similar to a strict contraction if and only if $r_\sigma(T) < 1$, which in turn is equivalent to uniform asymptotic stability (since $r_\sigma(T)^n = r_\sigma(T^n) \le \|T^n\| \quad \forall n \ge 0$). Therefore,

$$T^n \xrightarrow{u} 0 \quad\Longleftrightarrow\quad T \;\; is \;\; s.s.c.$$

In order to characterize weak asymptotic stability, it is natural to enquire whether the implication "$T^n \xrightarrow{u} 0 \Longrightarrow T$ *is s.s.c.*" can survive the relaxation of uniform asymptotic stability to weak asymptotic stability in one hand, and on the other hand, of similarity to a strict contraction to similarity to a contraction. Thus, the following

Question 1. $\qquad T^n \xrightarrow{w} 0 \quad\Longrightarrow\quad T \;\; is \;\; s.c.?$

has been raised in [7]. An affirmative answer to it would contribute not only to the characterization of weak asymptotic stability, but also to the characterization of similarity to a contraction, which is one of the problems posed in [5] that still remains unanswered. On the other hand, it has been noticed in [7] that there were partial evidences towards a negative answer to Question 1. Actually, had Question 1 an affirmative answer, then

Question 2. $\qquad T^n \xrightarrow{s} 0 \quad\Longrightarrow\quad T \;\; is \;\; s.c.?$

would obviously have an affirmative answer as well. However, an affirmative answer to it would lead us to a universal model for strong asymptotic stability (because, as it is readily verified by using the de Branges and Rovnyak theorem (cf. [2, p.23]), "$T \in B[H]$ is strongly asymptotically stable and similar to a contraction if and only if it is similar to part of a unilateral backward shift on $\ell_2(H)$"); which characterizes a partial evidence towards a negative answer even to Question 2. In fact, the answer to Question 2 is negative, so that it is also negative the answer to Question 1.

3. MAIN RESULT

Given an operator $F \in B[H]$, consider a unilateral backward shift of infinite multiplicity $T \in B[\ell_2(H)]$ constantly weighted by F. That is,

$$T\Sigma_{k=1}^{\infty} \oplus x_k = \Sigma_{k=1}^{\infty} \oplus F x_{k+1} \qquad\qquad \forall \Sigma_{k=1}^{\infty} \oplus x_k \in \ell_2(H).$$

We may identify T with the following infinite matrix of operators

$$T = \begin{pmatrix} 0 & F & & \\ & 0 & F & \\ & & 0 & F \\ & & & \ddots \end{pmatrix}.$$

Proposition. *Suppose H is a separable Hilbert space. T is similar to a contraction if and only if F is similar to a contraction.*

Proof. If F is similar to a contraction, then there exists $R \in G^+[H]$ such that

$$0 \leq R - F^*RF.$$

Set $D = \Sigma_{k=1}^{\infty} \oplus R \in G^+[\ell_2(H)]$, so that

$$0 \leq R \oplus \Sigma_{k=2}^{\infty} \oplus (R - F^*RF) = D - T^*DT.$$

Thus, T is similar to a contraction. On the other hand, if T is similar to a contraction, then there exists $Q \in G^+[\ell_2(H)]$ such that

$$0 \leq Q - T^*QT.$$

For each integer $k \geq 1$, let Q_k be the "kth block at the diagonal of Q". That is, set

$$Q_k = E_k Q E_k^*$$

in $G^+[H]$, where the bounded linear transformation $E_k : \ell_2(H) \to H$ is the co-isometry defined by

$$E_k \Sigma_{j=1}^{\infty} \oplus x_j = x_k \qquad \forall \Sigma_{j=1}^{\infty} \oplus x_j \in \ell_2(H).$$

Since $E_{k+1}T^* = F^*E_k$, we get

$$E_{k+1}(Q - T^*QT)E_{k+1}^* = Q_{k+1} - F^*Q_kF,$$

so that

$$0 \leq Q_{k+1} - F^*Q_kF$$

for every $k \geq 1$. Hence, by setting

$$R_m = \frac{1}{m} \sum_{k=1}^{m} Q_k$$

in $G^+[H]$, we have

$$\frac{1}{m}(Q_1 - Q_{m+1}) \leq R_m - F^*R_mF,$$

for every $m \geq 1$. Note that the sequence $\{R_m \in G^+[H]; m \geq 1\}$ has a weakly convergent subsequence, because H is separable and $\sup_{m\geq 1} \|R_m\| \leq \sup_{k\geq 1} \|Q_k\| \leq \|Q\|$ (e.g. see [1, p.99]). Since $B^+[H]$ is weakly closed in $B[H]$, the weak limit of any weakly convergent subsequence of $\{R_m \in G^+[H]; m \geq 1\}$ is in $B^+[H]$. Let $R_0 \in B^+[H]$ be the weak limit of a weakly convergent subsequence of $\{R_m \in G^+[H]; m \geq 1\}$. Thus, according to the above inequality,

$$0 \leq R_0 - F^* R_0 F,$$

because $\{Q_m \in G^+[H]; m \geq 1\}$ is bounded. Finally, note that $\{Q_k \in G^+[H]; k \geq 1\}$ is also bounded from below, since

$$\|Q^{-1}\|^{-1}\|x\|^2 \leq \langle QE_k^* x; E_k^* x\rangle = \langle Q_k x; x\rangle$$

for all $x \in H$ and every $k \geq 1$. Hence, $\|Q^{-1}\|^{-1}\|x\|^2 \leq \langle R_m x; x\rangle$ for all $x \in H$ and every $m \geq 1$, so that $R_0 \in G^+[H]$. Thus, F is similar to a contraction. ■

4. CONCLUSION

Given $F \in B[H]$, let $T \in B[\ell_2(H)]$ be defined as in the previous section. Since $\|T^n\| = \|F^n\|$ for every $n \geq 1$, it is readily verified that

$$\sup_{n\geq 1} \|F^n\| < \infty \quad \Longleftrightarrow \quad T^n \xrightarrow{s} 0.$$

Therefore, according to the above Proposition, in order to exhibit a strongly asymptotically stable operator $T \in B[\ell_2(H)]$ that is not similar to a contraction, all we need is to pick up a power bounded operator $F \in B[H]$ that is not similar to a contraction. An operator with such a property was given by Foguel [3]. Halmos has described it as follows [4]. Let K be a separable Hilbert space, and set $H = K \oplus K$. Let $S \in B[K]$ be a unilateral shift (of multiplicity 1) with respect to an orthonormal basis $\{e_k; k \geq 1\}$ for K, and let $P \in B[K]$ be the orthogonal projection with $range(P) = span\{e_j : j = k^3; k \geq 1\}$. The operator

$$F = \begin{pmatrix} S^* & P \\ 0 & S \end{pmatrix}$$

in $B[H]$ is power bounded but not similar to a contraction (cf. [3] and [4]). It is worth noticing that, by the very construction of F in [3], it is not weakly asymptotically stable (i.e. $F^n \overset{w}{\not\longrightarrow} 0$).

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